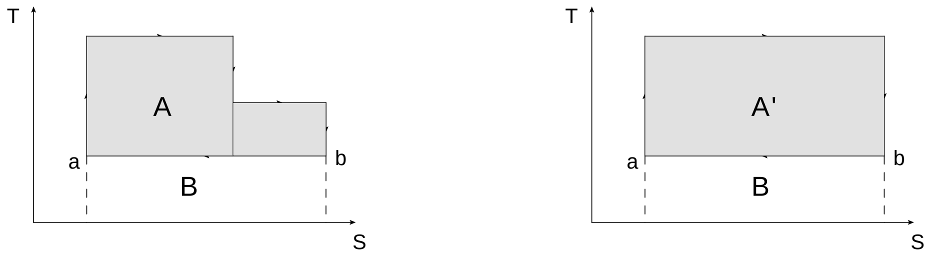
Phys. 807 — Statistical Mechanics

HW2 Solution

Huang 1.2 solution

The work done by the system going from point (a) to point (b) (see Fig. 1a) is

$$W_{a\to b} \;=\; Q_{a\to b} - E_b + E_a \;=\; \int_{a\to b} TdS - E_b + E_a \;=\; \text{Area (A + B)} - E_b + E_a$$



(a) (b)

The work done on the system in return $b \to a$ is

$$W_{b\to a} \;=\; -\int_{b\to a} TdS - E_a + E_b \;=\; -\,\text{Area B} - E_a + E_b$$

so the total work done by the system after a close cycle $a \to b \to a$ is

$$W \;=\; \oint TdS \;=\; \text{Area A + B} - \text{Area B} \;=\; \text{Area A}$$

The efficiency is

$$\eta \;=\; \frac{W}{Q_{a\to b}} \;=\; \frac{\text{Area A}}{\text{Area (A + B)}}$$

A pure Carnot cycle would correspond to rectangle shown in Fig. 1b so the efficiency is

$$\eta_C \;=\; \frac{W}{Q_{a\to b}} \;=\; \frac{\text{Area A}'}{\text{Area (A}' + \text{B)}} \;>\; \frac{\text{Area A}}{\text{Area (A + B)}} \;=\; \eta$$

Huang 1.5 solution

Using Eq. (3.71) (from lecture notes) we get

$$\left(\frac{\partial F}{\partial T}\right)_V = S \;\Rightarrow\; F(T,V) = F(T_0,V) - \int_{T_0}^{T} dT'\, S(T,V) = F(T_0,V) + \frac{RVT_0}{(a+1)V_0}\left[1 - \left(\frac{T}{T_0}\right)^{a+1}\right]$$

To find $F(T_0,V)$ we use the pressure $P = \frac{dW}{dV} = \frac{RT_0}{V}$ and the equation (3.71)

$$\left(\frac{\partial F}{\partial V}\right)_{T_0} = -P \;\Rightarrow\; F(T_0,V) = F(T_0,V_0) - \int_{V_0}^{V} dV'\, P(T_0,V') = F(T_0,V_0) - RT_0 \ln\frac{V}{V_0}$$

Neglecting the overall constant $F(T_0,V_0)$ we get

$$F(T,V) = RT_0 \ln\frac{V_0}{V} + \frac{RVT_0}{(a+1)V_0}\left[1 - \left(\frac{T}{T_0}\right)^{a+1}\right]$$

The pressure at arbitrary T is given by

$$P(V,T) = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{RT_0}{V} + \frac{RT_0}{(a+1)V_0}\left[1 - \left(\frac{T}{T_0}\right)^{a+1}\right]$$

so the equation of state is

$$PV = RT_0 + \frac{RT_0 V}{(a+1)V_0}\left[1 - \left(\frac{T}{T_0}\right)^{a+1}\right]$$

The work done at arbitrary T can be obtained as an integral

$$\int_{V_0}^{V} dV'\, P(T,V') = RT_0 \ln\frac{V}{V_0} + \frac{RT_0(V - V_0)}{(a+1)V_0}\left[1 - \left(\frac{T}{T_0}\right)^{a+1}\right]$$