Phys. 807 — Statistical Mechanics

HW8

Problem 1.

In the neighborhood of $z = 1$ the following expansion may be obtained

$$g_{5/2}(z) = 2.363\nu^{3/2} + 1.342 - 2.612\nu - 0.730\nu^2 + \dots \tag{1}$$

where $\nu = -\ln z$. From this, the corresponding expansions for $g_{3/2}, g_{1/2}$ and $g_{-1/2}$ may be obtained by the recursion formula $g_{n-1} = -\partial g_n/\partial\nu$. Using this expansion show that for the ideal Bose gas the discontinuity of $\partial C_V/\partial T$ at $T = T_c$ is given by

$$\left(\frac{\partial}{\partial T}\frac{C_V}{Nk}\right)_{T\to T_c^-} - \left(\frac{\partial}{\partial T}\frac{C_V}{Nk}\right)_{T\to T_c^+} = \frac{c}{T_c}\ .$$

and find constant c (numerically).

Solution

We start from Eq. (10.57) from the lecture notes rewritten as

$$\frac{C_V(T)}{Nk_B} = \begin{cases} \frac{15}{4}\,\frac{v}{\lambda^3}g_{5/2}(\alpha^*) - \frac{9}{4}\,\frac{g_{3/2}(\alpha^*)}{g_{1/2}(\alpha^*)} & T > T_c \\ \frac{15}{4}\,\frac{v}{\lambda^3}g_{5/2}(1) & T < T_c \end{cases} \tag{2}$$

where we used Eq. (10.34): $\frac{v}{\lambda^3} = g_{3/2}(\alpha^*)$.

Now

$$\left(\frac{\partial}{\partial T}\frac{C_V}{Nk}\right)_{T\to T_c^-} = \frac{15}{4}g_{5/2}(1)v\frac{\partial}{\partial T}\frac{1}{\lambda^3}\bigg|_{T=T_c}$$

and

$$\left(\frac{\partial}{\partial T}\frac{C_V}{Nk}\right)_{T\to T_c^+}$$
$$= \frac{15}{4}g_{5/2}(1)v\frac{\partial}{\partial T}\frac{1}{\lambda^3}|_{T=T_c} + \frac{\partial\alpha^*}{\partial T}(\frac{15}{4\alpha^*} - \frac{9}{4\alpha^*} + \frac{9}{4\alpha^*}\frac{g_{3/2}(\alpha_*)g_{-1/2}(\alpha_*)}{g_{1/2}^2(\alpha_*)})\Big|_{T\to T_c}$$

where we used $\alpha\frac{\partial}{\partial\alpha}g_n(\alpha) = g_{n-1}(\alpha)$.
Next, $\frac{d\alpha}{dT}$ is known from Eq. (10.55) from the lecture notes

$$\frac{d\alpha}{dT} = -\frac{3}{2T}\,g_{3/2}(\alpha)\,\frac{\alpha}{g_{1/2}(\alpha)} \tag{3}$$

so we get

$$\left(\frac{\partial}{\partial T}\frac{C_V}{Nk}\right)_{T\to T_c^-} - \left(\frac{\partial}{\partial T}\frac{C_V}{Nk}\right)_{T\to T_c^+}$$
$$= - \frac{9}{4T}\frac{g_{3/2}(\alpha^*)}{g_{1/2}(\alpha^*)}\Big|_{T\to T_c} - \frac{27}{8T}\frac{g_{3/2}^2(\alpha^*)g_{-1/2}(\alpha^*)}{g_{1/2}^3(\alpha^*)}\Big|_{T\to T_c}$$

Since $g_{3/2}(1) = 2.612$ and $g_{-1/2}(1) = \infty$ the first term vanishes as $T\to T_c$. To estimate the second term, we need the expansion of $g_{-1/2}$ and $g_{1/2}$ near $\alpha^*=1$ ($\equiv \nu\to 0$) which can be obtained from Eq. (1):

$$g_{1/2}(\alpha^*) = \frac{3}{4}(2.363)\nu^{-1/2} - 2(0.730) + \ldots$$
$$g_{-1/2}(\alpha^*) = \frac{3}{8}(2.363)\nu^{-3/2} + \ldots$$

We see that $\frac{g_{-1/2}(\alpha^*)}{g_{1/2}^3(\alpha^*)}\Big|_{\alpha^*\to 1} = \frac{8}{9(2.363)^2}$ so

$$\left(\frac{\partial}{\partial T}\frac{C_V}{Nk}\right)_{T\to T_c^-} - \left(\frac{\partial}{\partial T}\frac{C_V}{Nk}\right)_{T\to T_c^+} = -\frac{3}{T_c}(\frac{g_{3/2}(1)}{2.363})^2$$

and therefore

$$c \;=\; -3(\frac{2.612}{2.363})^2 \;\simeq\; -3.665$$

Problem 2.

Show that the equation of state of the ideal Bose gas phase can be written in the form of a virial expansion, i.e.,

$$\frac{Pv}{kT} = 1 + c_1\left(\frac{\lambda^3}{v}\right) + c_2\left(\frac{\lambda^3}{v}\right)^2 - \ldots\ .$$

and find constants c_1 and c_2.

Solution

The equation of state of an ideal Bose gas is given by Eq. (10.39) from the lecture notes (at $T > T_c$)

$$\frac{Pv}{k_BT} \;=\; \frac{v}{\lambda^3}g_{5/2}(\alpha_*) \;=\; \frac{v}{\lambda^3}[\alpha^* + \frac{\alpha_*^2}{2^{5/2}} + \frac{\alpha_*^3}{3^{5/2}} + ...]$$

On the other hand

$$\frac{\lambda^3}{v} \;=\; g_{3/2}(\alpha_*) \;=\; \alpha^* + \frac{\alpha_*^2}{2^{3/2}} + \frac{\alpha_*^3}{3^{3/2}} + ... \qquad (4)$$

so

$$\frac{Pv}{k_BT} \;=\; \frac{\alpha^* + \frac{\alpha_*^2}{2^{5/2}} + \frac{\alpha_*^3}{3^{5/2}} + ...}{\alpha^* + \frac{\alpha_*^2}{2^{3/2}} + \frac{\alpha_*^3}{3^{3/2}} + ...} \;=\; 1 - \frac{\alpha_*}{4\sqrt{2}} + \alpha_*^2(\frac{1}{16} - \frac{2}{9\sqrt{3}}) + O(\alpha_*^3) \qquad (5)$$

In order to get the equation of state in the form of virial expansion we must invert Eq. (4)

$$\alpha_* \;=\; \frac{\lambda^3}{v}[1 - \frac{1}{2\sqrt{2}}\frac{\lambda^3}{v} + O(\frac{\lambda^6}{v^2})]$$

and substitute it into Eq. (5). We get

$$\frac{Pv}{k_BT} \;=\; 1 - \frac{1}{4\sqrt{2}}\frac{\lambda^3}{v} + \frac{\lambda^6}{v^2}[\frac{1}{8} - \frac{2}{9\sqrt{3}}] \;+\; O(\frac{\lambda^9}{v^3})$$

so

$$c_1 \;=\; -\frac{1}{4\sqrt{2}} \qquad \text{and} \qquad c_2 \;=\; \frac{1}{8} - \frac{2}{9\sqrt{3}}$$