Phys. 807 — Statistical Mechanics

HW9 (due Tue Dec 6 at 4 p.m. in my mailbox).

Problem 1

Consider a two-dimensional electron gas in a magnetic field strong enough so that all particles can be accommodated in the lowest Landau level. Taking into account both orbital and spin paramagnetism, find the magnetization at absolute zero.

Solution

Magnetization due to orbital motion at zero temperature is given by Eq. (11.57) from the lecture notes. At $B > B_0$ we have

$$\mathcal{M}_{\text{Landau}} \;=\; -\mu N, \qquad \mu \;\equiv\; \frac{e\hbar}{2mc}$$

Magnetization due to Pauli paramagnetism is given by Eq. (11.76)

$$\mathcal{M} = \frac{1}{\beta}\frac{\partial}{\partial B}(\ln \mathcal{Z}_N) \;=\; N\frac{\partial}{\partial B}f(\bar{N}_+)$$

where $f(\bar{N}_+)$ is given by Eq. (11.75):

$$f(\bar{N}_+) \;=\; \frac{1}{N}\text{Max}\Big(\mu B(2N_+ - N) - F(N_+) - F(N - N_+)\Big)$$

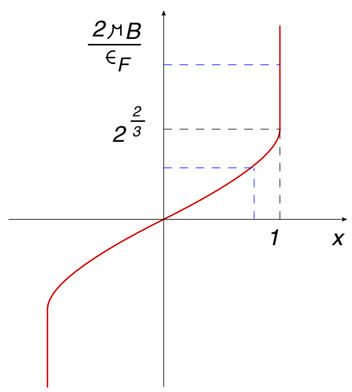
Here $F(N)$ is a free energy of the spinless gas. It obviously does not depend on magnetic field so at sufficiently large B maximum of $\mu B(2N_+ - N) - F(N_+) - F(N - N_+)$ occurs at $N_+ = N$ leading to

$$f(\bar{N}_+ = N) \;=\; \mu B$$

so the magnetization from spin effects is

$$\mathcal{M}_{\text{Pauli}} \;=\; \mu N \qquad \Rightarrow \qquad \mathcal{M}_{\text{Landau}} + \mathcal{M}_{\text{Pauli}} \;=\; 0$$

The graph on Fig. 48 for the equation for magnetization at arbitrary B looks like



Problem 2

The energy levels of an atom with a magnetic moment μ in the magnetic field B have the values

$$E_m = -mB\mu/j\,,$$

where j is a positive fixed integer number and m is an integer varying from $-j$ to $+j$. For n atoms per unit volume, find
(a) the magnetic moment M per unit volume of an ideal gas at the temperature T as a function of B
and
(b) the susceptibility

$$\chi = \frac{\partial M}{\partial B}\bigg|_{B=0} \,.$$

Solution

Similarly to Eq. (11.66) from the lecture notes we introduce the notations

$$\sum_{\vec{p}} n_{\vec{p},j} = N_j \quad , \sum_{\vec{p}} n_{\vec{p},j-1} = N_{j-1} \quad , \quad \sum_{\vec{p}} n_{\vec{p},-j} = N_{-j} \ .$$

so instead of Eq. (11.67) we get

$$E_N = \sum_{\vec{p}} \Big[\frac{p^2}{2M} \sum_{m=-j}^{j} n_{\vec{p},m}\Big] - \frac{1}{j}\mu B \sum_{m=-j}^{j} mN_m.$$

where M is the mass of the atom. If we assume that at realistic B the contribution from kinetic energies is small, we get

$$E_N = -\frac{1}{j}\mu B \sum_{m=-j}^{j} mN_m.$$

and the partition function takes the form

$$\mathcal{Z}_N = \sum_{N_j, N_{j-1}, .. N_{-j}} \delta_{N_j+N_{j-1}+..N_{-j}-N} \ \frac{N!}{N_j! N_{j-1}! ... N_{-j}!} e^{\frac{\beta}{j}\mu B \sum_{m=-j}^{j} mN_m}$$

where $\frac{N!}{N_j! N_{j-1}! ... N_{-j}!}$ is a combinatorial factor showing the number of ways to distribute N_j particles with spin projection j, N_{j-1} particles with spin projection $j-1$,... and N_{-j} particles with spin projection $-j$ among N particles. Using a well-known formula

$$(a_1 + a_2 + ... a_n)^N \ = \sum_{N_1, N_2, ... N_n} \frac{N!}{N_1! N_2! ... N_n!} \delta_{N_1+N_2+...N_n-N} a_1^{N_1} a_2^{N_2} .... a_N^{N_n}$$

we can reduce the partition function to

$$Z_N \ = \ \big(e^{\mu\beta B} + e^{\mu\beta B\frac{j-1}{j}} + e^{\mu\beta B\frac{j-2}{j}} + ... + e^{\mu\beta B\frac{1-j}{j}} + e^{-\mu\beta B}\big)^N \ = \ \left(\frac{\sinh\big(1+\frac{1}{2j}\big)\mu\beta B}{\sinh\frac{1}{2j}\mu\beta B}\right)^N$$

Actually, this formula can be derived in an easier way: the expression in parenthesis is the partition function for one atom (neglecting kinetic energy)

$$z_1 \;=\; e^{\mu\beta B} + e^{\mu\beta B\frac{j-1}{j}} + e^{\mu\beta B\frac{j-2}{j}} + ... + e^{\mu\beta B\frac{1-j}{j}} + e^{-\mu\beta B} \;=\; \frac{\sinh\left(1+\frac{1}{2j}\right)\mu\beta B}{\sinh\frac{1}{2j}\mu\beta B}$$

so for N non-interacting atoms we get

$$Z_N \;=\; z_1^N \;=\; \left(\frac{\sinh\left(1+\frac{1}{2j}\right)\mu\beta B}{\sinh\frac{1}{2j}\mu\beta B}\right)^N$$

The magnetization is given by Eq. (11.4) from the lecture notes

$$\mathcal{M} \;=\; \frac{1}{\beta V}\frac{\partial}{\partial B}\ln Z_N \;=\; \frac{\mu B}{\beta v}\left(\coth(1+\frac{1}{2j})\mu B\beta - \frac{1}{2j}\coth\frac{\mu B\beta}{2j}\right)$$

so the susceptibility takes the form

$$\chi = \left.\frac{\partial M}{\partial B}\right|_{B=0} \;=\; \frac{1}{\beta v}\left.\frac{\partial^2 \ln z_1}{\partial B^2}\right|_{B=0} \;=\; \frac{1}{\beta v}\left.\frac{\partial^2}{\partial B^2}\right|_{B=0}\ln\frac{1+\left(1+\frac{1}{2j}\right)^2\frac{\mu^2\beta^2B^2}{6}}{1+\frac{1}{4j^2}\frac{\mu^2\beta^2B^2}{6}} \;=\; \frac{j+1}{j}\frac{\mu^2\beta}{3v}$$