Due Thu Feb 19 at the lecture.

Problem 1.

A spin-$\frac{1}{2}$ particle is in an eigenstate of $\hat{S}_y$ with eigenvalue $\frac{\hbar}{2}$ at time $t = 0$. At that time it is placed in a constant magnetic field B in z direction. The spin is allowed to precess for a time T. At that instant, the magnetic field is switched very quickly to the x direction. After another time interval T, a measurement of the y component of the spin is made. What is the probability that the value $-\frac{\hbar}{2}$ will be found?

Solution.

The eigenstate of the operator S_y is $\Psi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$. Indeed,

$$\hat{\sigma}_y \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

The Hamiltonian for the first time interval T is $\hbar\omega_0\hat{\sigma}_z$ where $\omega_0 = \frac{egB}{4mc}$. The evolution of the state yields at time T

$$\Psi(T) = e^{-\frac{i}{\hbar}\hat{H}T}\Psi_0 = \begin{pmatrix} e^{-i\omega_0 T} & 0 \\ 0 & e^{i\omega_0 T} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega_0 T} \\ ie^{i\omega_0 T} \end{pmatrix}$$

The Hamiltonian for the second time interval is $\hbar\omega_0\hat{\sigma}_x$ so the evolution operator takes the form

$$e^{-\frac{i}{\hbar}\hat{H}T} = e^{-i\omega_0\hat{\sigma}_x T} = \cos\omega_0 T - i\sigma_x\sin\omega_0 T = \begin{pmatrix} \cos\omega_0 T & -i\sin\omega_0 T \\ -i\sin\omega_0 T & \cos\omega_0 T \end{pmatrix}$$

and therefore the state at time $2T$ is given by

$$\Psi(2T) = \begin{pmatrix} \cos\omega_0 T & -i\sin\omega_0 T \\ -i\sin\omega_0 T & \cos\omega_0 T \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega_0 T} \\ ie^{i\omega_0 T} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\omega_0 T e^{-i\omega_0 T} + \sin\omega_0 T e^{i\omega_0 T} \\ -i\sin\omega_0 T e^{-i\omega_0 T} + i\cos\omega_0 T e^{i\omega_0 T} \end{pmatrix}$$

The probability that the spin$-\frac{\hbar}{2}$ will be found is

$$\frac{1}{2}|\cos\omega_0 T e^{i\omega_0 T} - \sin\omega_0 T e^{-i\omega_0 T}|^2 = \frac{1}{4}(2 - \cos 4\omega_0 T)$$

Alternatively, at $t = T$ one can project onto eigenstates of σ_x operator:

$$\Psi(T) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega_0 T} \\ ie^{i\omega_0 T} \end{pmatrix} = \frac{1}{2\sqrt{2}}(e^{-i\omega_0 T} + ie^{i\omega_0 T})\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}(e^{-i\omega_0 T} - ie^{i\omega_0 T})\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The eigenstates of $\hat{\sigma}_x$ operator are evolved as

$$e^{-i\omega_0\hat{\sigma}_x T}\begin{pmatrix}1\\1\end{pmatrix} = e^{-i\omega_0 T}\begin{pmatrix}1\\1\end{pmatrix}$$

$$e^{-i\omega_0\hat{\sigma}_x T}\begin{pmatrix}1\\-1\end{pmatrix} = e^{i\omega_0 T}\begin{pmatrix}1\\-1\end{pmatrix}$$

so the state at $t = 2T$ is

$$\Psi(2T) = e^{-i\omega_0\hat{\sigma}_x T}\Psi(T) = \frac{1}{2\sqrt{2}}(e^{-2i\omega_0 T} + i)\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2\sqrt{2}}(1 - ie^{2i\omega_0 T})\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix}1 + i + e^{-2i\omega_0 T} - ie^{2i\omega_0 T}\\ i - 1 + e^{-2i\omega_0 T} + ie^{2i\omega_0 T}\end{pmatrix}$$

so the probability that the spin will be $-\frac{\hbar}{2}$ is

$$\frac{1}{8}|i - 1 + e^{-2i\omega_0 T} + ie^{2i\omega_0 T}|^2 = \frac{1}{4}(2 - \sin 4\omega_0 T)$$

Problem 2.

Two atoms with $j_1 = 1$ and $j_2 = 2$ are coupled, with an energy described by $\hat{H} = a\hat{\vec{J}}_1 \cdot \hat{\vec{J}}_2$ $(a > 0)$. Determine all of the energies and degeneracies for the coupled system. What are the eigenstates corresponding to maximal and minimal energy?

Solution.

The Hamiltonian can be rewritten as

$$\hat{H} = \frac{a}{2}(\hat{J}^2 - \hat{J}_1^2 - \hat{J}_2^2)$$

where $\hat{J} = \hat{J}_1 + \hat{J}_2$. The eigenstates of the system are $|j, m\rangle$ where $j_1 + j_2 \geq j \geq |j_1 - j_2|$ which gives $j = 1, 2, 3$ in our case. The corresponding degeneracy is $2j + 1$ so we have:

7 states with j=3 have highest energy $\frac{a}{2}(12 - 6 - 2) = 2a$,

5 states with j=2 have energy $\frac{a}{2}(6 - 6 - 2) = -a$, and

3 states with j=1 have lowest energy $\frac{a}{2}(2 - 2 - 6) = -3a$.

Check: the total number of states is $15 = (2j_1 + 1)(2j_2 + 1)$