HW5 solution

Problem. A three spin-$\frac{1}{2}$ particles are in the state

$$|\uparrow\rangle|\downarrow\rangle|\uparrow\rangle$$

(a) What are possible results and (their probabilities) for measurement of J^2 in this state?

(b) Same question about measurement of J_x.

Solution. (a)

Acting on the $|\uparrow\rangle|\uparrow\rangle|\uparrow\rangle$ state with $\hat{J}_-$ operator we get a state with $|j,m\rangle = |\frac{3}{2},\frac{1}{2}\rangle$

$$|\frac{3}{2},\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}(|\downarrow\rangle|\uparrow\rangle|\uparrow\rangle + |\uparrow\rangle|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle) \quad (1)$$

There are two orthogonal states which can be chosen as

$$\begin{aligned} |\frac{1}{2},\frac{1}{2}\rangle_1 &= \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle|\uparrow\rangle - |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle) \\ |\frac{1}{2},\frac{1}{2}\rangle_2 &= \frac{1}{\sqrt{6}}(-2|\downarrow\rangle|\uparrow\rangle|\uparrow\rangle + |\uparrow\rangle|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle) \quad (2) \end{aligned}$$

This choice of states corresponds to addition of spins of particle #2 and particle #3 with subsequent addition of spin of particle #1. Indeed, if we add spins of two last particles we get states (here it is convenient to trade $|\uparrow\rangle|\uparrow\rangle$ notations for $|\frac{1}{2},\frac{1}{2}\rangle$ ones)

$$|1,1\rangle = |\frac{1}{2},\frac{1}{2}\rangle|\frac{1}{2},\frac{1}{2}\rangle, \quad |1,0\rangle = \frac{1}{\sqrt{2}}(|\frac{1}{2},\frac{1}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle + |\frac{1}{2},-\frac{1}{2}\rangle|\frac{1}{2},\frac{1}{2}\rangle)$$

$$|1,-1\rangle = |\frac{1}{2},-\frac{1}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle, \quad |1,0\rangle = \frac{1}{\sqrt{2}}(|\frac{1}{2},\frac{1}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle - |\frac{1}{2},-\frac{1}{2}\rangle|\frac{1}{2},\frac{1}{2}\rangle)$$

Now we add spins $\frac{1}{2}$ and 1 and we are interested only in states with $m = \frac{1}{2}$. As usual, we start with

$$|\frac{3}{2},\frac{3}{2}\rangle = |\frac{1}{2},\frac{1}{2}\rangle|1,1\rangle$$

Acting on this state with lowering operator we get

$$\begin{aligned}\hat{J}_-|\frac{3}{2},\frac{3}{2}\rangle &= (J_-^{(1)}|\frac{1}{2},\frac{1}{2}\rangle)|1,1\rangle + |\frac{1}{2},\frac{1}{2}\rangle(J_-^{(2)}|1,1\rangle)\\ &= |\frac{1}{2},-\frac{1}{2}\rangle|1,1\rangle + \sqrt{2}|\frac{1}{2},\frac{1}{2}\rangle|1,0\rangle = \sqrt{3}|\frac{3}{2},\frac{1}{2}\rangle\end{aligned}$$

and therefore

$$|\frac{3}{2},\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}|\frac{1}{2},-\frac{1}{2}\rangle|1,1\rangle + \frac{\sqrt{2}}{\sqrt{3}}|\frac{1}{2},\frac{1}{2}\rangle|1,0\rangle$$

The state that we denoted by $|\frac{1}{2},\frac{1}{2}\rangle_2$

$$|\frac{1}{2},\frac{1}{2}\rangle_2 = -\frac{\sqrt{2}}{\sqrt{3}}|\frac{1}{2},-\frac{1}{2}\rangle|1,1\rangle + \frac{1}{\sqrt{3}}|\frac{1}{2},\frac{1}{2}\rangle|1,0\rangle$$

is obviously orthogonal to $|\frac{3}{2},\frac{1}{2}\rangle$. However, there is another state orthogonal to $|\frac{3}{2},\frac{1}{2}\rangle$: the result of the addition of particle 1 with spin $\frac{1}{2}$ and spin-0 combination of particles 2 and 3. Addition of spin 0 is trivial so

$$|\frac{1}{2},\frac{1}{2}\rangle_2 = |\frac{1}{2},\frac{1}{2}\rangle\frac{1}{\sqrt{2}}(|\frac{1}{2},-\frac{1}{2}\rangle - |-\frac{1}{2},\frac{1}{2}\rangle)$$

Returning to $\uparrow, \downarrow$ notations we can see from Eqs. (1) and (2) that

$$|\uparrow\rangle|\downarrow\rangle|\uparrow\rangle \;=\; \frac{1}{\sqrt{3}}|\frac{3}{2},\frac{1}{2}\rangle + \frac{1}{\sqrt{2}}|\frac{1}{2},\frac{1}{2}\rangle_1 + \frac{1}{\sqrt{6}}|\frac{1}{2},\frac{1}{2}\rangle_2$$

so the result of the measurement of J^2 is $j=\frac{3}{2}$ with probability $\frac{1}{3}$ and $j=\frac{1}{2}$ with probability

$$|_1\langle\frac{1}{2},\frac{1}{2}|\uparrow\rangle|\downarrow\rangle|\uparrow\rangle|^2 + |_2\langle\frac{1}{2},\frac{1}{2}|\uparrow\rangle|\downarrow\rangle|\uparrow\rangle|^2 \;=\; \frac{1}{2}+\frac{1}{6} \;=\; \frac{2}{3}$$

(b)

Let us turn out head on the angle $\frac{\pi}{2}$ around Y direction, then the problem looks like following: what are possible results of measurement of J_z (and probabilities) in the state $|-\frac{1}{2}\rangle_x|\frac{1}{2}\rangle_x|-\frac{1}{2}\rangle_x$ which we will denote as $|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle$. We will use the formula

$$|\rightarrow\rangle \;=\; \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle, \qquad |\leftarrow\rangle \;=\; \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$$

It is clear that the probability to get projection $\frac{3}{2}\hbar$ on z direction is

$$|\langle\uparrow|\langle\uparrow|\langle\uparrow|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 \;=\; |\langle\uparrow|\leftarrow\rangle_x|^2\,|\langle\uparrow|\rightarrow\rangle|^2\,|\langle\uparrow\leftarrow\rangle|^2 \;=\; \frac{1}{8}$$

The probability to have projection of spin $s_z=\frac{1}{2}\hbar$ is

$$|_1\langle\frac{3}{2},\frac{1}{2}|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 + |_1\langle\frac{1}{2},\frac{1}{2}|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 + |_2\langle\frac{1}{2},\frac{1}{2}|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2$$
$$=\; \frac{1}{3}|(\langle\downarrow|\langle\uparrow|\langle\uparrow| + \langle\uparrow|\langle\downarrow|\langle\uparrow| + \langle\uparrow|\langle\uparrow|\langle\downarrow|)|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2$$

$$
\begin{aligned}
&+ \frac{1}{2}|(\langle\uparrow|\langle\downarrow|\langle\uparrow| - \langle\uparrow|\langle\uparrow|\langle\downarrow|)|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 \\
&+ \frac{1}{6}|(-2\langle\downarrow|\langle\uparrow|\langle\uparrow| + \langle\uparrow|\langle\downarrow|\langle\uparrow| + \langle\uparrow|\langle\uparrow|\langle\downarrow|)|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 \\
&= \frac{1}{24} + \frac{1}{4} + \frac{1}{12} = \frac{3}{8}
\end{aligned} \tag{3}
$$

To find the probability of $s_z = -\frac{1}{2}$ we need to know the states

$$
\begin{aligned}
|\frac{3}{2}, \frac{1}{2}\rangle &= \frac{1}{\sqrt{3}}(|\downarrow\rangle|\downarrow\rangle|\uparrow\rangle + |\downarrow\rangle|\uparrow\rangle|\downarrow\rangle + |\uparrow\rangle|\downarrow\rangle|\downarrow\rangle) \\
|\frac{1}{2}, -\frac{1}{2}\rangle_1 &= \frac{1}{\sqrt{2}}(|\downarrow\rangle|\downarrow\rangle|\uparrow\rangle - |\downarrow\rangle|\uparrow\rangle|\downarrow\rangle) \\
|\frac{1}{2}, -\frac{1}{2}\rangle_2 &= \frac{1}{\sqrt{6}}(2|\uparrow\rangle|\downarrow\rangle|\downarrow\rangle - |\downarrow\rangle|\downarrow\rangle|\uparrow\rangle - |\downarrow\rangle|\uparrow\rangle|\downarrow\rangle)
\end{aligned} \tag{4}
$$

so the probability to have projection of spin $s_z = -\frac{1}{2}\hbar$ is

$$
\begin{aligned}
&|_1\langle\frac{3}{2}, -\frac{1}{2}|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 + |_1\langle\frac{1}{2}, -\frac{1}{2}|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 + |_2\langle\frac{1}{2}, -\frac{1}{2}|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 \\
&= \frac{1}{3}|(\langle\downarrow|\langle\downarrow|\langle\uparrow| + \langle\uparrow|\langle\downarrow|\langle\downarrow| + \langle\downarrow|\langle\uparrow|\langle\downarrow|)|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 \\
&+ \frac{1}{2}|(\langle\downarrow|\langle\downarrow|\langle\uparrow| - \langle\downarrow|\langle\uparrow|\langle\downarrow|)|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 \\
&+ \frac{1}{6}|(2\langle\uparrow|\langle\downarrow|\langle\downarrow| - \langle\downarrow|\langle\downarrow|\langle\uparrow| - \langle\downarrow|\langle\uparrow|\langle\downarrow|)|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 \\
&= \frac{1}{24} + \frac{1}{4} + \frac{1}{12} = \frac{3}{8}
\end{aligned} \tag{5}
$$

and finally the probability to have $s_z = -\frac{3}{2}\hbar$ is

$$
|\langle\downarrow|\langle\downarrow|\langle\downarrow||\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 = \frac{1}{8}
$$