

Problem. A three spin- $\frac{1}{2}$ particles are in the state

$$|\uparrow\rangle|\downarrow\rangle|\uparrow\rangle$$

(a) What are possible results and (their probabilities) for measurement of J^2 in this state?

(b) Same question about measurement of J_x .

Solution. (a)

Acting on the $|\uparrow\rangle|\uparrow\rangle|\uparrow\rangle$ state with \hat{J}_- operator we get a state with

$$|j, m\rangle = \left|\frac{3}{2}, \frac{1}{2}\right\rangle$$

$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(|\downarrow\rangle|\uparrow\rangle|\uparrow\rangle + |\uparrow\rangle|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle) \quad (1)$$

There are two orthogonal states which can be chosen as

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle_1 = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle|\uparrow\rangle - |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle)$$

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle_2 = \frac{1}{\sqrt{6}}(-2|\downarrow\rangle|\uparrow\rangle|\uparrow\rangle + |\uparrow\rangle|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle|\uparrow\rangle|\downarrow\rangle) \quad (2)$$

This choice of states corresponds to addition of spins of particle #2 and particle #3 with subsequent addition of spin of particle #1.

Indeed, if we add spins of two last particles we get states (here it is convenient to trade $|\uparrow\rangle|\uparrow\rangle$ notations for $|\frac{1}{2}, \frac{1}{2}\rangle$ ones)

$$|1, 1\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle\left|\frac{1}{2}, \frac{1}{2}\right\rangle, \quad |1, 0\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, \frac{1}{2}\right\rangle\left|\frac{1}{2}, -\frac{1}{2}\right\rangle + \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\left|\frac{1}{2}, \frac{1}{2}\right\rangle\right)$$

$$|1, -1\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle|\frac{1}{2}, -\frac{1}{2}\rangle, \quad |1, 0\rangle = \frac{1}{\sqrt{2}}(|\frac{1}{2}, \frac{1}{2}\rangle|\frac{1}{2}, -\frac{1}{2}\rangle - |\frac{1}{2}, -\frac{1}{2}\rangle|\frac{1}{2}, \frac{1}{2}\rangle)$$

Now we add spins $\frac{1}{2}$ and 1 and we are interested only in states with $m = \frac{1}{2}$. As usual, we start with

$$|\frac{3}{2}, \frac{3}{2}\rangle = |\frac{1}{2}, \frac{1}{2}\rangle|1, 1\rangle$$

Acting on this state with lowering operator we get

$$\begin{aligned} \hat{J}_-|\frac{3}{2}, \frac{3}{2}\rangle &= (J_-^{(1)}|\frac{1}{2}, \frac{1}{2}\rangle)|1, 1\rangle + |\frac{1}{2}, \frac{1}{2}\rangle(J_-^{(2)}|1, 1\rangle) \\ &= |\frac{1}{2}, -\frac{1}{2}\rangle|1, 1\rangle + \sqrt{2}|\frac{1}{2}, \frac{1}{2}\rangle|1, 0\rangle = \sqrt{3}|\frac{3}{2}, \frac{1}{2}\rangle \end{aligned}$$

and therefore

$$|\frac{3}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{3}}|\frac{1}{2}, -\frac{1}{2}\rangle|1, 1\rangle + \frac{\sqrt{2}}{\sqrt{3}}|\frac{1}{2}, \frac{1}{2}\rangle|1, 0\rangle$$

The state that we denoted by $|\frac{1}{2}, \frac{1}{2}\rangle_2$

$$|\frac{1}{2}, \frac{1}{2}\rangle_2 = -\frac{\sqrt{2}}{\sqrt{3}}|\frac{1}{2}, -\frac{1}{2}\rangle|1, 1\rangle + \frac{1}{\sqrt{3}}|\frac{1}{2}, \frac{1}{2}\rangle|1, 0\rangle$$

is obviously orthogonal to $|\frac{3}{2}, \frac{1}{2}\rangle$. However, there is another state orthogonal to $|\frac{3}{2}, \frac{1}{2}\rangle$: the result of the addition of particle 1 with spin $\frac{1}{2}$ and spin-0 combination of particles 2 and 3. Addition of spin 0 is trivial so

$$|\frac{1}{2}, \frac{1}{2}\rangle_2 = |\frac{1}{2}, \frac{1}{2}\rangle\frac{1}{\sqrt{2}}(|\frac{1}{2}, -\frac{1}{2}\rangle - |-\frac{1}{2}, \frac{1}{2}\rangle)$$

Returning to \uparrow, \downarrow notations we can see from Eqs. (1) and (2) that

$$|\uparrow\rangle|\downarrow\rangle|\uparrow\rangle = \frac{1}{\sqrt{3}}|\frac{3}{2}, \frac{1}{2}\rangle + \frac{1}{\sqrt{2}}|\frac{1}{2}, \frac{1}{2}\rangle_1 + \frac{1}{\sqrt{6}}|\frac{1}{2}, \frac{1}{2}\rangle_2$$

so the result of the measurement of J^2 is $j = \frac{3}{2}$ with probability $\frac{1}{3}$ and $j = \frac{1}{2}$ with probability

$$|{}_1\langle\frac{1}{2}, \frac{1}{2}|\uparrow\rangle|\downarrow\rangle|\uparrow\rangle|^2 + |{}_2\langle\frac{1}{2}, \frac{1}{2}|\uparrow\rangle|\downarrow\rangle|\uparrow\rangle|^2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

(b)

Let us turn out head on the angle $\frac{\pi}{2}$ around Y direction, then the problem looks like following: what are possible results of measurement of J_z (and probabilities) in the state $|-\frac{1}{2}\rangle_x|\frac{1}{2}\rangle_x|-\frac{1}{2}\rangle_x$ which we will denote as $|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle$. We will use the formula

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle, \quad |\leftarrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$$

It is clear that the probability to get projection $\frac{3}{2}\hbar$ on z direction is

$$|\langle\uparrow|\langle\uparrow|\langle\uparrow|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 = |\langle\uparrow|\leftarrow\rangle_x|^2 |\langle\uparrow|\rightarrow\rangle|^2 |\langle\uparrow|\leftarrow\rangle|^2 = \frac{1}{8}$$

The probability to have projection of spin $s_z = \frac{1}{2}\hbar$ is

$$\begin{aligned} & |{}_1\langle\frac{3}{2}, \frac{1}{2}|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 + |{}_1\langle\frac{1}{2}, \frac{1}{2}|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 + |{}_2\langle\frac{1}{2}, \frac{1}{2}|\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 \\ &= \frac{1}{3}(|\langle\downarrow|\langle\uparrow|\langle\uparrow| + \langle\uparrow|\langle\downarrow|\langle\uparrow| + \langle\uparrow|\langle\uparrow|\langle\downarrow||\leftarrow\rangle|\rightarrow\rangle|\leftarrow\rangle|^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} |(\langle \uparrow | \langle \downarrow | \langle \uparrow | - \langle \uparrow | \langle \uparrow | \langle \downarrow |) | \leftarrow \rangle | \rightarrow \rangle | \leftarrow \rangle|^2 \\
& + \frac{1}{6} |(-2 \langle \downarrow | \langle \uparrow | \langle \uparrow | + \langle \uparrow | \langle \downarrow | \langle \uparrow | + \langle \uparrow | \langle \uparrow | \langle \downarrow |) | \leftarrow \rangle | \rightarrow \rangle | \leftarrow \rangle|^2 \\
& = \frac{1}{24} + \frac{1}{4} + \frac{1}{12} = \frac{3}{8}
\end{aligned} \tag{3}$$

To find the probability of $s_z = -\frac{1}{2}$ we need to know the states

$$\begin{aligned}
|\frac{3}{2}, \frac{1}{2}\rangle &= \frac{1}{\sqrt{3}}(|\downarrow\rangle|\downarrow\rangle|\uparrow\rangle + |\downarrow\rangle|\uparrow\rangle|\downarrow\rangle + |\uparrow\rangle|\downarrow\rangle|\downarrow\rangle) \\
|\frac{1}{2}, -\frac{1}{2}\rangle_1 &= \frac{1}{\sqrt{2}}(|\downarrow\rangle|\downarrow\rangle|\uparrow\rangle - |\downarrow\rangle|\uparrow\rangle|\downarrow\rangle) \\
|\frac{1}{2}, -\frac{1}{2}\rangle_2 &= \frac{1}{\sqrt{6}}(2|\uparrow\rangle|\downarrow\rangle|\downarrow\rangle - |\downarrow\rangle|\downarrow\rangle|\uparrow\rangle - |\downarrow\rangle|\uparrow\rangle|\downarrow\rangle) \quad (4)
\end{aligned}$$

so the probability to have projection of spin $s_z = -\frac{1}{2}\hbar$ is

$$\begin{aligned}
& |{}_1\langle \frac{3}{2}, -\frac{1}{2} | \leftarrow \rangle | \rightarrow \rangle | \leftarrow \rangle|^2 + |{}_1\langle \frac{1}{2}, -\frac{1}{2} | \leftarrow \rangle | \rightarrow \rangle | \leftarrow \rangle|^2 + |{}_2\langle \frac{1}{2}, -\frac{1}{2} | \leftarrow \rangle | \rightarrow \rangle | \leftarrow \rangle|^2 \\
& = \frac{1}{3} |(\langle \downarrow | \langle \downarrow | \langle \uparrow | + \langle \uparrow | \langle \downarrow | \langle \downarrow | + \langle \downarrow | \langle \uparrow | \langle \downarrow |) | \leftarrow \rangle | \rightarrow \rangle | \leftarrow \rangle|^2 \\
& + \frac{1}{2} |(\langle \downarrow | \langle \downarrow | \langle \uparrow | - \langle \downarrow | \langle \uparrow | \langle \downarrow |) | \leftarrow \rangle | \rightarrow \rangle | \leftarrow \rangle|^2 \\
& + \frac{1}{6} |(2\langle \uparrow | \langle \downarrow | \langle \downarrow | - \langle \downarrow | \langle \downarrow | \langle \uparrow | - \langle \downarrow | \langle \uparrow | \langle \downarrow |) | \leftarrow \rangle | \rightarrow \rangle | \leftarrow \rangle|^2 \\
& = \frac{1}{24} + \frac{1}{4} + \frac{1}{12} = \frac{3}{8}
\end{aligned} \tag{5}$$

and finally the probability to have $s_z = -\frac{3}{2}\hbar$ is

$$|\langle \downarrow | \langle \downarrow | \langle \downarrow | | \leftarrow \rangle | \rightarrow \rangle | \leftarrow \rangle|^2 = \frac{1}{8}$$