Due Thu Apr 2 at the lecture.

Problem 1.

Two particles of masses m_1 and m_2 move in one dimension and are not subject to any external forces. The potential energy of interaction between the particles is given by

$$V(x_1, x_2) \;=\; \begin{cases} 0 & \text{if} \quad |x_1 - x_2| < a \\ \infty & \text{if} \quad |x_1 - x_2| > a \end{cases}$$

Obtain expressions for the energy eigenvalues and eigenfunctions of this system if its total momentum is P.

Problem 2.

Repeat the calculation done in Problem 1 for the case where the two particles have the same mass m and are (i) indistinguishable spin-zero bosons and (ii) indistinguishable spin-half fermions.

Solution

1.

The wavefunction is a product of free wave function

$$e^{i\frac{p}{\hbar}(x_1+x_2)}$$

of motion of center of mass and the wavefunction of "effective particle" with mass $\mu = \frac{m_1 m_2}{m_1+m_2}$, see Eq. (15.20) from the lecture notes. The potential for "effective particle" is

$$V(x) \;=\; \begin{cases} 0 & \text{if} \quad |x| < a \\ \infty & \text{if} \quad |x| > a \end{cases}$$

where $x \equiv x_1 - x_2$. This is an infinite well with width $2a$ so the eigenfunctions are (see Eq. (5.33) from the lecture notes)

$$\psi_n(x) \;=\; \theta(a - |x_1 - x_2|)\frac{1}{\sqrt{a}} \begin{cases} \cos\frac{n\pi x}{2a} & \text{for } n \text{ odd} \\ \sin\frac{n\pi x}{2a} & \text{for } n \text{ even} \end{cases} \tag{1}$$

with energies

$$E_{p,n} \;=\; \frac{p^2}{2(m_1+m_2)} + \frac{\hbar^2 k_n^2}{2\mu} \;=\; \frac{p^2}{2(m_1+m_2)} + \frac{\hbar^2\pi^2 n^2}{8\mu a^2}\,. \tag{2}$$

Thus, the eigenfunctions are

$$\psi(x_1, x_2) \;=\; \theta(a - |x_1 - x_2|)e^{i\frac{p}{\hbar}(x_1+x_2)}\frac{1}{\sqrt{a}} \begin{cases} \cos\frac{n\pi(x_1-x_2)}{2a} & \text{for } n \text{ odd} \\ \sin\frac{n\pi(x_1-x_2)}{2a} & \text{for } n \text{ even} \end{cases}$$

2.

If the particles are indistinguishable spin-0 bosons with $m_1 = m_2 = m$, the wavefunction must be symmetric with replacement $1 \leftrightarrow 2$ so only even $n = 2k$ are allowable and the wavefunction is

$$\psi_{p,k}(x_1, x_2) \;=\; \theta(a - |x_1 - x_2|)e^{i\frac{p}{\hbar}(x_1+x_2)}\frac{1}{\sqrt{a}}\sin\frac{k\pi(x_1 - x_2)}{2a}, \qquad k = 1, 2, 3, ...$$

If the particles are indistinguishable spinors with spin $\frac{1}{2}$ the product of coordinate wavefunction and spin wavefunction must be antisymmetric with respect to $1 \leftrightarrow 2$ replacement, so we need even $n = 2k$ for the total spin 0 and odd $n = 2k + 1$ for the total spin 1. The wavefunctions are

$$\psi_{p,k}(x_1, x_2; 1, m) \;=\; \theta(a-|x_1-x_2|)e^{i\frac{p}{\hbar}(x_1+x_2)}\frac{1}{\sqrt{a}}\sin\frac{k\pi(x_1 - x_2)}{a}\begin{cases} |\uparrow\rangle|\uparrow\rangle & m = 1 \\ \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle) & m = 0 \\ |\downarrow\rangle|\downarrow\rangle & m = -1 \end{cases}, \qquad k = 1, 2, 3, ...$$

and

$$\psi_{p,k}(x_1, x_2; 0, 0) \;=\; \theta(a-|x_1-x_2|)e^{i\frac{p}{\hbar}(x_1+x_2)}\frac{1}{\sqrt{a}}\cos\frac{(2k + 1)\pi(x_1 - x_2)}{2a}(|\uparrow\rangle|\downarrow\rangle-|\downarrow\rangle|\uparrow\rangle), \qquad k = 1, 2, 3, ...$$