Problems and solutions

Problem 1.

Consider a system of spin $\frac{1}{2}$. What are the eigenstates and eigenvalues of the operator S_x+S_y? Suppose a measurement of this quantity is made, and the system is found to be in the eigenstate with the larger eigenvalue. What is the probability that a subsequent measurement of S_y yields $\frac{\hbar}{2}$?

Solution

For spin $\frac{1}{2}$ $J_+|\downarrow\rangle = \hbar|\uparrow\rangle$ and $J_-|\uparrow\rangle = \hbar|\downarrow\rangle$ so the equation for eigenstates is

$$(J_x+J_y)(a|\uparrow\rangle + b\downarrow\rangle) \;=\; \Big(\frac{1-i}{2}J_+ + \frac{1+i}{2}J_-\Big)(a|\uparrow\rangle + b|\downarrow\rangle)$$
$$=\; \hbar\Big(\frac{1-i}{2}b|\uparrow\rangle + \frac{1+i}{2}a|\downarrow\rangle\Big) \;=\; \lambda\hbar(a|\uparrow\rangle + b|\downarrow\rangle)$$

which gives

$$\frac{1-i}{2}b = \lambda a, \qquad \frac{1+i}{2}a = \lambda b \quad \Rightarrow \quad \lambda^2 = \frac{1}{2} \;\Leftrightarrow\; \lambda = \pm\frac{1}{\sqrt{2}}$$

The eigenstate for the eigenvalue $\frac{\hbar}{\sqrt{2}}$ is

$$\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1+i}{2}|\downarrow\rangle$$

and for the eigenvalue $-\frac{\hbar}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1+i}{2}|\downarrow\rangle$$

The eigenstate of S_y with eigenvalue $\frac{\hbar}{2}$ is $\frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle)$ so the probability to have $S_y = \frac{\hbar}{2}$ in the state with wave function $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1+i}{2}|\downarrow\rangle$ is

$$|\frac{1}{\sqrt{2}}\langle(\uparrow| - i\langle\downarrow|)(\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1+i}{2}|\downarrow\rangle)|^2 = \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

Problem 2.

Particles with angular momentum 1 are passed through a Stern-Gerlach apparatus which separates them according to the z-component of their angular momentum. Only the $m = -1$ component is allowed to pass through the apparatus. A second apparatus separates the beam according to its angular momentum component along the u-axis. The u-axis and the z-axis are both perpendicular to the beam direction but have an angle θ between them. Find the relative intensities of the three beams separated in the second apparatus.

Solution

The problem can be formulated like that:

A beam of particles with spin $\hbar$ pointing in the direction making angle $\pi - \theta$ with z-axis goes through the Stern-Gerlach apparatus which separates them according to the z-component of their angular momentum. Find the relative intensities of the three beams.

First step is to find state $|\psi\rangle$ of particles in the beam before the apparatus. If the beam is going in the y-direction the averages of angular momentum operator are

$$\langle\psi|J_z|\psi\rangle \;=\; -\hbar\cos\theta, \qquad \langle\psi|J_x|\psi\rangle \;=\; \hbar\sin\theta, \qquad \langle\psi|J_y|\psi\rangle \;=\; 0$$

Writing the state ψ as $a|1,1\rangle + b|1,0\rangle + c|1,-1\rangle$ we get (see Eqs. (9.166) and (9.168) from the lecture notes)

$$\hbar(a^\dagger,\; b^\dagger,\; c^\dagger)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \;=\; -\hbar\cos\theta$$

and

$$\frac{\hbar}{\sqrt{2}}(a^\dagger,\; b^\dagger,\; c^\dagger)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \;=\; \hbar\sin\theta$$

This yields system of 3 equations

$$\begin{aligned} a^2+b^2+c^2 &= 1 \\ a^2-c^2 &= -\cos\theta \\ b(a+c) &= \tfrac{\sin\theta}{\sqrt{2}} \end{aligned}$$

$\Rightarrow a = \frac{1-\cos\theta}{2},\; b = \frac{\sin\theta}{\sqrt{2}},\; c = \frac{1+\cos\theta}{2}$

Another solution: the spin in $\pi - \theta$ direction is given by the operator

$$\vec{J}\cdot\vec{n} \;=\; -\cos\theta J_z + \sin\theta J_x \;=\; \frac{\hbar}{\sqrt{2}}\begin{pmatrix} -\sqrt{2}\cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \sin\theta \\ \sqrt{2}\cos\theta & \sin\theta & 0 \end{pmatrix}$$

so the equation for the eigenstate of $\vec{J}\cdot\vec{n}$ operator takes the form

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} -\sqrt{2}\cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \sin\theta \\ 0 & \sin\theta & \sqrt{2}\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \hbar\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and we get a system of equations

$$\begin{aligned} a^2+b^2+c^2 &= 1 \\ \sqrt{2}a(1+\cos\theta) &= b\sin\theta \\ (a+c)\sin\theta &= b\sqrt{2} \\ \sqrt{2}c(1-\cos\theta) &= b\sin\theta \end{aligned}$$

$\Rightarrow a = \frac{1-\cos\theta}{2},\ b = \frac{\sin\theta}{\sqrt{2}},\ c = \frac{1+\cos\theta}{2}$

Thus, the $|\psi\rangle$ state is

$$|\psi\rangle \;=\; \frac{1-\cos\theta}{2}|1,1\rangle + \frac{\sin\theta}{\sqrt{2}}|1,0\rangle + \frac{1+\cos\theta}{2}|1,-1\rangle$$

and the relative wight of three beams is

$(1-\cos\theta)^2 : 2\sin^2\theta : (1+\cos\theta)^2$

Problem 3.

Let h_0 be the Hamiltonian of a particle. Assume that the operator h_0 acts only on the orbital variables and has three equidistant levels of energies 0, $\hbar\omega_0$, and $2\hbar\omega_0$ ($\omega_0 > 0$) which are non-degenerate in the orbital state space . (In the total space, the degeneracy of each level would be $2s+1$ where s is the spin of the particle). From the point of view of orbital variables, we are concerned only with the subspace spanned by three corresponding eigenstates of h_0.

(a)

Consider a system of three independent electrons whose Hamiltonian can be written as

$$\hat{H} \;=\; \hat{h}_0(1) + \hat{h}_0(2) + \hat{h}_0(3)$$

Find the energy levels of $\hat{H}$ and their degeneracies.

(b)

Same question for a system of three identical bossns of spin 0.