HW assignment 2. Due Feb 25 on the lecture

Find the Feynman, retarded and advanced propagators in a massless KG theory.

Solution
Feynman propagator

$$
\begin{aligned}
\langle 0|\mathrm{T}\{\hat\phi(x)\hat\phi(0)\}|0\rangle \;&=\; i\int đ^4p\ \frac{e^{-ipx}}{p^2+i\epsilon} \;=\; i\int đ^3p\ e^{i\vec p\cdot\vec x}\int đp_0\ \frac{e^{-ip_0x_0}}{(p_0-|\vec p|+i\epsilon)(p_0+|\vec p|-i\epsilon)}\\
&=\; \theta(x_0)\int\frac{đ^3p}{2|\vec p|}\ e^{i\vec p\cdot\vec x-i|\vec p|x_0} \;+\; \theta(-x_0)\int\frac{đ^3p}{2|\vec p|}\ e^{i\vec p\cdot\vec x+i|\vec p|x_0}\\
&\overset{r\equiv|\vec x|}{=}\; \frac{\theta(x_0)}{16\pi^2}\int_0^\infty pdp\int_0^\pi \sin\theta\ d\theta\ e^{-ipx_0+ipr\cos\theta} \;+\; \frac{\theta(-x_0)}{8\pi^2}\int_0^\infty pdp\int_0^\pi \sin\theta\ d\theta\ e^{ipx_0+ipr\cos\theta}\\
&=\; -\frac{i\theta(x_0)}{8\pi^2 r}\int_0^\infty dp\ \left(e^{-ipx_0+ipr}-e^{-ipx_0-ipr}\right) - i\frac{\theta(-x_0)}{8\pi^2 r}\int_0^\infty dp\ \left(e^{ipx_0+ipr}-e^{ipx_0-ipr}\right)\\
&=\; -\frac{i\theta(x_0)}{8\pi^2 r}\int_0^\infty dp\ \left(e^{-ip(x_0-r-i\epsilon)}-e^{-ip(x_0+r-i\epsilon)}\right) - i\frac{\theta(-x_0)}{8\pi^2 r}\int_0^\infty dp\ \left(e^{ip(x_0+r+i\epsilon)}-e^{ip(x_0-r+i\epsilon)}\right)\\
&=\; -\frac{\theta(x_0)}{8\pi^2 r}\Big[\frac{1}{x_0-r-i\epsilon}-\frac{1}{x_0+r-i\epsilon}\Big] + \frac{\theta(-x_0)}{8\pi^2 r}\Big[\frac{1}{x_0+r+i\epsilon}-\frac{1}{x_0-r+i\epsilon}\Big]\\
&=\; -\frac{\theta(x_0)}{4\pi^2(x_0^2-r^2-i\epsilon x_0)} - \frac{\theta(-x_0)}{4\pi^2(x_0^2-r^2+i\epsilon x_0)} \;=\; -\frac{1}{4\pi^2(x_0^2-r^2-i\epsilon)} \;=\; \frac{1}{4\pi^2(-x^2+i\epsilon)}
\end{aligned} \tag{1}
$$

Retarded propagator

$$
\begin{aligned}
\theta(x_0)\langle[\hat\phi(x),\hat\phi(0)]\rangle \;&=\; i\int đ^4p\ \frac{e^{-ipx}}{p^2+i\epsilon p_0} \;=\; i\int đ^3p\ e^{i\vec p\cdot\vec x}\int đp_0\ \frac{e^{-ip_0x_0}}{(p_0-|\vec p|+i\epsilon)(p_0+|\vec p|+i\epsilon)}\\
&=\; \theta(x_0)\int\frac{đ^3p}{2|\vec p|}\ e^{i\vec p\cdot\vec x-i|\vec p|x_0} \;-\; \theta(x_0)\int\frac{đ^3p}{2|\vec p|}\ e^{i\vec p\cdot\vec x+i|\vec p|x_0}\\
&=\; -\frac{\theta(x_0)}{8\pi^2 r}\Big[\frac{1}{x_0-r-i\epsilon}-\frac{1}{x_0+r-i\epsilon}\Big] - \frac{\theta(x_0)}{8\pi^2 r}\Big[\frac{1}{x_0+r+i\epsilon}-\frac{1}{x_0-r+i\epsilon}\Big]\\
&=\; -\frac{\theta(x_0)}{4\pi^2(x_0^2-r^2-i\epsilon)} + \frac{\theta(x_0)}{4\pi^2(x_0^2-r^2+i\epsilon)} \;=\; -2\pi i\delta(x_0^2-r^2)\theta(x_0) \;=\; -2\pi i\delta(x^2)\theta(x_0)
\end{aligned} \tag{2}
$$

Advanced propagator

$$
\begin{aligned}
\theta(-x_0)\langle[\hat\phi(0),\hat\phi(x)]\rangle \;&=\; i\int đ^4p\ \frac{e^{-ipx}}{p^2-i\epsilon p_0} \;=\; i\int đ^3p\ e^{i\vec p\cdot\vec x}\int đp_0\ \frac{e^{-ip_0x_0}}{(p_0-|\vec p|-i\epsilon)(p_0+|\vec p|-i\epsilon)}\\
&=\; -\theta(-x_0)\int\frac{đ^3p}{2|\vec p|}\ e^{i\vec p\cdot\vec x-i|\vec p|x_0} \;+\; \theta(-x_0)\int\frac{đ^3p}{2|\vec p|}\ e^{i\vec p\cdot\vec x+i|\vec p|x_0}\\
&=\; \frac{\theta(-x_0)}{8\pi^2 r}\Big[\frac{1}{x_0-r-i\epsilon}-\frac{1}{x_0+r-i\epsilon}\Big] + \frac{\theta(-x_0)}{8\pi^2 r}\Big[\frac{1}{x_0+r+i\epsilon}-\frac{1}{x_0-r+i\epsilon}\Big]\\
&=\; \frac{\theta(-x_0)}{4\pi^2(x_0^2-r^2+i\epsilon)} - \frac{\theta(-x_0)}{4\pi^2(x_0^2-r^2-i\epsilon)} \;=\; -2\pi i\delta(x_0^2-r^2)\theta(-x_0) \;=\; -2\pi i\delta(x^2)\theta(-x_0)
\end{aligned} \tag{3}
$$