HW assignment 5. Due Thu April 14 on the lecture

Find the total cross section of $e^+e^- \to \mu^+\mu^-$ annihilation (averaged over the polarizations of incoming electron and positron) at high energies (assume $s \gg m^2, \mu^2$). From QED viewpoint the μ-meson is just a massive electron with mass μ.

Solution

The matrix element of $\mathcal{M}$-matrix is

$$\mathcal{M}(p_1, s_1; p_1', s_1' \to p_2, s_2; p_2', s_2') = \bar{U}(p_2, s_2)\gamma_\nu V(p_2', s_2') \frac{g^{\mu\nu}}{(p_1 + p')_1^2} \bar{v}(p_1', s_1')\gamma_\mu u(p_1, s_1)$$

where spinors U and V describe muon and anti-muon, respectively. For the total cross section one gets:

$$\begin{aligned}
\Big(\frac{d\sigma}{d\Omega}\Big)_{\text{unpolarized}} &= \frac{e^4}{64\pi^2 s}\sqrt{\frac{s-4\mu^2}{s-4m^2}}\frac{1}{4}\sum_{s_1, s_1', s_2, s_2'} A(p_2, s_2; p_2', s_2' \leftarrow p_1, s_1; p_1', s_1') A^\dagger(p_2, s_2; p_2', s_2' \leftarrow p_1, s_1; p_1', s_1') \\
&= \frac{e^4}{256\pi^2 s^3}\sqrt{\frac{s-4\mu^2}{s-4m^2}}\sum_{s_1, s_1', s_2, s_2'} \bar{U}(p_2, s_2)\gamma^\mu V(p_2', s_2')\bar{V}(p_2', s_2')\gamma_\nu U(p_2, s_2)\bar{v}(p_1', s_1')\gamma_\mu u(p_1, s_1)\bar{u}(p_1, s_1)\gamma^\nu u(p_1', s_1') \\
&= \frac{e^4}{256\pi^2 s^3}\sqrt{\frac{s-4\mu^2}{s-4m^2}}\mathrm{Tr}\{\gamma^\mu(\not{p}_2' - \mu)\gamma_\nu(\not{p}_2 + \mu)\}\mathrm{Tr}\{\gamma_\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_1' - m)\} \\
&= \frac{e^4}{16\pi^2 s^3}\sqrt{\frac{s-4\mu^2}{s-4m^2}}[p_2^\mu p_2'^\nu + p_2^\nu p_2'^\mu - (\mu^2 + p_2\cdot p_2')g^{\mu\nu}][p_{1\mu}p_{1\nu}' + p_{1\nu}p_{1\mu}' - (m^2 + p_1\cdot p_1')g_{\mu\nu}] \\
&= \frac{e^4}{16\pi^2 s^3}\sqrt{\frac{s-4\mu^2}{s-4m^2}}\big[2(p_1\cdot p_2)(p_1'\cdot p_2') + 2(p_1\cdot p_2')(p_1'\cdot p_2) - 2[\mu^2 + p_2\cdot p_2']p_1\cdot p_1' - 2[m^2 + p_1\cdot p_1']p_2\cdot p_2' + 4(m^2 + p_1\cdot p_1')(\mu^2 + p_2\cdot p_2')\big] \\
&= \frac{e^4}{8\pi^2 s^3}\sqrt{\frac{s-4\mu^2}{s-4m^2}}\big[(p_1\cdot p_2)(p_1'\cdot p_2') + (p_1\cdot p_2')(p_1'\cdot p_2) + \mu^2 p_1\cdot p_1' + m^2 p_2\cdot p_2' + 2m^2\mu^2\big] \\
&= \frac{e^4}{32\pi^2 s^3}\sqrt{\frac{s-4\mu^2}{s-4m^2}}[t^2 + u^2 + 4s(m^2+\mu^2) - 2(m^2+\mu^2)] = \frac{e^4}{32\pi^2 s^3}\sqrt{\frac{s-4\mu^2}{s-4m^2}}[s^2 - 2ut + 2(m^2+\mu^2)] \qquad (1)
\end{aligned}$$

Formulas in the c.m. frame

$$\begin{aligned}
t &= m^2 + \mu^2 - \frac{s}{2} - \frac{1}{2}\sqrt{s-4\mu^2}\sqrt{s-4m^2}\cos\theta \\
u &= m^2 + \mu^2 - \frac{s}{2} + \frac{1}{2}\sqrt{s-4\mu^2}\sqrt{s-4m^2}\cos\theta \qquad (2)
\end{aligned}$$

and therefore

$$\Big(\frac{d\sigma}{d\Omega}\Big)_{\text{unpolarized}} = \frac{e^4}{64\pi^2 s^3}\sqrt{\frac{s-4\mu^2}{s-4m^2}}\big[s^2 + 4s(m^2+\mu^2) + (s-4m^2)(s-4\mu^2)\cos^2\theta\big] \qquad (3)$$

For the total cross section one gets:

$$\sigma_{\text{tot}} = \int d\Omega\Big(\frac{d\sigma}{d\Omega}\Big)_{\text{unpolarized}} = \frac{e^4}{16\pi s^3}\sqrt{\frac{s-4\mu^2}{s-4m^2}}\big[s^2 + 4s(m^2+\mu^2) + \frac{1}{3}(s-4m^2)(s-4\mu^2)\big] \qquad (4)$$

At large $s \gg m^2, \mu^2$ one can use directly Eq. (1) and approximate

$$s \simeq 4p^2, \quad t \simeq -4p^2\sin^2\frac{\theta}{2} \simeq -s\sin^2\frac{\theta}{2}, \quad u \simeq -4p^2\cos^2\frac{\theta}{2} \simeq -s\cos^2\frac{\theta}{2}$$

(where $p = |\vec{p}_1| \simeq |\vec{p}_2|$) so

$$\Big(\frac{d\sigma}{d\Omega}\Big)_{\text{unpolarized}} = \frac{e^4}{64\pi^2 s}\big(1 + \cos^2\theta\big)$$

and the total cross section is

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{e^4}{12\pi s} = \frac{16\pi\alpha^2}{3\pi s}$$