Quantization of the electromagnetic field.

Maxwell's eqs in a free space

$$\vec{\nabla}\times\vec{E} = -\dot{\vec{B}} \qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad \text{„first pair"} \qquad (565)$$

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{B} = \dot{\vec{E}} \qquad \text{„second pair"} \qquad (566)$$

↑ Gauss law

Potentials:

$$\vec{E} = -\vec{\nabla}\phi - \dot{\vec{A}} \qquad \phi \text{ - scalar potential}$$
$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad \vec{A} \text{ - vector potential} \qquad (567)$$

$A^\mu = (\phi, \vec{A})$ - 4-vector potential („electromagnetic field")

Rel.-inv. form of eq. (567):

$$F_{\mu\nu}(x) = \frac{\partial}{\partial x^\mu} A_\nu(x) - \frac{\partial}{\partial x^\nu} A_\mu(x) \qquad (568)$$

where

$$\begin{array}{cc} & \overset{\nu}{\rightarrow} \\ \mu\downarrow & \end{array} \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \leftarrow \text{„field strength tensor"} \qquad (569)$$

With (568), the first pair of Maxwell eqs. (565) is fulfilled automatically.

The second pair (566) can be written down as

$$\frac{\partial}{\partial x_\mu} F_{\mu\nu}(x) = 0 \qquad (570)$$

Gauge invariance

$$A_\mu(x) \to A_\mu(x) + \frac{\partial}{\partial x^\mu}\Lambda(x) \Rightarrow F_{\mu\nu}(x) \to F_{\mu\nu}(x) \qquad (571)$$

arbitrary (scalar) function

Electric and magnetic fields do not change ⇒
⇒ physics is the same.

Coulomb gauge : $\vec{\nabla}\cdot\vec{A}(\vec{x},t)=0$, $A_0(\vec{x},t)=0$ (572)

Explicit form

$$A^i(\vec{x},t) = \int_{-\infty}^{t} dt'\, F^{0i}(\vec{x},t') = -\int_{-\infty}^{t} dt'\, E^i(\vec{x},t') \qquad (573) \quad \Rightarrow$$

> $A_0=0$ trivial ; $\vec{\nabla}\cdot\vec{A}(\vec{x},t) = -\int_{-\infty}^{t} dt'\, \underbrace{\vec{\nabla}\cdot\vec{E}(\vec{x},t')}_{=0 \text{ (Gauss law)}} = 0$

Proof of eq. (573) : we must prove that the usual rule $F_{\mu\nu}=\partial_\mu A_\nu - \partial_\nu A_\mu$ reproduces the original $\vec{E}$ and $\vec{B}$

$$F^{0i} = \partial^0 A^i - \cancel{\partial^i \underbrace{A^0}_{=0}} = \frac{\partial}{\partial t}(-1)\int_{-\infty}^{t} dt'\, \vec{E}^i(\vec{x},t') = -E^i(\vec{x},t)$$

$$\frac{1}{2}\epsilon_{ijk}F^{jk} = \epsilon_{ijk}\frac{\partial}{\partial x_j}A^k = \int_{-\infty}^{t} dt'\, \epsilon_{ijk}\frac{\partial}{\partial x^j}E^k(\vec{x},t') = +\int_{-\infty}^{t} dt'\,(\vec{\nabla}\times\vec{E})_i =$$
$$= -\int_{-\infty}^{t} dt'\, \dot{\vec{B}}_i(\vec{x},t') = -\vec{B}_i(\vec{x},t) \qquad (574)$$

which agrees with the table (569).

Expansion in the plane waves

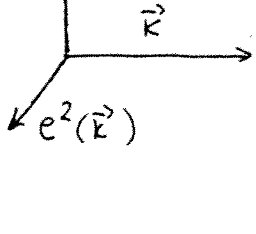
Maxwell's eqs $\Leftrightarrow$ $\partial_0 F^{0k} + \partial_j F^{jk} = 0 \Rightarrow \partial_0(\partial^0 A^k - \cancel{\partial^k A^0}_{A_0=0}) + \partial_j(\partial^j A^k - \partial^k A^j)$
$$= \partial_0^2 A^k + \partial_j\partial^j A^k + \cancel{\partial^k \underbrace{(\vec{\nabla}\cdot\vec{A})}_{=0}} = \partial^2 A^k = 0 \qquad (575)$$

$\partial^2 A_k = 0$ → three Klein-Gordon eqs. (plus additional condition $\vec{\nabla}\cdot\vec{A}=0$).

Solution of eq. (575)

$$A_i(\vec{x},t) = \int \frac{d^3k}{\sqrt{2E_k}} \sum_{\lambda=1,2} e_i^\lambda(\vec{k})\,(a_k^\lambda e^{-ikx} + a_k^{\lambda *} e^{ikx})\Big|_{k_0=E_k\equiv|\vec{k}|} \qquad (576)$$

$\vec{\nabla}\cdot\vec{A}=0 \Leftrightarrow \vec{k}\cdot\vec{e}^\lambda$ → two transverse polarizations



$e^1(\vec{k})$, $e^2(\vec{k})$ – „polarization vectors"

$e^i(\vec{k}) \perp \vec{k}$

(see AQM)

Lagrangian

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) \qquad (577)$$

Check: Euler-Lagrange eqs. for (572) are Maxwell's eqs:

$$\underbrace{\frac{\partial \mathcal{L}}{\partial A_\mu}}_{=0} = \frac{d}{dx^\nu} \underbrace{\frac{\partial \mathcal{L}}{\partial \partial_\nu A_\mu}}_{= F_{\nu\mu}} \Rightarrow 0 = \frac{d}{dx_\nu} F_{\nu\mu}$$

$A_\mu(x)$ - canonical coordinates

Canonical momenta:

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \qquad (578)$$

$$\pi^k = \frac{\partial \mathcal{L}}{\partial \dot{A}_k} = -\dot{A}^k + \frac{\partial}{\partial x_k} A_0 = E^k \qquad (579)$$

Hamiltonian

$$\mathcal{H} = \sum \pi^k \dot{A}_k - \mathcal{L} = -\vec{\pi} \cdot \dot{\vec{A}} - \mathcal{L} = \vec{\pi} \cdot (\vec{E} + \vec{\nabla}\phi) - \frac{1}{2}(\vec{E}^2 - \vec{B}^2) =$$
$$= \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \vec{E} \cdot \vec{\nabla}\phi$$

$$H = \int d^3x\, \mathcal{H}(\vec{x}, t) = \int d^3x\, \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \underbrace{\int d^3x\, \vec{E} \cdot \vec{\nabla}\phi}_{\text{by parts}}$$

$$\overset{\text{by parts}}{=} - \int d^3x\, \phi (\vec{\nabla} \cdot \vec{E}) = 0$$

$$\Rightarrow H = \int d^3x\, \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \quad \text{- classical Hamiltonian} \qquad (580)$$

Quantization: $A_\mu(x) \to \hat{A}_\mu(\vec{x})$, $\pi_k(x) \to \hat{\pi}_k(\vec{x})$ + can. comm. rel

We would like to have canonical commutation relations in the form

$$[\hat{A}_\mu(\vec{x}), \hat{A}_\nu(\vec{y})] = [\hat{\pi}_k(\vec{x}), \hat{\pi}_l(\vec{y})] = 0 \qquad (581)$$

$$[\hat{\pi}_k(\vec{x}), \hat{A}_0(\vec{y})] = 0 \Rightarrow \hat{A}_0 \text{ commutes with all operators} \Rightarrow A_0 \text{ is a c-number} \qquad (582)$$

$$[\hat{\pi}_j(\vec{x}), \hat{A}_k(\vec{y})] \equiv [\hat{E}_j(x), \hat{A}_k(y)] = i\delta_{jk}\, \delta^3(\vec{x} - \vec{y}) \qquad (583)$$

Problem: eq. (578) contradicts to Gauss' law. Indeed, we would like to have $\vec{\nabla}\cdot\hat{\vec{E}}(\vec{x})=0$ (just as in the classical field theory), but

$$[\hat{E}_j(\vec{x}),\hat{A}_k(\vec{y})] = i\delta_{jk}\,\delta(\vec{x}-\vec{y}) \Rightarrow [\vec{\nabla}\cdot\hat{\vec{E}}(\vec{x}),\hat{A}_k(\vec{y})] = i\frac{\partial}{\partial x_k}\delta(\vec{x}-\vec{y}) \neq 0 \qquad (584)$$

Way out (Bjorken & Drell textbook):

$$[\pi_i(\vec{x}), A_j(\vec{y})] = i\,\delta^{tr}_{ij}(\vec{x}-\vec{y}) \qquad (585)$$

$$\delta^{tr}_{ij}(\vec{x}-\vec{y}) \overset{def}{\equiv} \int d^3k\,\left(\delta_{ij}-\frac{k_i k_j}{\vec{k}^2}\right)e^{i\vec{k}(\vec{x}-\vec{y})} \qquad (5\ldots$$

The comm. relation (580) is consistent with Gauss law:

$$[\vec{\nabla}\cdot\vec{E}(\vec{x}), A_k(\vec{y})] = i\frac{\partial}{\partial x_j}\delta^{tr}_{jk}(\vec{x}-\vec{y}) = -\int d^3k\; k_j\left(\delta_{jk}-\frac{k_j k_k}{\vec{k}^2}\right)e^{ik(\vec{x}-\vec{y})} = 0$$

In principle, there are many other ways to quantize the electromagnetic field („quantization of the constrained systems", Dirac).

A property: $\vec{\nabla}\cdot\vec{A}$ is a c-number (just like A_0) (587)

Proof:

$$[\vec{\nabla}\cdot\hat{\vec{A}}(\vec{x}),\hat{A}_k(\vec{y})] = \frac{\partial}{\partial x_j}[\hat{A}_j(\vec{x}),\hat{A}_k(\vec{y})] = 0$$

$$[\vec{\nabla}\cdot\hat{\vec{A}}(\vec{x}),\hat{\pi}_k(\vec{y})] = \frac{\partial}{\partial x_j}[\hat{A}_j(\vec{x}),\hat{\pi}_k(y)] = \frac{\partial}{\partial x_j}(-i)\,\delta^{tr}_{jk}(\vec{x}-\vec{y}) = 0$$

Since both A_0 and $\vec{\nabla}\cdot\vec{A}$ c-numbers, the gauge condition (572) at $t=0$

$$\hat{A}_0(\vec{x})=0\,,\quad \vec{\nabla}\cdot\vec{A}(\vec{x})=0$$

is consistent with the canonical commutation relations (581), (582), and (584).

Quantization in the Coulomb gauge

As usual, we

(i) fix the commutation relations $[\pi(\vec{x}), A(\vec{y})]$ at $t=0$

(ii) define the ladder operators

(iii) construct Heisenberg operators $\hat{A}_i(\vec{x},t) = e^{i\hat{H}t}\hat{A}_i(\vec{x})e^{-i\hat{H}t}$

Canonical comm. relations at t. 0

$$[\pi_i(\vec{x}), \pi_k(\vec{y})] = [A_i(\vec{x}), A_k(\vec{y})] = 0$$
$$[\pi_i(\vec{x}), A_k(\vec{y})] = i\,\delta^{tr}_{ik}(\vec{x}-\vec{y}) \qquad (588)$$

Ladder operators

$$(576) \Rightarrow \quad \hat{A}_i(\vec{x}) = \int \frac{d^3k}{\sqrt{2E_k}} \sum_{\lambda=1,2} e^\lambda_i(\vec{k}) (\hat{a}^\lambda e^{i\vec{k}\vec{x}} + \hat{a}^{\lambda+}_k e^{-i\vec{k}\vec{x}}). \qquad (589)$$

Property:

$$\left.\begin{aligned} &[\hat{a}^\lambda_k, \hat{a}^{+\lambda'}_{k'}] = (2\pi)^3 \delta_{\lambda\lambda'} \delta(\vec{k}-\vec{k}) \\ (590) \quad &[a_k, a_{k'}] = [a^+_k, a^+_{k'}] = 0 \end{aligned}\right\} \Rightarrow (588)$$

$\Rightarrow$ eq. (590) is a right guess.

Proof:

$$\pi_i(\vec{x},t) = -\dot{\vec{A}}_i(\vec{x},t) = \int \frac{d^3k}{\sqrt{2E_k}} (+iE_k) \sum_\lambda e^\lambda(\vec{k}) (a^\lambda_k e^{-ikx} - a^{*\lambda}_k e^{ikx})$$

$$\Rightarrow \hat{\vec{\pi}}(\vec{x}) = \int \frac{d^3k}{\sqrt{2E_k}} (+iE_k) \sum_\lambda \vec{e}^\lambda(\vec{k}) (\hat{a}^\lambda_k e^{i\vec{k}\vec{x}} - \hat{a}^{+\lambda}_k e^{-i\vec{k}\vec{x}}) \qquad (591)$$

$$[\hat{\pi}_i(\vec{x}), \hat{A}_j(\vec{y})] = \int \frac{d^3k}{\sqrt{2E_k}} \frac{d^3k'}{\sqrt{2E'_k}} (+iE_k) \sum_{\lambda,\lambda'} e^\lambda_i(\vec{k}) e^{\lambda'}_j(\vec{k}') [a^\lambda_k e^{+i\vec{k}\vec{x}} -$$
$$- \hat{a}^{\lambda+}_k e^{-i\vec{k}\vec{x}}, \hat{a}^{\lambda'}_{k'} e^{i\vec{k}\vec{y}} + \hat{a}^{+\lambda'}_{k'} e^{-i\vec{k}\vec{y}}] = \int \frac{d^3k}{\sqrt{2E_k}} \frac{d^3k'}{\sqrt{2E'_k}} (+iE_k) \sum_{\lambda,\lambda'} e^\lambda_i(k) e^{\lambda'}_j(\vec{k}') \cdot$$
$$\cdot (2\pi)^3 \delta_{\lambda\lambda'} \delta(\vec{k}-\vec{k}') \quad i\vec{k} \quad + e^{-i\vec{k}(\vec{x}-\vec{y})} = i \int d^3k \sum_\lambda e^\lambda_i(\vec{k}) e^\lambda_j(\vec{k}) e^{-ik(\vec{x}-\vec{y})}$$

It is easy to check that

$$\sum_{\lambda=1,2} e^\lambda_i(\vec{k}) e^\lambda_j(\vec{k}) = \delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \qquad (592)$$

so $[\hat{\pi}_i(\vec{x}), A_j(\vec{y})] = i\,\delta^{tr}_{ij}(\vec{x}-\vec{y})$

Hamiltonian (580) $\Rightarrow$

$$\hat{H} = \frac{1}{2}\int d^3x (\vec{E}^2(\vec{x}) + \vec{B}^2(\vec{x})) \qquad (593)$$

In terms of ladder operators

$$\left.\begin{aligned} &\hat{\vec{E}}(\vec{x}) = \hat{\vec{\pi}}(\vec{x}) = (591) \\ &\hat{\vec{B}}(\vec{x}) = \vec{\nabla}\times\hat{\vec{A}}(\vec{x}) = \int \frac{d^3k}{\sqrt{2E_k}}\, i \sum_\lambda \vec{k}\times\vec{e}^\lambda(\vec{k}) (\hat{a}^\lambda_k e^{i\vec{k}\vec{x}} - \hat{a}^{+\lambda}_k e^{-i\vec{k}\vec{x}}) \end{aligned}\right\} \Rightarrow$$

$$\frac{1}{2}\int d^3x (\hat{\vec{E}}^2(\vec{x}) + \hat{\vec{B}}^2(x)) = \int d^3k\, E_k \sum_\lambda \hat{a}^{+\lambda}_k \hat{a}^\lambda_k \qquad (594)$$

$$\hat{H} = \int d^3k\ E_k \sum_\lambda \hat{a}_k^{\lambda +} \hat{a}_k^\lambda \quad \Rightarrow \tag{595}$$

$$[\hat{H}, a_k^{+\lambda}] = E_k a_k^{\lambda +} \quad ; \quad [\hat{H}, a_k^\lambda] = -E_k a_k^\lambda \tag{596}$$

$|0\rangle$ - vacuum state

(explicit form is $\Psi_0(\vec{A}) \sim e^{-\int d^3x\, \vec{B}(\vec{x}) \frac{1}{W} \vec{B}(\vec{x})}$ where $W = \sqrt{-\vec{\nabla}^2}$, cf. eqs. (91) and (92)).

$$a_k^\lambda |0\rangle = 0 \tag{597}$$

$$a_k^{\lambda +}|0\rangle \sqrt{2E_k} = |k, \lambda\rangle \quad \text{- one-photon state} \tag{598}$$

Energy of this state is $E_k = |k|$ because

$$\hat{H}\, a_k^{\lambda +}|0\rangle = [H, a_k^{+\lambda}]|0\rangle = E_k\, a_k^{\lambda +}|0\rangle \tag{599}$$

We must check also that the momentum of this one-photon state is indeed $\vec{k}$ (i.e., that $a_k^{+\lambda}|0\rangle$ is an eigenstate of momentum operator with eigenvalue $\vec{k}$)

Momentum operator

General formula for tensor of energy-momentum in classical field theory

$$T^{\mu\nu} = \sum_\phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial^\nu \phi - g^{\mu\nu}\mathcal{L} \tag{600}$$

(cf. eqs (136-138)). For the electromagnetic field we get

$$T^{\mu\nu} = \sum_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\alpha} \partial^\nu A_\alpha - g^{\mu\nu}\mathcal{L} =$$

$$= -F^{\mu\alpha}\partial^\nu A_\alpha - g^{\mu\nu}\mathcal{L} = -F^{\mu\alpha}F^\nu{}_\alpha + \frac{g^{\mu\nu}}{4} F_{\xi\eta}F^{\xi\eta} \cancel{- \partial_\alpha (F^{\mu\alpha} A^\nu)}$$

does not contribute to H or P_i

$$\Rightarrow T^{\mu\nu} = -F^{\mu\alpha}F^\nu{}_\alpha + \frac{g^{\mu\nu}}{4} F^{\xi\eta}F_{\xi\eta} \tag{601}$$

Momentum of classical e.m. field:

$$P^i(t) = \int d^3x\, T^{0i}(\vec{x},t) = \int d^3x \{-\dot{A}^k(\partial^i A_k - \underbrace{\cancel{\partial_k A^i}}_{\substack{=0 \text{ by parts} \\ (\text{since } \partial_k A^k = 0)}})\} =$$

$$= -\int d^3x\, \dot{A}^k(\vec{x},t)\,\partial^i A_k(\vec{x},t)$$

$$\Rightarrow \vec{P}(\vec{x},t) = -\int d^3x\, \dot{A}^k(\vec{x},t)\,\vec{\nabla} A_k(\vec{x},t) = \quad (602)$$

Let us prove that eq. (602) coincides with the familiar form of ~~momentum of e.m.~~ field in terms of Pointing vector $\vec{E}\times\vec{B}$

$$\vec{P}(t) = \int d^3x\, \vec{E}\times\vec{B} = \int d^3x\, \vec{E}\times(\vec{\nabla}\times\vec{A}) \quad (603)$$

Proof:

$$P^i(t) = \int d^3x\, (\vec{E}\times(\vec{\nabla}\times\vec{A}))_i = \epsilon_{ijk}\int d^3x\, \vec{E}_j\, \epsilon_{jmn}\,\partial_m \vec{A}_n = \int d^3x\, (\underbrace{\vec{E}_j \vec{\partial}_j \vec{A}_i}_{\substack{=0 \text{ by parts} \\ (\text{since } \partial_j E_j = 0)}} +$$

$$+ \vec{E}_j \vec{\partial}_i \vec{A}_j) = \int d^3x\, (-E^j \partial^i A^j) = 602$$

In quantum theory

$$\hat{\vec{P}} = \int d^3x\, \hat{\vec{E}}(\vec{x})\times\hat{\vec{B}}(\vec{x}) \quad (604)$$

It is easy to check that

$$\hat{P}^i = \int d^3k\, k^i \sum_\lambda a_k^{\lambda+} a_k^\lambda \quad (605)$$

(cf. eq. (144)) $\Rightarrow$

$$\Rightarrow [\hat{P}^i, \hat{a}_k^{\lambda+}] = k^i \hat{a}_k^{\lambda+} \quad ; \quad [\hat{P}^i, \hat{a}_k^\lambda] = -k^i \hat{a}_k^\lambda \quad (606)$$

Using eqs. (596) and (606) it is easy to prove that

$$e^{i\hat{P}x}\,\hat{a}_k^{+\lambda}\,e^{-i\hat{P}x} = a_k^{+\lambda} e^{-ipx} \quad , \quad e^{iPx}\, a_k^\lambda\, e^{-iPx} = a_k^\lambda e^{-ipx} \quad (607)$$

where $\hat{P}^\mu = (\hat{H}, \hat{P}^i)$ (cf. eqs. (155), (156)) $\Rightarrow$

$$\Rightarrow \hat{A}_i(\vec{x},t) = e^{i\hat{H}t}\,\hat{A}_i(\vec{x})\,e^{-i\hat{H}t} = e^{i\hat{P}x}\,\hat{A}_i(0)\,e^{-i\hat{P}\hat{x}} =$$

$$= \int \frac{d^3k}{\sqrt{2E_k}} \sum_\lambda (a_k^\lambda e^{-ikx} + a_k^{+\lambda} e^{ikx})\, e_i^\lambda(\vec{k})\Big|_{k_0 = E_k} \quad (608)$$

cf. eq. (153))

Spin of the photon

$$a^R_k \equiv \frac{1}{\sqrt{2}}(a^1_k - ia^2_k) \Rightarrow a^{R+}_k = \frac{1}{\sqrt{2}}(a^{1+}_k + ia^{2+}_k)$$
$$a^L_k \equiv \frac{1}{\sqrt{2}}(a^1_k + ia^2_k) \Rightarrow a^{L+}_k = \frac{1}{\sqrt{2}}(a^{+}_k - ia^{2+}_k) \qquad (609)$$

$a^{R+}_k|0\rangle\sqrt{2E_k} \equiv |\vec{k}, right\rangle$ – right-handed photon

$a^{L+}_k|0\rangle\sqrt{2E_k} \equiv |\vec{k}, left\rangle$ – left-handed photon

Let us prove that the helicity (≡ projection of the spin on the direction of the momentum) of the right-handed photon is +1 (and that of left-handed photon is (−1)).

Angular momentum of the electromagnetic field

The general f-la for angular momentum tensor is

$$M^{\alpha\mu\nu} = x^\mu T^{\alpha\nu} - x^\nu T^{\alpha\mu} \qquad (610)$$

It is easy to check that $\partial_\alpha M^{\alpha\mu\nu} = x_\mu \partial_\alpha T^{\alpha\nu} - \mu \leftrightarrow \nu = 0$ because $\partial_\alpha T^{\alpha\beta} = 0$ (which is a manifestation of conservation of energy-momentum).

$$\partial_\alpha M^{\alpha\mu\nu} = 0 \Rightarrow \frac{d}{dt}\int d^3x\, M^{0\mu\nu}(\vec{x},t) = 0 \Rightarrow \qquad (611)$$

$\Rightarrow \int d^3x\, M^{0\mu\nu}(\vec{x},t)$ conserves (conservation of angular mometum

Vector of angular momentum of a classical field

$$\vec{J}_i(t) = \frac{1}{2}\epsilon_{ijk}\int d^3x\, M^{0jk}(\vec{x},t) = \epsilon_{ijk}\int d^3x\, x^j \cdot T^{0k}(\vec{x},t) \qquad (612)$$

For electromagnetic field $T^{0k} = F^{0m}F_m{}^k$ (see eq. (601)) so

$$\vec{J}_i(t) = \epsilon_{ijk}\int d^3x\, \vec{E}_m F_{mk}\vec{x}_j = \epsilon_{ijk}\int d^3x\, \vec{x}_j \vec{E}_m(-\epsilon_{mkn}\vec{B}_n) =$$
$$= \int d^3x\, \epsilon_{ijk}\vec{x}_j(\vec{E}\times\vec{B})_k \Rightarrow$$
$$\Rightarrow \vec{J}_i(t) = \int d^3x\, \vec{x}\times\underbrace{(\vec{E}\times\vec{B})}_{\text{Pointing vecter}} \qquad (613)$$

It is convenient to rewrite eq. (613) in a following way

$$J_i(T) = \int d^3x\, (\vec{x}\times(\vec{E}\times(\vec{\nabla}\times\vec{A})))_i = \epsilon_{ijk}\int d^3x\, \vec{x}_j \vec{E}_m(\vec{\partial}_k\vec{A}_m - \vec{\partial}_m\vec{A}_k) =$$

$$= \epsilon_{ijk}\int d^3x\, (\vec{x}_j\vec{E}_m\vec{\partial}_k\vec{A}_m - \underbrace{\vec{x}_j\vec{E}_m\vec{\partial}_m\vec{A}_k}_{\overset{\text{by parts}}{=} \vec{E}_j\vec{A}_k}) = \epsilon_{ijk}\int d^3x\, (\vec{E}_j\vec{A}_k + \vec{x}_j\vec{E}_m\vec{\partial}_k\vec{A}_m)$$

$$\Rightarrow \vec{J}(t) = \int d^3x\, (\underbrace{\vec{E}\times\vec{A}}_{\text{spin}} + \underbrace{E_k(\vec{x}\times\vec{\nabla})A_k}_{\text{orbital momentum}}) \qquad (614)$$

(density of the momentum is $E_k\vec{\nabla}A_k$, see eq. (602))

The corresponding quantum operator is

$$\hat{J} = \int d^3x\, (\hat{\vec{E}}(\vec{x})\times\vec{A}(\vec{x}) + E_k(\vec{x})(x\times\vec{\nabla})A_k(\vec{x})) \qquad (615)$$

et us consider the right-handed photon moving in z irection and prove that $\hat{J}_3|k\uparrow\uparrow oz,\text{right}\rangle = |k\uparrow\uparrow oz,\text{right}\rangle$

$$[\hat{J}_3, a_k^{R+}] = [\int d^3x\,(\hat{\vec{E}}_1\hat{\vec{A}}_2 - \hat{\vec{E}}_2\hat{\vec{A}}_1 + \vec{E}_j(\vec{x}_1\frac{\partial}{\partial\vec{x}_2} - \vec{x}_2\frac{\partial}{\partial\vec{x}_1})\vec{A}_j), a_k^{+R}] \qquad (616)$$

$$\left.\begin{aligned} [A_j(\vec{x}), a_k^{+R}] &= \frac{e_j^R(k)}{\sqrt{2\omega_k}} e^{i|\vec{k}|x_3} \qquad (617)\\ [E_j(\vec{x}), a_k^{+R}] &= i\sqrt{\frac{\omega_k}{2}}\, e_j^R(k) e^{i|\vec{k}|x_3} \end{aligned}\right\} \text{see eqs (589), (590) and (591)} \qquad (618)$$

where $\vec{e}^R(k) = \frac{1}{\sqrt{2}}(\vec{e}^1 + i\vec{e}^2)$, see AQM. (As usual $\omega_k \equiv E_k = |\vec{k}|$)

It is easy to see that

$$\int d^3x\, [E_j(\vec{x}_1\frac{\partial}{\partial\vec{x}_2} - \vec{x}_2\frac{\partial}{\partial\vec{x}_1})A_j, a_k^{+R}] = \int d^3x\, E_j(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1})[A_j(\vec{x}), a_k^{+R}]$$

$$+ \int d^3x\, [E_j(\vec{x}), a_k^{+R}](\vec{x}_1\frac{\partial}{\partial\vec{x}_2} - \vec{x}_2\frac{\partial}{\partial\vec{x}_1})A_j(\vec{x}) = 0 \qquad (619)$$

(eq. (617) ⇒ first term = 0 ; eq. (618) + integration by parts ⇒ second term = 0)

HW 4 : Show that

$$\int d^3x\, [\hat{\vec{E}}_1(\vec{x})\hat{\vec{A}}_2(\vec{x}) - \hat{E}_2(\vec{x})\hat{A}_1(\vec{x}), a_k^{+R}] = a_k^{+R}$$

$$\Rightarrow [\hat{J}_3, a_k^{R+}] = a_k^{R+} \qquad (620)$$

$$\Rightarrow \hat{J}_3 a_k^{R+}|0\rangle = [\hat{J}_3, a_k^{R+}]|0\rangle + \cancel{a_k^{R+}\underbrace{\hat{J}_3|0\rangle}_{0}} = a_k^{R+}|0\rangle$$

⇒ $a_k^{R+}|0\rangle$ is an eigenstate of spin operator with eigenvalue 1. Similarly, one can prove that

$[\hat{J}_3, a_k^{L+}] = -a_k^{L+}$

so the state $|\vec{k},\text{left}\rangle$ has helicity (−1)

QED

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$$

$D_\mu \equiv \partial_\mu + ieA_\mu$ – „covariant derivative"
$\overleftarrow{D}_\mu \equiv \overleftarrow{\partial}_\mu - ieA_\mu$ (621)

Roughly

$$\mathcal{L} = \underbrace{-\frac{1}{4} F_{\mu\nu}F^{\mu\nu}}_{\text{free e.m. Lagrangian}} + \underbrace{\bar{\psi}(i\not{\partial} - m)\psi}_{\text{free Dirac L-n}} - \underbrace{e\bar{\psi}\not{A}\psi}_{\text{interaction Lagrangian}}$$

Euler-Lagrange eqs

22) $\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \frac{d}{dx^\mu}\frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}} \Rightarrow (i\not{D} - m)\psi = 0$ ← Dirac eqs. in the external field $A_\mu(x)$

23) $\frac{\partial \mathcal{L}}{\partial \psi} = \frac{d}{dx^\mu}\frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} \Rightarrow -e\bar{\psi}\not{A} - m\bar{\psi} = i\partial_\mu \bar{\psi}\gamma^\mu = \bar{\psi}(i\overleftarrow{\not{D}} + m) = 0$

24) $\frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{d}{dx^\mu}\frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} \Rightarrow -e\bar{\psi}\gamma^\nu\psi = -\partial_\mu F^{\mu\nu} \Rightarrow \partial_\mu F^{\mu\nu} = e\bar{\psi}\gamma^\nu\psi$

↓ Maxwell's eqs. with a source term

Gauge invariance

$\psi(x) \to e^{i\alpha(x)}\psi(x)$
$\bar{\psi}(x) \to e^{-i\alpha(x)}\bar{\psi}(x)$
$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$

$\alpha(x)$ - arbitrary function of x (625)
($\alpha \equiv -\frac{i}{e}\Lambda$ from previous lecture)

Property: $F_{\mu\nu}(x) \to F_{\mu\nu}(x)$: trivial (626)

$D_\mu \psi(x) \to e^{i\alpha}\psi(x)$: $(\partial_\mu + ieA_\mu)\psi(x) \to$ (627)
$\to (\partial_\mu + ieA_\mu - i(\partial_\mu \alpha))e^{i\alpha(x)}\psi =$
$= e^{i\alpha}(\partial_\mu + ieA_\mu)\psi$

$\Rightarrow \bar{\psi}\gamma_\mu D^\mu\psi \to \bar{\psi}e^{-i\alpha}\gamma_\mu e^{i\alpha}D^\mu\psi = \bar{\psi}\gamma_\mu D^\mu\psi \Rightarrow$ (628)

$\Rightarrow \mathcal{L}_{QED}$ is gauge-invariant

Conserved current

$$j_\mu = e\bar{\psi}\gamma_\mu\psi \tag{629}$$

Check:

$$\frac{\partial}{\partial x_\mu} j_\mu(x) = \frac{\partial}{\partial x_\mu}\bar{\psi}(x)\gamma_\mu\psi(x) = \bar{\psi}(x)(+\overleftarrow{\not\partial} + \not\partial)\psi = \tag{630}$$

$$= \bar{\psi}(x)(\overleftarrow{\not\partial} - ie\not{A} - im + \not\partial + ie\not{A} + im)\psi(x) = \bar{\psi}(x)(\overleftarrow{\not{D}} - im + \not{D} + im)\psi(x) = 0$$

Conservation of charge:

$$\int d^3x\, j_0(\vec{x},t) \qquad \frac{dQ}{dt} = \int d^3x\,(-\vec{\nabla}\cdot\vec{j}(x,t)) = 0 \tag{631}$$

(cf. eq. (520))

Coulomb gauge:

$$\vec{\nabla}\cdot\vec{A} = 0 \tag{632}$$

NB: It is impossible to satisfy both conditions $\vec{\nabla}\cdot\vec{A} = 0$ and $A_0 = 0$ in the interacting theory, so we can choose eigther $\vec{\nabla}\cdot\vec{A} = 0$ (Coulomb gauge) or $A_0 = 0$ (temporal gauge).

Dynamical variables (canonical coordinates)

A_i, ψ

$A_0 \equiv \phi$ is <u>not</u> a dynamical variable:

$$\partial_i F^{i0} = \partial_i(\partial^i\phi - \partial^0 A^i) = -\vec{\nabla}^2\phi - \partial^0\underbrace{\cancel{(\partial_i A^i)}}_{=0} = e\bar{\psi}\gamma^0\psi \quad \text{(Maxwell's eqn (624))}$$

$\Rightarrow$

$$\vec{\nabla}^2\phi(\vec{x},t) = -e\,\psi^+(\vec{x},t)\psi(\vec{x},t) \qquad \psi^+(\vec{x},t)\psi(\vec{x},t) \equiv \rho(\vec{x},t) \tag{633}$$

charge density

$$\Rightarrow \phi(\vec{x},t) = e\int d^3x'\,\frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|} \tag{634}$$

(potential due to the continuously distributed charge)

$\Rightarrow \phi = A^0$ is not an independent dynamical variable.

Canonical momenta

$$\frac{\partial L}{\partial \dot{\psi}(\vec{x},t)} \equiv \pi(\vec{x},t) = i\psi^+(\vec{x},t) \tag{635}$$

$$\frac{\partial L}{\partial \dot{A}_0(\vec{x},t)} \equiv \pi^0 = 0 \tag{636}$$

$$\frac{\partial L}{\partial \dot{A}_k(\vec{x},t)} \equiv \pi^k(\vec{x},t) = -\dot{A}^k(\vec{x},t) - \partial^k A^0(\vec{x},t) = E^k(\vec{x},t) \tag{637}$$

Quantization (Heisenberg picture) $\psi(\vec{x},t) \to \hat{\psi}(\vec{x})$, $A_i(\vec{x},t) \to \hat{A}$

Canonical commutation relations at $t=0$ + Heisenber equations for time-dependent field operators.

Can. comm. relations

$$\{\hat{\psi}_\xi(\vec{x}), \psi_\eta(\vec{y})\} = \delta_{\xi\eta}\,\delta(\vec{x}-\vec{y}) \tag{638}$$

$$\{\hat{\psi}(\vec{x}), \hat{\psi}(\vec{y})\} = \{\psi^+(\vec{x}), \psi^+(\vec{y})\} = 0 \tag{639}$$

$$[\hat{\psi}(\vec{x}), \hat{A}_i(\vec{y})] = [\psi^+(\vec{x}), A_i(y)] = 0 \quad \Rightarrow \tag{640}$$

$$\Rightarrow [\hat{\Phi}(\vec{x}), \hat{A}_i(\vec{y})] = 0 \quad , \text{ see eq. (634) } \left(\hat{\Phi} = e\int d^3x' \,\frac{\hat{\varrho}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) \tag{641}$$

$$[\hat{\pi}_j(\vec{x}), \hat{A}_k(\vec{y})] = i\,\delta^{tr}_{jk}(\vec{x}-\vec{y}) \tag{642}$$

$$[\hat{A}_i(\vec{x}), \hat{A}_j(\vec{y})] = [\hat{\pi}_i(\vec{x}), \hat{\pi}_j(\vec{y})] = 0 \tag{643}$$

Ladder operators: as usual

$$\hat{\psi}(\vec{x}) = \int \frac{d^3p}{\sqrt{2E_p}} \sum_s \left(u(p,s)\,\hat{a}^s_p\, e^{+i\vec{p}\vec{x}} + v(p,s)\,\hat{b}^{+s}_p e^{-i\vec{p}\vec{x}}\right)$$

$$\hat{\bar{\psi}}(\vec{x}) = \int \frac{d^3p}{\sqrt{2E_p}} \sum_s \left(\bar{v}(p,s)\,\hat{b}^s_p\, e^{i\vec{p}\vec{x}} + \bar{u}(p,s)\,\hat{a}^{+s}_p e^{-i\vec{p}\vec{x}}\right) \tag{644}$$

$$\hat{A}_i(\vec{x}) = \int \frac{d^3k}{\sqrt{2\omega_k}} \sum_\lambda e^\lambda_i(k)\left(\hat{a}^\lambda_k e^{i\vec{k}\vec{x}} + \hat{a}^{+\lambda}_k e^{-i\vec{k}\vec{x}}\right)$$

$$\hat{\pi}_i(\vec{x}) = \int \frac{d^3k}{\sqrt{2\omega_k}}\, i\omega_k \sum_\lambda e^\lambda_i(k)\left(a^\lambda_k e^{i\vec{k}\vec{x}} - \hat{a}^{+\lambda}_k e^{-i\vec{k}\vec{x}}\right).$$

Commutation relations for ladder operators: as usual

$$[a^\lambda_k, a^{+\lambda'}_{k'}] = (2\pi)^3\,\delta(\vec{k}-\vec{k}')\,\delta_{\lambda\lambda'}$$

$$\{a^s_p, a^{+s'}_{p'}\} = (2\pi)^3\,\delta(\vec{p}-\vec{p}')\,\delta_{ss'} \qquad \{b^s_p, b^{+s'}_{p'}\} = (2\pi)^3\,\delta(\vec{p}-\vec{p}')\,\delta_{ss'} \tag{645}$$

All other (anti) commutators vanish. We have checked that the rules (645) lead to the canonical commutation relations (638) - (643).

Hamiltonian

In classical electrodynamics

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \dot{\psi} + \frac{\partial \mathcal{L}}{\partial \dot{A}_j} \dot{A}_j - \mathcal{L} = \bar{\psi}(-i\gamma^k \partial_k + m)\psi + \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \vec{E}\cdot\vec{\nabla}\phi + e A^\mu \bar{\psi}\gamma_\mu \psi \tag{646}$$

$$H(t) = \int d^3x \, \mathcal{H}(\vec{x}, t)$$

Let us divide $\vec{E}$ into longitudinal and transverse parts

$$\vec{E} = \vec{E}_l + \vec{E}_t = \underbrace{-\vec{\nabla}\phi}_{\text{longitudinal}} \underbrace{-\dot{\vec{A}}}_{\text{transverse because } \partial_i A^i = 0} \tag{647}$$

Then

$$\frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \vec{E}\cdot\vec{\nabla}\phi = \frac{1}{2}(\vec{E} + \vec{\nabla}\phi)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}B^2 =$$
$$= \frac{1}{2}(\dot{\vec{A}}^2 + \vec{B}^2) - \frac{1}{2}(\vec{\nabla}\phi)^2 \tag{648}$$

so

$$\mathcal{H} = \bar{\psi}(-i\gamma^k \partial_k + m)\psi + \frac{1}{2}(\dot{\vec{A}}^2 + \vec{B}^2) + e\bar{\psi}\not{A}\psi - \frac{1}{2}(\vec{\nabla}\phi)^2 \tag{649}$$

$$H(t) = \int d^3x \, \bar{\psi}(-i\gamma^k\partial_k + m)\psi + \int d^3x \, \frac{1}{2}(\dot{\vec{A}}^2 + \vec{B}^2) + \int d^3x \, e\bar{\psi}\not{A}\psi -$$
$$- e^2 \int d^3x \, d^3y \, \bar{\psi}\gamma_0\psi(\vec{x},t) \frac{1}{4\pi|\vec{x}-\vec{y}|} \bar{\psi}\gamma_0\psi(\vec{y},t) \tag{650}$$

Quantum operator:

$$\hat{H} = \int d^3x \, \hat{\bar{\psi}}(\vec{x})(-i\gamma^k\partial_k + m)\hat{\psi}(\vec{x}) + \frac{1}{2}\int d^3x \, (\dot{\hat{\vec{A}}}{}^2(\vec{x}) + \hat{\vec{B}}^2(\vec{x})) + e\int d^3x \, \hat{\bar{\psi}}\hat{\not{A}}\hat{\psi}(\vec{x})$$
$$- e^2 \int d^3x \, d^3y \, \hat{\bar{\psi}}\gamma_0\hat{\psi}(\vec{x}) \frac{1}{4\pi|\vec{x}-\vec{y}|} \hat{\bar{\psi}}\gamma_0\hat{\psi}(\vec{y}) =$$
$$= \hat{H}_0^{Dirac} + \hat{H}_0^{e.m.} + \hat{H}_{int} + \hat{H}_{Coulomb} \tag{651}$$

It is easy to check that

$$\hat{H}_0^{e.m.} = \sum_\lambda \int d^3k \, \omega_k a_k^{+\lambda} a_k^\lambda \tag{652}$$

(without using the condition $A_0 = 0$ we would get $\frac{1}{2}\int d^3\vec{x} \, (\dot{\vec{A}}^2 + \vec{B}^2)$ instead of $\frac{1}{2}\int d^3x \, (\vec{E}^2 + \vec{B}^2)$ for the free $H_0^{e.m.}$)

Interaction representation

Hamiltonian

$$\hat{H} = \hat{H}_0^{Dirac} + \hat{H}_0^{e.m.} + \hat{H}_{int} + \hat{H}_{Coulomb} \tag{653}$$

$$\hat{H}_0^{Dirac} = \int d^3x\, \hat{\bar\psi}(\vec{x})(+i\vec{\gamma}\cdot\vec{\nabla}+m)\psi(\vec{x}) = \int d^3p\, E_p \sum_s (a_p^{s+}a_p^s + b_p^{s+}b_p^s)$$

$$\hat{H}_0^{e.m.} = \int d^3x\, \frac{1}{2}\left(\hat{\dot{\vec{A}}}^2(\vec{x}) + \hat{\vec{B}}^2(\vec{x})\right) = \int d^3k\, E_k \sum_{\lambda=1,2} a_k^{\lambda+} a_k^\lambda$$

$$\hat{H}_{int} = e\int d^3x\, \hat{\bar\psi}(\vec{x})\hat{\not{A}}(\vec{x})\hat{\psi}(\vec{x})$$

$$\hat{H}_{Coulomb} = -e^2\int d^3x\,d^3y\, \bar\psi\gamma_0\psi(\vec{x})\frac{1}{4\pi|\vec{x}-\vec{y}|}\bar\psi\gamma_0\psi(\vec{y})$$

Operators in the interaction representation

$$\hat\psi_I(\vec{x},t) \equiv e^{i(\hat{H}_0^{Dir}+\hat{H}_0^{e.m})t}\hat\psi(\vec{x})e^{-i(\hat{H}_0^{Dir}+\hat{H}_0^{e.m})t} = e^{i\hat{H}_0^{Dir}t}\hat\psi(\vec{x})e^{-i\hat{H}_0^{Dir}t}$$

$$\bar\psi_I(\vec{x},t) = e^{i\hat{H}_0^{Dir}t}\hat{\bar\psi}(\vec{x})e^{-i\hat{H}_0^{Dir}t} \tag{654}$$

$$\hat{A}_I^\mu(\vec{x},t) \equiv e^{i(\hat{H}_0^{Dir}+\hat{H}_0^{e.m.})t}\hat{A}_I^\mu(x)e^{-i(\hat{H}_0^{Dir}+\hat{H}_0^{e.m.})t} = e^{i\hat{H}_0^{em.}t}\hat{A}_i(\vec{x})e^{-i\hat{H}_0^{e.m.}t}$$

In terms of ladder operators this reads (see eqs. (533), (608))

$$\hat\psi_I(x) = \int\frac{d^3p}{\sqrt{2E_p}}\sum_s\left(u(p,s)a_p^s e^{-ipx} + v(p,s)b_p^{+s}e^{ipx}\right)\Big|_{p_0=E_p\equiv\sqrt{m^2+\vec{p}^2}} \tag{655}$$

$$\hat{\bar\psi}_I(x) = \int\frac{d^3p}{\sqrt{2E_p}}\sum_s\left(\bar v(p,s)b_p^s e^{-ipx} + \bar u(p,s)a_p^{+s}e^{ipx}\right)\Big|_{p_0=E_p} \tag{656}$$

$$\hat{A}_\mu^I(x) = \int\frac{d^3k}{\sqrt{2\omega_k}}\sum_\lambda e_\mu^\lambda(k)\left(a_k^\lambda e^{-ikx} + a_k^{+\lambda}e^{ikx}\right)\Big|_{k_0=\omega_k\equiv|\vec{k}|} \tag{657}$$

Calculation of the Green functions

$$\langle\Phi|T\{\psi(x_1)\dots\psi(x_m)\bar\psi(y_1)\dots\bar\psi(y_n)A(z_1)\dots A(z_n)\}|\Phi\rangle \tag{658}$$

$$\left.\begin{aligned}\psi(\vec{x}\ t) &\equiv e^{i\hat{H}t}\psi(\vec{x})e^{-iHt}\\ A_i(\vec{x},t) &\equiv e^{i\hat{H}t}A_i(\vec{x})e^{-iHt}\end{aligned}\right\}\ \text{Heisenberg operators}$$

$\downarrow$ „physical vacuum" – ground state of $\hat{H}$

As usual, if we define $|0\rangle$ („perturbative vacuum") – ground state of $\hat{H}_0 = \hat{H}_0^{Dirac} + \hat{H}_0^{e.m.}$ such that

$$a_k^\lambda|0\rangle = 0 \tag{659}$$

$$a_p^s|0\rangle = b_p^s|0\rangle = 0$$

and use the property (266)

$$e^{-iHT}|0\rangle \xrightarrow[T\to\infty(1-i\epsilon)]{} e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle$$

we can repeat the steps (262) - (274) and derive the interaction-representation expression for the Green function (658):

$$\langle\Omega|T\{\hat{\psi}(x_1)\ldots\hat{\bar{\psi}}(y_n)\hat{A}(z_1)\ldots\hat{A}(z_\ell)\}|\Omega\rangle = \frac{\langle 0|T\{\hat{\psi}_I(x_1)\ldots\hat{\bar{\psi}}_I(y_n)\hat{A}^I(z_1)\ldots\hat{A}^I(z_\ell)\, e^{-i\int_{-\infty}^{\infty}dt\,(\hat{H}_I(t)+\hat{H}_{Coul}(t))}\}|0\rangle}{\langle 0|T\{\exp -i\int_{-\infty}^{\infty}dt\,(\hat{H}_I(t)+\hat{H}_{Coulomb}(t))\}|0\rangle}$$

where (660)

$$\hat{H}_I(t) \equiv e^{i\hat{H}_0t}\hat{H}_{int}e^{-i\hat{H}_0t} = e\int d^3x\; \hat{\bar{\psi}}_I(\vec{x},t)\hat{\not{A}}_I(\vec{x},t)\hat{\psi}_I(\vec{x},t) \tag{661}$$

$$\hat{H}_{Coulomb}(t) \equiv e^{i\hat{H}_0t}\hat{H}_{Coulomb}e^{-i\hat{H}_0t} = -e^2\int d^3x\,d^3y\;\hat{\bar{\psi}}_I\gamma_0\hat{\psi}_I(\vec{x},t)\frac{1}{4\pi|\vec{x}-\vec{y}|}\hat{\bar{\psi}}_I\gamma_0\hat{\psi}_I(\vec{y},t) \tag{662}$$

Now we can expand the r.h.s. of eq. (660) in powers of e ($\Leftrightarrow$ in powers of $H_I + H_{Coul}$) and use Wick's theorem for calculation of all possible Feynman diagrams. The contractions are

$$\psi_I(x)\bar{\psi}_I(y) \equiv \langle 0|T\{\psi_I(x)\bar{\psi}_I(y)\}|0\rangle = \int\frac{d^4p}{i}\,\frac{e^{-ip(x-y)}}{m^2-p^2-i\epsilon}(m+\not{p}) \tag{663}$$

Feynman propagator for Dirac particle

and

$$A_I^i(x)A_I^j(y) \equiv \langle 0|T\{\hat{A}_I^i(x)\hat{A}_I^j(y)\}|0\rangle = \int\frac{d^4k}{i}\,\frac{e^{-ik(x-y)}}{-k^2-i\epsilon}\sum_{\lambda=1,2}e^{\lambda\,i}(k)e^{\lambda\,j}(k)$$

$$= \int\frac{d^4k}{i}\,e^{-ik(x-y)}\frac{1}{-k^2-i\epsilon}\left(-g^{ij}+\frac{k^ik^j}{\vec{k}^2}\right) = \int\frac{d^4k}{i}\,\frac{e^{-ik(x-y)}}{k^2+i\epsilon}\left(g^{ij}+\frac{k^ik^j}{\vec{k}^2}\right) \equiv$$

$$\equiv D^{ij}_{tr}(x-y) \tag{664}$$

which is the propagator of transverse photons (recall that $\sum_{\lambda=1,2}\vec{e}^{\,\lambda}_i\,\vec{e}^{\,\lambda}_j = \delta_{ij} - \frac{\vec{k}_i\vec{k}_j}{\vec{k}^2}$)

We can rewrite the transverse propagator (664) as follows ($\eta \equiv (1,0,0,0)$ - unit 4-vector in time direction):

$$D^{tr}_{\mu\nu} = \int \frac{d^4k}{i}\, e^{-ik(x-y)}\, \frac{1}{k^2+i\epsilon}\left(g_{\mu\nu} + \frac{k_\mu k_\nu}{\vec{k}^2} - \frac{k_0(k_\mu\eta_\nu + k_\nu\eta_\mu)}{\vec{k}^2} - \frac{k^2\eta_\mu\eta_\nu}{\vec{k}^2}\right) = \qquad (665)$$

$$= \underbrace{\int\frac{d^4k}{i}\,\frac{e^{-ik(x-y)}}{k^2+i\epsilon}}_{\text{Feynman propagator}} + \underbrace{\int \frac{d^4k}{i}\,\frac{e^{-ik(x-y)}}{k^2+i\epsilon}\left(\frac{k_\mu k_\nu}{\vec{k}^2} - \frac{k_0}{\vec{k}^2}(k_\mu\eta_\nu+k_\nu\eta_\mu)\right)}_{\text{vanishes due to the Ward identity}} + \underbrace{i\,\frac{\eta_\mu\eta_\nu\,\delta(x_0-y_0)}{4\pi|\vec{x}-\vec{y}|}}_{\text{instantaneous propagation}}$$

It can be demonstrated, that the instantaneous term in the propagator cancels with the contributions due to $\hat{H}_{Coulomb}$. For simplicity, consider the four-fermion Green function

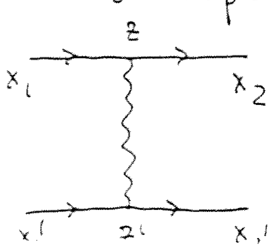
$$\langle\Omega|T\{\psi(x_1)\bar\psi(x_2)\psi(x_1')\bar\psi(x_2')\}|\Omega\rangle = \frac{\langle 0|T\{\psi(x_1)\bar\psi(x_2)\psi(x_1')\bar\psi(x_2')\, e^{-i\int dt(H_I + H_{Coulomb})}\}|0\rangle}{\langle 0|T\exp -i\int dt\,(H_I + H_{Coulomb})\}|0\rangle} \qquad (666)$$

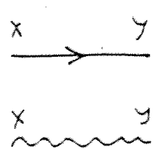
In the first nontrivial order in pert. theory we get (as usual, the denominator in eq. (666) cancels vacuum bubbles

$$\langle 0|T\{\psi(x_1)\bar\psi(x_2)\psi(x_1')\bar\psi(x_2')\Big(-\frac{e^2}{2}\int dz\,dz'\,\bar\psi \not{A}\psi(z)\,\bar\psi\not{A}\psi(z') + ie^2\int dz\,dz'\,\bar\psi\gamma_\mu\psi(z)\cdot$$
$$\cdot\bar\psi\gamma_\nu\psi(z')\,\frac{\eta_\mu\eta_\nu\,\delta(z_0-z_0')}{4\pi|\vec{z}-\vec{z}'|}\Big]\}|0\rangle =$$

$$= -e^2\int dz\,dz'\; \psi(x_1)\bar\psi\not{A}\psi(z)\bar\psi(x_2)\;\; \psi(x_1')\bar\psi\not{A}\psi(z')\bar\psi(x_2') + \qquad (667)$$
$$+ ie^2\int dz\,dz'\; \psi(x_1)\bar\psi\not{\eta}\psi(z_1)\bar\psi(x_2)\;\; \psi(x_1')\bar\psi\not{\eta}\psi(z')\bar\psi(x_2')\;\frac{\delta(z_0-z_0')}{4\pi|z-z'|} =$$
$$= -e^2\int dz\,dz'\,(S_F(x_1-z)\gamma^\mu S_F(z-x_2))\,(S_F(x_1'-z')\gamma^\nu S_F(z'-x_2'))\,\{D^F_{\mu\nu}(z-z') +$$
$$+ \cancel{\frac{i\eta_\mu\eta_\nu}{4\pi|\vec{z}-\vec{z}'|}\delta(z_0-z_0')} + \text{terms vanishing due to Ward identity}\} +$$
$$+ ie^2\int dz\,dz'\,(S_F(x_1-z)\gamma^\mu S_F(z-x_2))\,\cancel{(S_F(x_1'-z')\gamma^\nu S_F(z'-x_2'))\,\frac{\eta_\mu\eta_\nu\,\delta(z_0-z_0')}{4\pi|\vec{z}-\vec{z}'|}}$$
$$= -e^2\int dz\,dz'\,(S_F(x_1-z)\gamma^\mu S_F(z-x_2))\,(S_F(x_1'-z')\gamma^\nu S_F(z'-x_2'))\,D^F_{\mu\nu}(z-z')$$

the corresponding Feynman diagram is





$= S_F(x-y)$

$= D^F_{\mu\nu}(x-y)$

} Feynman propagators.

This cancellation between the contributions of $H_{Coulomb}$ and instantaneous term in $D^{tr}_{\mu\nu}$ can be proved in arbitrary order in pert. theory so we get

$$\langle\Omega|T\{\psi(x_1)\bar\psi(x_2)\psi(x_1')\bar\psi(x_2')\}|\Omega\rangle = \frac{\langle 0|T\{\psi(x_1)\bar\psi(x_2)\psi(x_1')\bar\psi(x_2')\, e^{-ie\int dz\, \bar\psi \not{A}\psi(z)}\}|0\rangle}{\langle 0|T\{\exp -ie\int dz\, \bar\psi\not{A}\psi(z)\}|0\rangle} \quad (668)$$

where the contraction of photon fields should be understood as Feynman propagator

$$\overbrace{A_\mu(x)\, A_\nu}(y) = \int\frac{dk}{i}\, e^{-ik(x-y)}\, \frac{g_{\mu\nu}}{k^2+i\epsilon} = D^F_{\mu\nu}(x-y) \quad (669)$$

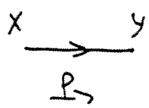
Similar formula can be written down for a general Green function (660)

$$\langle\Omega|T\{\psi(x_1)\ldots\bar\psi(y_n)A(z_1)\ldots A(z_\ell)\}|\Omega\rangle = \frac{\langle 0|T\{\psi(x_1)\ldots\bar\psi(y_n)A(z_1)\ldots A(z_\ell)\, e^{-ie\int dz\, \bar\psi\not{A}\psi(z)}\}|0\rangle}{\langle 0|T\{\exp -ie\int dz\, \bar\psi\not{A}\psi(z)\}|0\rangle} \quad (670)$$

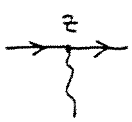
with the contraction of both fermion and photon fields given by Feynman propagators.

This leads to the following diagram technique:

I. Set of Feynman rules for QED in coordinate space

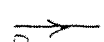
(•)  $S_F(x-y) = \int\frac{dp}{i}\, e^{-ip(x-y)}\, \frac{m+\not{p}}{m^2-p^2-i\epsilon}$ – fermion propagator

(••)  $D_F^{\mu\nu}(x-y) = \int\frac{dk}{i}\, e^{-ik(x-y)}\, \frac{g_{\mu\nu}}{k^2+i\epsilon}$ – photon propagator

(•••)  $-ie\gamma^\mu$ – vertex (and $\int dz$ is assumed)

II Set of Feynman rules in the momentum space for usual Green functions

$$G(p_1,\ldots p_m; k_1,\ldots k_n) = \int \Pi dx_i\, \Pi dz_i\, e^{i\Sigma p_i x_i + \Sigma k_i z_i}\langle\Omega|T\{\psi(x_1)\ldots\bar\psi(x_m)A(z_1)\ldots A(z_n)\}|\Omega\rangle \quad (671)$$

(*)  $\frac{m+\not{p}}{i(m^2-p^2-i\epsilon)}$ $\frac{g_{\mu\nu}}{i(k^2+i\epsilon)}$ – propagators

(**)  $-ie\gamma_\mu(2\pi)^4\delta(p_1-p_2-k)$ – vertex

(***) (−1) for each fermion loop (see eq. (550))

VB: In QED there is no symmetry factors

III Set of Feynman rules for $\mathcal{G}(p_1,\dots p_m;k_1,\dots k_n)$

(*) $\underset{p}{\longrightarrow} = \frac{m+\not{p}}{m^2-p^2-i\epsilon}$ $\quad$ $\sim\!\sim\!\sim = \frac{g_{\mu\nu}}{p^2+i\epsilon}$ - propagators

(**) $p \longrightarrow p-k$, k $= -e\gamma_\mu$ - vertex

(***) $\int\frac{d^4p}{i}$ for each undetermined loop momentum p + + extra (−1) for each fermion loop

Relation between $\mathcal{G}$ and G is

$$G(p_1,\dots p_m;k_1,\dots k_n) \quad i^{m+n-1}(2\pi)^4\delta(\Sigma p_i+\Sigma k_i)\,\mathcal{G}(p_1,\dots p_m;k_1,\dots k_n) \qquad (672))$$

Next step is to get Feynman rules for matrix elements of S-matrix ($\Leftrightarrow$ cross sections)

LSZ theorem for photons.

Maxwell's eqn:

$$\partial_\mu F^{\mu\nu}(x)=ej^\nu(x) \;\Rightarrow\; \underbrace{\partial_0 F^{0i}}_{\partial_0A^i-\partial^iA^0}+\partial_k \overbrace{F^{ki}}^{\partial^kA^i-\partial^iA^k}=ej^i \Rightarrow \qquad (673)$$

$$\Rightarrow(\partial_0^2+\partial_k\partial^k)A_i-\partial_0\partial^iA^0=ej^i \;\Rightarrow\; \partial^2A^i=ej^i+\partial^i\partial^0\phi$$

$A_0 \equiv$ scalar potential (see eq. (634))

Define $\quad j_i^{tr}(x)=j_i(x)+\frac{1}{e}\partial_i\partial^0\phi(x)$

$$\Rightarrow \partial^2A_i(x)=ej_i^{tr}(x) \qquad (674)$$

Similarly to the scalar case, we introduce the „in" and „out" field corresponding to free particles at $t\to-\infty$ and $t\to+\infty$

Quantum version of eq. (674)

$$\partial^2 \hat{A}_i(x) = e\hat{j}_i^{tr}(x) \tag{675}$$

where $\hat{A}_i(x)$ and $\psi_i(x)$ ($\bar{\psi}_i(x)$) are Heisenberg operators

$$\hat{A}_i(\vec{x},t) \equiv e^{i\hat{H}t} A_i(\vec{x}) e^{-i\hat{H}t}$$

$$\psi_i(\vec{x},t) \equiv e^{i\hat{H}t}\psi_i(\vec{x}) e^{-i\hat{H}t}, \quad \bar{\psi}(\vec{x},t) = e^{i\hat{H}t}\bar{\psi}(\vec{x}) e^{-i\hat{H}t} \tag{676}$$

(cf. eq. (196))

Proof:

$$\partial_0^2 \hat{A}_i(x) = \left(\frac{\partial}{\partial t}\right)^2 e^{i\hat{H}t}\hat{A}_i(\vec{x})e^{-iHt} = -e^{i\hat{H}t}[\hat{H}[\hat{H},A_i(\vec{x})]]e^{-iHt} \tag{677}$$

$$[H,A_i(\vec{x})] = [H_0^{e.m.},A_i(\vec{x})] + \underbrace{\cancel{[H^{Dirac},A_i(x)]}}_{=0} + \underbrace{\cancel{[\hat{H}_{int},A_i(x)]}}_{=0 \text{ since } [A_i(\vec{x}),A_j(\vec{y})]=0} + \overset{=0}{\cancel{[H_{Coulomb},A_i(x)]}}$$

$$= \left[H_0, \int\frac{d^3k}{\sqrt{2\omega_k}}\sum_\lambda (a_k^\lambda e^{+i\vec{k}\vec{x}} + a_k^{+\lambda}e^{-i\vec{k}\vec{x}})e_i^\lambda\right] = -\int\frac{d^3k}{\sqrt{2\omega_k}}\,\omega_k\sum_\lambda e_i^\lambda(k)(\hat{a}_k^\lambda e^{i\vec{k}\vec{x}} - \hat{a}_k^{+\lambda}e^{-i\vec{k}\vec{x}}) \tag{678}$$

$$\Rightarrow [\hat{H}[\hat{H},\hat{A}_i(x)]] = \left[\hat{H}_0^{e.m.}, \int\frac{d^3k}{\sqrt{2\omega_k}}\,\omega_k\sum_\lambda e_i^\lambda(k)(-\hat{a}_k^\lambda e^{i\vec{k}\vec{x}} + \hat{a}_k^{+\lambda}e^{-i\vec{k}\vec{x}})\right] +$$

$$+ e\int d^3y\, \bar{\psi}\gamma^\mu\psi(\vec{y})\left[A_\mu(\vec{y}), \int\frac{d^3k}{\sqrt{2\omega_k}}\,\omega_k\sum_\lambda e_i^\lambda(k)(-a_k^\lambda e^{i\vec{k}\vec{x}} + a_k^{+\lambda}e^{-i\vec{k}\vec{x}})\right] =$$

$$= \int\frac{d^3k}{\sqrt{2\omega_k}}\,\omega_k^2\sum_\lambda e_i^\lambda(k)(\hat{a}_k^\lambda e^{i\vec{k}\vec{x}} + \hat{a}_k^{+\lambda}e^{-i\vec{k}\vec{x}}) +$$

$$+ e\int d^3y\, j^\mu(\vec{y})\left[\int\frac{d^3k'}{\sqrt{2\omega_{k'}}}\sum_{\lambda'} e_\mu^{\lambda'}(k')(a_{k'}^{\lambda'}e^{i\vec{k}'\vec{y}} + a_{k'}^{+\lambda'}e^{-i\vec{k}'\vec{y}}), \int\frac{d^3k}{\sqrt{2\omega_k}}\,\omega_k\sum e_i^\lambda(k)(-a_k\cdot\right.$$

$$\left.\cdot e^{i\vec{k}\vec{x}} + a_k^{+\lambda}e^{-i\vec{k}\vec{x}})\right] =$$

$$= \int\frac{d^3k}{\sqrt{2\omega_k}}\,\vec{k}^2\sum_\lambda e_i^\lambda(k)(\hat{a}_k^\lambda e^{i\vec{k}\vec{x}} + \hat{a}_\lambda^{+k}e^{-i\vec{k}\vec{x}}) +$$

$$+ e\int d^3y\, j^\mu(\vec{y})\int\frac{d^3k}{2\cancel{\omega_k}}(e^{i\vec{k}(\vec{x}-\vec{y})} + e^{-i\vec{k}(\vec{x}-\vec{y})})\underbrace{\sum_\lambda e_\mu^\lambda(k)e_i^\lambda(k)}_{-g_{\mu i} + \frac{k_\mu k_i}{\vec{k}^2}}\cancel{\omega_k}$$

$$= -\vec{\nabla}^2 A_i(\vec{x}) - ie\int d^3y\, \hat{j}^n(\vec{y})\int d^3k\, e^{ik(x-y)}\left(g_{in} - \frac{k_i k_n}{\vec{k}^2}\right) =$$

$$= -\vec{\nabla}^2 A_i(\vec{x}) - ie\int d^3y\, \hat{j}^{tr\,n}(\vec{y})\,\delta_{in}^{tr}(\vec{x}-\vec{y}) = -\vec{\nabla}^2 A_i(\vec{x}) - e j_i^{tr}(\vec{x}) \tag{679}$$

Therefore

$$\partial_0^2\hat{A}_i(x) - \vec{\nabla}^2\hat{A}_i(x) = -e^{i\hat{H}t}(\cancel{-\vec{\nabla}^2 A_i(\vec{x})} - e\hat{j}_i^{tr}(\vec{x}))e^{-i\hat{H}t} - \cancel{\vec{\nabla}^2 e^{iHt}\hat{A}_i(\vec{x})e^{-i\hat{H}t}}$$

$$= e\, e^{i\hat{H}t}\hat{j}_i^{tr}(\vec{x})e^{-i\hat{H}t} = e\hat{j}_i^{tr}(x) \Rightarrow (675)$$

(675) $\Rightarrow$ Maxwell's eqn for Heisenberg operators

$$\partial_\mu \hat{F}^{\mu\nu}(x) = e\hat{\bar{\psi}}\gamma^\nu\psi(x) \tag{680}$$

LSZ for photons

Define

$$\sqrt{Z_3}\,\hat{A}^i_{in}(x) = \hat{A}^i(x) + ie\int dz\, \mathcal{D}_R^{ik}(x-z)\, j_k^{tr}(z) \tag{681}$$

$$\sqrt{Z_3}\,\hat{A}^i_{out}(x) = \hat{A}^i(x) + ie\int dz\, \mathcal{D}_A^{ik}(x-z)\, j_k^{tr}(z) \tag{682}$$

where

$$D_R^{\mu\nu}(x-z) \equiv \int\frac{dk}{i}\,\frac{g_{\mu\nu}}{k^2 - i\epsilon k_0}\,e^{-ik(x-z)} \quad \text{- retarded Green function}$$

$$D_A^{\mu\nu}(x-z) \equiv \int\frac{dk}{i}\,\frac{g_{\mu\nu}}{k^2 + i\epsilon k_0}\,e^{-ik(x-z)} \quad \text{- advanced Green function}$$

(683)

(cf. eq. (180)). (The proportionality constant is called Z_3 for historical reasons)

$\hat{A}_{in}(x)$ and $\hat{A}_{out}(x)$ satisfy the free-field equations

$$\sqrt{Z_3}\,\partial^2\hat{A}^i_{in}(x) = \partial^2\hat{A}^i(x) + ie\int dz\, \underbrace{\partial^2\mathcal{D}_R^{ik}(x-z)}_{i\delta(x-z),\text{ see eq. (683)}}\, j_k^{tr}(z) =$$

$$= \partial^2\hat{A}^i(x) - e\hat{j}^i_{tr}(x) = 0 \quad \text{(see eq. (675))} \Rightarrow$$

$$\Rightarrow \partial^2\hat{A}^i_{in}(x) = 0 \tag{684}$$

Similarly $\partial^2\hat{A}^i_{out}(x) = 0$ (685)

$D_R^{\mu\nu}(x-z) = 0$ when $x_0 < z_0 \Rightarrow$

$$\Rightarrow \lim_{t\to-\infty}\sqrt{Z_3}\,\hat{A}_{in}(\vec{x},t) = A(\vec{x},t). \tag{686}$$

Similarly $\lim_{t\to\infty}\sqrt{Z_3}\,\hat{A}_{out}(x,t) = A(\vec{x},t)$ (687)

Since $\hat{A}_{in}(x)$ and $\hat{A}_{out}(x)$ satisfy the free-field equations (684) and (685) they can be expanded in plane waves:

$$\vec{A}_{in}(x) = \int\frac{d^3k}{\sqrt{2\omega_k}}\,\vec{e}^{\lambda}(k)\left(a^{\lambda}_{in}(k)\,e^{-ikx} + a^{+\lambda}_{in}(k)\,e^{ikx}\right)\Big|_{k_0=\omega_k=|\vec{k}|} \tag{688}$$

$$\vec{A}_{out}(x) = \int\frac{d^3k}{\sqrt{2\omega_k}}\,\vec{e}^{\lambda}(k)\left(a^{\lambda}_{out}(k)\,e^{-ikx} + a^{+\lambda}_{out}(k)\,e^{ikx}\right)\Big|_{k_0=\omega_k} \tag{689}$$

„In"- vacuum $|\Omega\rangle_{in}$ is defined as a state annihilated by a_{in} and „out"- vacuum $|\Omega\rangle_{out}$ as a state annihilated by a_{out}

$$a^{\lambda}_{in}(k)|\Omega\rangle_{in} = 0 \quad , \quad a^{\lambda}_{out}(k)|\Omega\rangle_{out} = 0$$

Usual assumption is $|\Omega_{in}\rangle = |\Omega_{out}\rangle = |\Omega\rangle$ (see eq. (215)) – – nothing happens with vacuum state

$$a^{+\lambda_1}_{in}(k_1)\, a^{+\lambda_1'}_{in}(k_1') \ldots a^{+\lambda_m}_{in}(k_1^{(m)})\,|\Omega\rangle \quad \text{– free-particle „in" states}$$

$$a^{+\lambda_2}_{out}(k_2)\, a^{+\lambda_2'}_{out}(k_2') \ldots a^{+\lambda_n}_{out}(k_2^{(n)})\,|\Omega\rangle \quad \text{– free-particle „out" state}$$

Let us prove that

$$_{out}\langle k_2,\lambda_2;Y|k_1,\lambda_1;X\rangle = -\, e^{\lambda_2}_{\mu}(k_2)\, e^{\lambda_1}_{\nu}(k_1)\int dy\, dx\, e^{ik_2y - ik_1x}\,\frac{1}{Z_3} \cdot \partial_x^2\, \partial_y^2\; _{out}\langle Y|T\{\hat{A}^{\mu}(y)\,\hat{A}^{\nu}(x)\}|X\rangle_{in} \qquad (690)$$

where

$$|k_1,\lambda_1;X\rangle_{in} \equiv \sqrt{2\omega_{k_1}}\;\hat{a}^{+\lambda_1}_{in}(k_1)\,|X_{in}\rangle$$

$$|k_2,\lambda_2;Y\rangle_{out} \equiv \sqrt{2\omega_{k_2}}\;\hat{a}^{+\lambda_2}_{out}(k_2)\,|Y_{out}\rangle$$

and $|X\rangle_{in}$ and $|Y\rangle_{out}$ are arbitrary „in" and „out" states

Proof:

First,

$$\sqrt{2\omega_k}\;\hat{a}^{\lambda}_{out}(k) = -i\int d^3y\; e^{i\omega_k y_0 - i\vec{k}\vec{y}}\, \overleftrightarrow{\partial_0}\, \hat{A}^i_{out}(y)\, e^{\lambda}_i(k) =$$

$$= \lim_{y_0\to\infty} \frac{(-i)}{\sqrt{Z_3}}\int d^3y\; e^{i\omega_k y_0 - i\vec{k}\vec{y}}\, \overleftrightarrow{\partial_0}\, \hat{A}^i(y)\, e^{\lambda}_i(k) =$$

$$= \frac{-i}{\sqrt{Z_3}}\int d^4y\; e^{\lambda}_i(k)\,\frac{\partial}{\partial y_0}\Big(\underbrace{e^{i\omega_k y_0 - i\vec{k}\vec{y}}}_{e^{iky}|_{k_0=\omega_k}}\, \overleftrightarrow{\partial_0}\, \hat{A}^i(y)\Big) - i\lim_{y_0\to-\infty}\int d^3y\; e^{\lambda}_i(k)\,\frac{e^{iky}}{\sqrt{Z_3}}\,\overleftrightarrow{\partial_0}\,\hat{A}^i(y)$$

$$= \frac{-i}{\sqrt{Z_3}}\int d^4y\; e^{\lambda}_i(k)\, e^{iky}\Big|_{k_0=\omega_k}(\omega_k^2 + \partial_0^2)\,\hat{A}^i(y) \;+\; \sqrt{2\omega_k}\;\hat{a}_{in}(k)$$

$$= \frac{-i}{\sqrt{Z_3}}\int dy\; e^{\lambda}_{\mu}(k)\, e^{iky}\,\partial^2 \hat{A}^{\mu}(y)\Big|_{k_0=\omega_k} \;+\; \sqrt{2\omega_k}\;\hat{a}_{in}(k) \qquad (691)$$

Therefore

$$ {}_{out}\langle k_2,\lambda_2;Y|k_1,\lambda_1;X\rangle_{in} = {}_{out}\langle Y|\hat{a}^{\lambda_2}_{out}(k_2)|k_1,\lambda_1;X\rangle_{in} = \tag{692}$$

$$\frac{-i}{\sqrt{Z_3}}\int dy\; e^{\lambda_2}_{\mu}(k_2)\, e^{ik_2y}\, {}_{out}\langle Y|\partial^2\hat{A}^{\mu}(y)|k_1,\lambda_1;X\rangle_{in} + \underbrace{\cancel{{}_{out}\langle Y|a^{\lambda_2}_{in}(k_2)|k_1,\lambda_1;X\rangle_{in}}}_{=0}$$

0 if $k_2\neq k_1$ and $k_2\neq$ any other momentum in $|X\rangle_{in}$

Next, we repeat (691) for the incoming photon

$$\sqrt{2\omega_k}\,\hat{a}^{+\lambda}_{in}(k) = i\int dx\; e^{-ikx}\overleftrightarrow{\partial_0}\hat{A}^{i}_{in}(x)\,e^{\lambda}_{i}(k)\big|_{k_0=\omega_k} \Rightarrow$$

$$\Rightarrow \langle Y_{out}|\partial^2\hat{A}^{\mu}(y)\,\hat{a}^{+\lambda_1}_{in}(k_1)|X_{in}\rangle = i\int d^3x\; e^{\lambda_1}_{\nu}(k_1)\,e^{-ik_1x}\langle Y_{out}|\partial^2\hat{A}^{\mu}(y)\overleftrightarrow{\frac{\partial}{\partial x_0}}\cdot$$

$$\cdot\,\hat{A}^{\nu}_{in}(x)|X_{in}\rangle\big|_{k_0=\omega_{k_1}} = \text{take } x_0\to-\infty =$$

$$=\frac{i}{\sqrt{Z_3}}\int d^3x\; e^{\lambda_1}_{\nu}(k_1)(\exp -ik_1x)\;{}_{out}\langle Y|\partial^2\hat{A}^{\mu}(y)\overleftrightarrow{\frac{\partial}{\partial x_0}}\hat{A}^{\nu}(x)|X\rangle_{in}\big|_{x_0\to-\infty} =$$

$$= (x_0=+\infty) - \{(x_0=+\infty)-(x_0=-\infty)\} =$$

$$=\frac{i}{\sqrt{Z_3}}\int d^3x\; e^{\lambda_1}_{\nu}(k_1)\,e^{-ik_1x}\langle Y_{out}|\overleftrightarrow{\frac{\partial}{\partial x_0}}\hat{A}^{\nu}(x)\,\partial^2\hat{A}_{\mu}(y)|X_{in}\rangle\big|_{x_0\to+\infty} -$$

$$-\frac{i}{\sqrt{Z_3}}\int d^4x\; e^{\lambda_1}_{\nu}(k_1)\frac{\partial}{\partial x_0}\Big[e^{-ik_1x}\langle Y_{out}|T\{\partial^2\hat{A}^{\mu}(y)\overleftrightarrow{\frac{\partial}{\partial x_0}}\hat{A}^{\nu}(x)\}|X_{in}\rangle\Big]\Big|_{k_0=\omega_{k_1}}$$

$$= \underbrace{\cancel{\langle Y_{out}|a^{+\lambda_1}_{out}(k_1)\partial^2A^{\mu}(y)|X_{in}\rangle}}_{=0} - i\int d^4x\; e^{\lambda_1}_{\nu}(k_1)\,e^{-ik_1x}\left(\omega^2_{k_1}+\frac{\partial^2}{\partial x_0^2}\right)\frac{1}{\sqrt{Z_3}}\cdot$$

0 if $k_1\neq$ any of the momenta in $|Y_{out}\rangle$

$$\cdot\langle Y_{out}|T\{\partial^2\hat{A}^{\mu}(y)\,\hat{A}^{\nu}(x)\}|X_{in}\rangle =$$

$$= -\frac{i}{\sqrt{Z_3}}\int d^4x\; e^{\lambda_1}_{\nu}(k_1)\,e^{-ik_1x}\,\partial_x^2\partial_y^2\langle Y_{out}|T\{\hat{A}^{\mu}(y)\hat{A}^{\nu}(x)\}|X_{in}\rangle \tag{693}$$

Substituting (693) into eq. (692), we get (690):

$$ {}_{out}\langle k_2,\lambda_2;Y|k_1,\lambda_1;X\rangle_{in} = \frac{(-i)^2}{Z_3}\int dx\,dy\; e^{\lambda_2}_{\mu}(k_2)\,e^{\lambda_1}_{\nu}(k_1)\,e^{-ik_1x+ik_2y}\cdot$$

$$\cdot\,\partial_x^2\partial_y^2\langle Y_{out}|T\{\hat{A}^{\nu}(x)\,\hat{A}^{\mu}(y)\}|X_{in}\rangle$$

In general,

$$\text{out}\langle k_2,\lambda_2;\dots k_2^{(n)},\lambda_2^{(n)}|k_1,\lambda_1;k_1',\lambda_1';\dots k_1^{(m)},\lambda_1^{(m)}\rangle_{in} =$$

$$= \left(\frac{-i}{\sqrt{Z_3}}\right)^{m+n} \int \prod_{i=1}^{m} dx_i\, e^{\lambda_1^{(i)}}_{\ i}(k_1)\, e^{-ik_1^{(i)}\cdot} \prod_{i=1}^{n} dy_i\, e^{\lambda_2^{(i)}}_{\mu_i}(k_2)\, e^{ik_2^{(i)}y_i} \tag{694}$$

$$\prod_1^n \frac{\partial^2}{\partial y_i^2} \prod_1^m \frac{\partial^2}{\partial x_i^2} \langle\Omega|T\{A^{\mu_1}(y_1)\dots A^{\mu_n}(y_n)\,A^{\nu_1}(x_1)\dots A^{\nu_m}(x_m)\}|\Omega\rangle$$

If we neglect $Z_3 - 1 = O(e^2)$,

(694) $\Rightarrow$

Peskin's rule

$$\text{out}\langle \underbrace{p_2,s_2;\dots p_2^{(n_2)},s_2^{(n_2)}}_{\text{electrons \& positrons}};\underbrace{k_2,\lambda_2;\dots k_2^{(m_2)},\lambda_2^{(m_2)}}_{\text{photons}}|p_1,s_1;\dots p_1^{(n_1)},s_1^{(n_1)};k_1,\lambda_1;\dots k_1^{(m_1)},\lambda_1^{(m_1)}\rangle_{in}$$

$$= \langle p_2,s_2;\dots k_2^{(m_2)},\lambda_2^{(m_2)}|T\exp\left(-ie\int dz\,\bar\psi(z)\not{A}(z)\psi(z)\right)|p_1,s_1;\dots k_1^{(m_1)},\lambda_1^{(m_1)}\rangle \tag{695}$$

where

$$|p_1,s_1;\dots k_1^{(m_1)},\lambda_1^{(m_1)}\rangle = \prod\sqrt{2E_i}\; a_{p_1}^{+s_1}\dots a_{k_1^{(m_1)}}^{+\lambda_1^{(m_1)}}|0\rangle$$

$$\langle p_2,s_2;\dots k_2^{(m_2)},\lambda_2^{(m_2)}| = \prod\sqrt{2E_i}\;\langle 0|a_{p_2}^{s_2}\dots a_{k_2^{(m_2)}}^{\lambda_2^{(m_2)}} \tag{696}$$

- states in the interaction representation.

The contractions are Feynman propagators

$$\psi(x)\bar\psi(y) = S_F(x-y) = \int \frac{dp}{i}\, e^{-ip(x-y)}\,\frac{m+\not{p}}{m^2-p^2-i\epsilon}$$

$$A_\mu(x)A_\nu(y) = D^F_{\mu\nu}(x-y) = \int\frac{dk}{i}\,e^{-ik(x-y)}\,\frac{g_{\mu\nu}}{k^2+i\epsilon}$$

and also

$$A_\mu(x)\;a_k^{+\lambda}\sqrt{2\omega_k} = e_\mu^\lambda(k)\,e^{-ikx}$$

$$\sqrt{2\omega_k}\,a_k^\lambda\;A_\mu(x) = e_\mu^\lambda\,e^{ikx} \tag{697}$$

$$\psi(x)\;a_p^{+s}\sqrt{2E_p} = u(p,s)\,e^{-ipx}$$

$$\bar\psi(x)\;b_p^{+s}\sqrt{2E_p} = \bar v(p,s)\,e^{-ipx}$$

$$\sqrt{2E_p}\;a_p^s\;\bar\psi(x) = \bar u(p,s)\,e^{ipx}$$

$$\sqrt{2E_p}\;b_p^s\;\psi(x) = v(p,s)\,e^{ipx}$$

For matrix elements of S (and M) matrix $\quad S = 1 + (2\pi)^4 \delta(\,) i M$

integer number

$$M(p_1, .. p_1^{(m_1)}, k_1, .. k_1^{(n_1)}; p_2, .. p_2^{(m_2)}; k_2, .. k_2^{(n_2)}) = (-1)^{\#} G^{amp}_{conn}(p_1, ... k_1^{(n_1)}; p_2, ... k_2^{(n_2)}) \cdot$$

- $\left\{ \begin{array}{l} u(p_1, s_1) \text{ for incoming fermion} \\ \bar{u}(p_2, s_2) \text{ -"- outgoing -"-} \\ \bar{v}(p_1, s_1) \text{ for incoming antifermion} \\ v(p_2, s_2) \text{ -"- outgoing -"-} \\ \left.\begin{array}{l} e^{\lambda}_{\mu}(p_2) \\ e^{\lambda}_{\mu}(p_1) \end{array}\right\} \text{for incoming or outgoing photon} \end{array} \right.$