Renormalization in terms of "renormalized fields"

$$\mathcal{L} = -\frac{1}{4} F^{(0)}_{\mu\nu} F^{(0)\mu\nu} + \bar{\Psi}_0 (i\not{\partial} + e_0 \not{A} - m_0)\Psi_0 \qquad e_0, m_0 \text{ - bare charge and mass}$$

Feynman diagrams are UV divergent $\Rightarrow$ cutoff μ

Renormalization $\mathcal{L} = \mathcal{L}_b + \mathcal{L}_c$ (Peskin ch. 10.3)

$$\mathcal{L}_b = -\frac{1}{4} F_{\mu\nu}^2 + \bar{\Psi}(i\not{\partial} - m_{ph} + e_{ph}\not{A})\Psi \qquad (780)$$

$$\mathcal{L}_c = -\frac{1}{4}(Z_3 - 1) F_{\mu\nu}^2 + (Z_2 - 1)\bar{\Psi}(i\not{\partial} - m_{ph})\Psi + (Z_1 - 1) e_{ph} \bar{\Psi}\not{A}\Psi + Z_2 \delta m \bar{\Psi}\Psi$$

e_{ph}, m_{ph} - physical charge and mass

$\Psi = Z_2^{-1/2}\Psi_0$, $A = Z_3^{-1/2} A_{(0)}$ - renormalized fields.

Z_2 and $\underbrace{Z_3}_{\text{(and } \delta m)}$ are found as a series in e_{ph}^2 using the conditions

$$(*) \quad G(p) \xrightarrow[p^2 \to m_{ph}^2]{} \frac{1}{m_{ph} - \not{p}} \qquad (781)$$

$$(**) \quad \mathcal{D}_{\mu\nu}(p) \xrightarrow[p^2 \to 0]{} \frac{g_{\mu\nu}}{p^2} \qquad (782)$$

Feynman diagrams with this prescription are finite if one computes them in terms of e_{ph} and m_{ph}.

*): 1. m must be preserved as the position of the pole in exact Green function $\Rightarrow$ $G(x,y) \equiv \langle \Omega | T\{\Psi(x)\bar{\Psi}(y)\} | \Omega \rangle$

$$\Rightarrow \frac{1}{m - \not{p} + \Sigma(p) - \delta m} \longrightarrow \frac{1}{m - \not{p}} \Rightarrow \delta m = \Sigma(p)\Big|_{\not{p} = m}$$

In the second order we had $\delta m =$  $= \frac{3\alpha}{4\pi} \frac{m}{(2 - d/2)} + \frac{3\alpha}{4\pi} m \ln \mu^2/m^2 \qquad (783)$

2. Residue in the pole $m^2 \to p^2$ must be preserved as 1.

$(\Sigma(p) = -$  $)$

$$\frac{1}{m - \not{p} + \Sigma(\not{p}) - \Sigma(m)} \to \frac{1}{m - \not{p}} \Rightarrow \frac{\partial \Sigma}{\partial \not{p}}\Big|_{\not{p} = m} = 0$$

$\frac{\partial}{\partial \not{p}}(-$  $- (Z_2 - 1)$  $) \Rightarrow (Z_2 - 1) = + \frac{\partial}{\partial \not{p}}(-$  $)$

$-$  $= \frac{\alpha}{4\pi} \frac{m - \not{p}}{2 - d/2} + \frac{\alpha}{4\pi}(m - \not{p}) \ln \frac{\mu^2}{m^2}$ + (mass structure)

$$\Rightarrow (Z_2 - 1) = -\frac{\alpha}{4\pi} \frac{1}{2 - d/2} - \frac{\alpha}{4\pi} \ln \frac{\mu^2}{m^2} \qquad (784)$$

(**) Residue at the photon pole must be also 1 in each order in α.

$$\mathcal{D}_{\mu\nu}(q) = \frac{g_{\mu\nu}}{q^2(1+\Pi(q^2))} \xrightarrow{q^2\to 0} \frac{g_{\mu\nu}}{q^2} \qquad \Pi(0) = 0$$

$$\Pi(q^2) = \quad + \underset{(Z_3 - 1)}{} \quad = \frac{\alpha}{3\pi}\frac{1}{2-d/2} + \frac{\alpha}{3\pi}\ln\frac{\mu^2}{m^2} + (Z_3 - 1) = 0$$

$$\Rightarrow Z_3 - 1 = -\frac{\alpha}{3\pi}\frac{1}{2-d/2} - \frac{\alpha}{3\pi}\ln\frac{\mu^2}{m^2} \qquad (785)$$

4. Physical charge (Coulomb potential between two well separated electrons) must be e_{ph}
Due to Ward identity $Z_1 = Z_2$, (4) follows from (2) and (3)

Corresponding counterterm is (see eq. (784) and $Z_1 = Z_2$)

$$(Z_1 - 1)\gamma_\mu = \left(-\frac{\alpha}{4\pi}\frac{1}{2-d/2} - \frac{\alpha}{4\pi}\ln\frac{\mu^2}{m^2}\right) \qquad (786)$$

Thus, the correction to $\Gamma_\mu(m,m)$ vanishes (due to Ward identity)

$$\underset{p\to m}{} \quad - \frac{\alpha}{4\pi}\frac{\gamma_\mu}{2-d/2} - \frac{\alpha\gamma_\mu}{4\pi}\ln\frac{\mu^2}{m^2} = 0. \qquad (787)$$

This means that the answer for the three-point Green function at arbitrary momenta

$$\underset{p_1 \quad p_2}{} \quad - \frac{\alpha}{4\pi}\frac{1}{2-d/2}\gamma_\mu - \frac{\alpha}{4\pi}\gamma_\mu\ln\frac{\mu^2}{m^2} = \underset{p_1 \quad p_2}{} \; - \; \boxed{\underset{m \quad m}{}} =$$

$$= \text{finite}$$

Check:

$$\underset{p_1 \quad p_2}{\overset{p_1+p \;\; p_2+p}{}} = e^3\int\frac{d^dp}{(2\pi)^4 i}\frac{\gamma_\alpha(m+\not p_1+\not p)\gamma_\mu(m+\not p_2+\not p)\gamma_\alpha \quad M^{4-d}}{p^2(m^2-(p_1+p)^2-i\epsilon)(m^2-(p_2+p)^2-i\epsilon)} \rightarrow$$

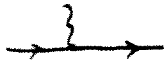
$$\xrightarrow[p\to\infty]{} e^3\int\frac{d^dp}{i}\frac{\gamma_\alpha\not p\gamma_\mu\not p\gamma_\alpha M^{4-d}}{p^2(p^2-m^2+i\epsilon)^2} \simeq \frac{e^3}{16\pi^2}\int_0^\infty d^dp\frac{\gamma_\mu M^{4-d}}{(m^2-p^2-i\epsilon)^2} \longrightarrow \frac{\alpha e}{4\pi}\left(\frac{1}{2-d/2} + \ln\frac{\mu^2}{m^2}\right)$$

(787a)

So, the final set of Feynman rules is

 $\dfrac{1}{m-\not p-i\epsilon}$

 $\dfrac{1}{k^2+i\epsilon}$

 $e\gamma_\mu$

m - physical mass
e - physical charge
LSZ: S-matrix is the Green function (amputated without any coefficients)

$-(Z_2-1)(m-\not p)$

 $Z_2-1 = -\frac{\alpha}{4\pi}\frac{1}{2-d/2} - \frac{\alpha}{4\pi}\ln\frac{M^2}{m^2} + O(\alpha)$

δm

 - mass couterterm

$-(Z_3-1)k^2$

 $Z_3-1 = -\frac{\alpha}{3\pi}\frac{1}{2-d/2} - \frac{\alpha}{3\pi}\ln\frac{M^2}{m^2} + O(\alpha^4)$

 $Z_1-1 = Z_2-1 = -\frac{\alpha}{4\pi(2-d/2)} - \frac{\alpha}{4\pi}\ln\frac{M^2}{m^2} + O(\alpha^4)$

$(Z_1-1)\gamma_\mu$

} counterterm

Order by order, these counterterms will eliminate the UV divergencies in the Feynman diagrams

Another renormalization scheme: subtractions at the Euclidean point $p^2=-M^2$

$$\mathcal{L} = \mathcal{L}_b + \mathcal{L}_c \tag{788}$$

$$\mathcal{L}_b = -\frac{1}{4}F_{\mu\nu}^2 + \bar\psi(i\partial - m(M) + e(M)\not A)\psi$$

$$\mathcal{L}_c = -\frac{1}{4}(Z_3(M)-1)F_{\mu\nu}^2 + (Z_2(M)-1)\bar\psi(i\not\partial - m(M))\psi \overbrace{}^{-Z_2(M)\delta m(M)\bar\psi\psi} + (Z_1(M)-1)e(M)\bar\psi\not A\psi$$

$e(M)$, $m(M)$ - parameters

$$e(M) = e_{ph}(1 + c_1 e_{ph}^2 + c_2 e_{ph}^4 + \dots)$$
$$m(M) = m_{ph}(1 + c_1' e_{ph}^2 + c_2' e_{ph}^4 + \dots)$$

$c_1, c_2, \dots c_1', c_2', \dots$ are <u>finite</u>

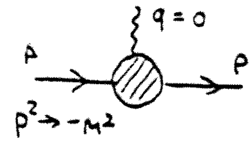
Again, $Z_2(M)$ and $Z_3(M)$ can be found using the two requirements (*) and (**)

(*) $\mathcal{G}(p) \xrightarrow[p^2\to -M^2]{} \dfrac{1}{m(M) - \not{p}}$

(**) $\mathcal{D}_{\mu\nu}(p) \xrightarrow[p^2\to -M^2]{} \dfrac{g_{\mu\nu}}{p^2}$

} this enables us to find $Z_2(M)$ and $Z_3(M)$

Ward identity $\Rightarrow$ third condition



$= \gamma_\mu e(M)$

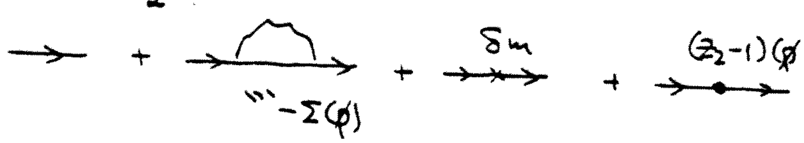
Renormalized charge

$$e = Z_1^{-1} Z_2 Z_3^{1/2} e_0 = Z_3^{1/2} e_0 \tag{789}$$

How this scheme works (at one-loop level):

1. Calculate Z_2

$\mathcal{G}(p) =$



$+ \cdots =$

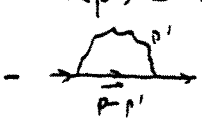
$$= \frac{1}{m - \not{p} + \Sigma(p) - \delta m + (Z_2 - 1)(m - \not{p})} \tag{790}$$

$\Sigma(p) = A(p^2)(m - \not{p}) + m B(p^2) \Rightarrow \mathcal{G}(p) = \dfrac{1}{(m-\not{p})(1 + A(p^2) + (Z_2 - 1)) + mB(p^2) - \delta}$

$\mathcal{G}(p) \xrightarrow[p^2\to -M^2]{} \dfrac{1}{m - \not{p}} \Rightarrow \begin{array}{l} A(M^2) + (Z_2 - 1) = 0 \\ m B(M^2) = \delta m \end{array} \Bigg\} \Rightarrow \begin{array}{l} Z_2 - 1 = - A(M^2) \\ \delta m = m B(M^2) \end{array}$

Result of the calculation

$\Sigma(p) =$



$$= - e^2 \mu^{4-d} \int \frac{d^d p'}{i} \frac{\gamma_\alpha (m + \not{p} - \not{p}')\gamma^\alpha}{(m^2 - (p-p')^2 - i\epsilon)(p'^2 + i\epsilon)} \xrightarrow[d\to 4]{} \frac{e^2 (4m - \not{p})}{16\pi^2 (2 - d/2)} +$$

$$+ \frac{e^2}{16\pi^2} \int_0^1 d\beta \left[(4m - 2\not{p}\bar\beta) \ln \frac{\mu^2}{m^2\beta - p^2 \bar\beta\beta - i\epsilon} + \not{p} - 2m \right]$$

$\Rightarrow A(p^2) = \dfrac{e^2}{16\pi^2(2 - d/2)} + \dfrac{e^2}{8\pi^2} \displaystyle\int_0^1 d\beta \left\{ \bar\beta \ln \frac{\mu^2}{m^2\beta - p^2\bar\beta\beta - i\epsilon} - \frac{1}{2} \right\}$ $\qquad \bar\beta \equiv 1 - \beta$

$$B(p^2) = \frac{3e^2}{16\pi^2(2 - d/2)} + \frac{e^2}{8\pi^2} \int_0^1 d\beta \left[(2m - \bar\beta) \ln \frac{\mu^2}{m^2\beta - p^2\bar\beta\beta - i\epsilon} - \frac{1}{2} \right] \tag{791}$$

$$\Rightarrow Z_2 - 1 = - A(M^2) = - \frac{e^2}{16\pi^2(2 - d/2)} - \frac{e^2}{8\pi^2} \int_0^1 d\beta \left[\bar\beta \ln \frac{\mu^2}{m^2\beta + M^2\bar\beta\beta} - \frac{1}{2} \right] \tag{792}$$

$$\delta m = m B(-M^2) = \frac{3e^2 m}{16\pi^2(2 - d/2)} + \frac{e^2 m}{8\pi^2} \int_0^1 d\beta \left[(2m - \bar\beta) \ln \frac{\mu^2}{m^2\beta + M^2\bar\beta\beta} - \frac{1}{2} \right] \tag{793}$$

2. Calculate Z_3

$\mathcal{D}_{\mu\nu}(p) =$ $+$  $+$  $+ \dots =$

$(Z_3-1)(p_\mu p_\nu - g_{\mu\nu} p^2)$

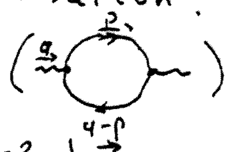
$$= \frac{g_{\mu\nu}}{p^2} + \frac{1}{p^2}\Pi(p^2)(p_\mu p_\nu - p^2 g_{\mu\nu})\frac{1}{p^2} + \frac{1}{p^2}(Z_3-1)(p_\mu p_\nu - g_{\mu\nu}p^2)\frac{1}{p^2} + \dots$$

$$= \frac{g_{\mu\nu} - p_\mu p_\nu/p^2}{p^2(1+\Pi(p^2)+(Z_3-1))} + \frac{p_\mu p_\nu}{p^4} = \frac{g_{\mu\nu}}{p^2(1+\Pi(p^2)+(Z_3-1))} + p_\mu p_\nu \cdot \text{sm} \qquad (794)$$

$$\mathcal{D}_{\mu\nu}(p) \xrightarrow[p^2\to -M^2]{} \frac{g_{\mu\nu}}{p^2} + \text{longitudinal terms} \Rightarrow \Pi(-M^2) + (Z_3 - 1) = 0$$

$$\Rightarrow Z_3 - 1 = -\Pi(-M^2)$$

Result of the calculation:

$\Pi(q^2) = \frac{1}{q_\mu q_\nu - q^2 g_{\mu\nu}}\Big($  $\Big) = \frac{e^2\mu^{4-d}}{q_\mu q_\nu - q^2 g_{\mu\nu}} \int \frac{dp}{i} \frac{\mathrm{Tr}\{\gamma_\mu\, m+\not p\, \gamma_\nu\, m+\not p -}{(m^2-p^2)(m^2-(q-p}$

$$\xrightarrow[d\to 4]{} \frac{e^2}{12\pi^2(2-d/2)} + \frac{e^2}{2\pi^2}\int_0^1 d\beta\, \bar\beta\beta \ln\frac{\mu^2}{m^2 - q^2\bar\beta\beta + i\epsilon} \qquad (795)$$

$$\Rightarrow Z_3 - 1 = -\Pi(-M^2) = -\frac{e^2}{12\pi^2(2-d/2)} - \frac{e^2}{2\pi^2}\int_0^1 d\beta\, \bar\beta\beta \ln\frac{\mu^2}{m^2 + M^2\bar\beta\beta}$$

After that, Green functions of renormalized fields are finite (796)

Connection between $e(M)$ and e_{ph}

$$e_0 = Z_3^{-1/2} e_{ph} \qquad Z_3(m_{ph}, e_{ph}) = 1 - \frac{e_{ph}^2}{12\pi^2(2-d/2)} - \frac{e_{ph}^2}{2\pi^2}\int_0^1 d\beta\, \bar\beta\beta \ln\frac{\mu^2}{m_{ph}^2(1-\bar\beta\beta)}$$

$$e_0 = Z_3^{-1/2} e(M) \qquad Z_3(m(M), e(M)) = 1 - \frac{e^2(M)}{12\pi^2(2-d/2)} - \frac{e^2(M)}{2\pi^2}\int_0^1 d\beta\, \bar\beta\beta \ln\frac{\mu^2}{m^2 + M^2\bar\beta\beta}$$

$$\Rightarrow \quad e(M)\, Z_3^{-1/2}(m(M), e(M), \mu) = e_{ph}\, Z_3^{-1/2}(m_{ph}, e_{ph}, \mu) \Rightarrow \tag{797}$$

$$\Rightarrow e(M) = e_{ph}\left(\frac{Z_3(m(M), e(M), \mu)}{Z_3(m_{ph}, e_{ph}, \mu)}\right)^{1/2}$$

$$\frac{Z_3(m(M), e(M), \mu)}{Z_3(m_{ph}, e_{ph}, \mu)} = 1 + \frac{e_{ph}^2}{12\pi^2(2-d/2)} + \frac{e_{ph}^2}{2\pi^2}\int_0^1 d\beta\, \bar\beta\beta \ln\frac{\mu^2}{m_{ph}^2(1-\bar\beta\beta)} - \frac{e^2(M)}{12\pi^2(2-d/2)}$$

$$- \frac{e^2(M)}{2\pi^2}\int_0^1 d\beta\, \bar\beta\beta \ln\frac{\mu^2}{m^2 + M^2\bar\beta\beta} \simeq 1 + \frac{e_{ph}^2}{2\pi^2}\int_0^1 d\beta\, \bar\beta\beta \ln\frac{m^2 + M^2\bar\beta\beta}{m^2(1-\bar\beta\beta)} \tag{798}$$

If $M \gg m$ $$\frac{Z_3(m(M), e(M), \mu)}{Z_3(m_{ph}, e_{ph}, \mu)} = 1 + \frac{e_{ph}^2}{12\pi^2}\ln\frac{M^2}{m^2} \Rightarrow$$

$$\Rightarrow e(M) = e_{ph}\left(1 + \frac{e_{ph}^2}{24\pi^2}\ln\frac{M^2}{m^2}\right) \Rightarrow \alpha(M) = \alpha\left(1 + \frac{\alpha}{12\pi}\ln\frac{M^2}{m^2}\right) \tag{799}$$