Outline

1. Introduction: BFKL pomeron in hign-energy pQCD
   - Regge limit in QCD.
   - Perturbative QCD at high energies.
   - BFKL and collider physics
2. High-energy scattering and Wilson lines
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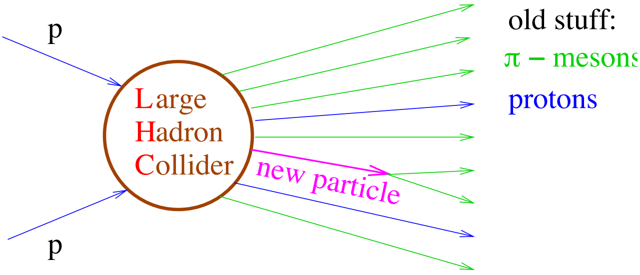
   - Evolution equation for color dipoles.
   - Light-ray vs Wilson-line operator expansion.
   - Rescaling in the Regge limit.
   - Propagators in a shock-wave background.
   - Leading order: BK equation.

Heisenberg uncertainty principle: $\Delta x = \frac{\hbar}{p} = \frac{\hbar c}{E}$

LHC: E=7 $\rightarrow$ 14 TeV $\Leftrightarrow$ distances $\sim 10^{-18}$ cm

(Planck scale is 10^{-33} cm - a long way to go!)



To separate a “new physics signal” from the “old” background one needs to understand the behavior of QCD cross sections at large energies

Strong interactions at asymptotic energies: Froissart bound

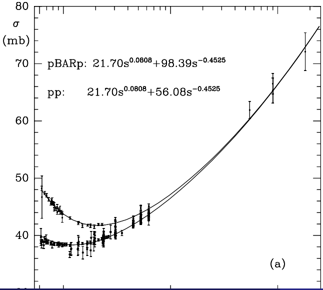
Regge limit: $E \gg$ everything else

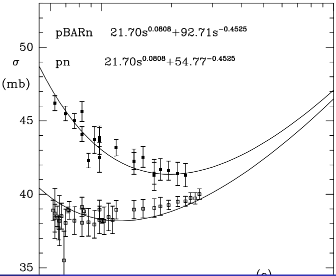
$$\left.\begin{array}{c}\text{Causality}\\ \text{Unitarity}\end{array}\right\} \quad \Rightarrow \quad \sigma_{\text{tot}} \overset{E\to\infty}{\leq} \ln^2 E$$

Froissart, 1962

Long-standing problem - not explained in any quantum field theory (or string theory) in 50 years!

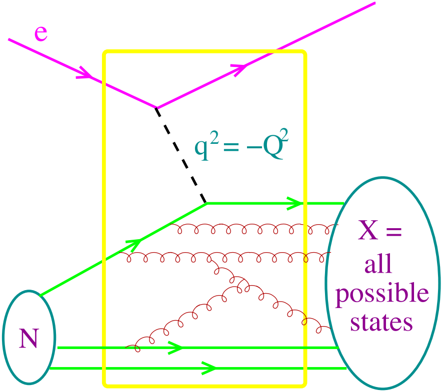
Experiment: $\sigma_{\text{tot}} \sim s^{0.08}$ $(s \equiv 4E^2_{\text{c.m.}})$. Numerically close to $\ln^2 E$.



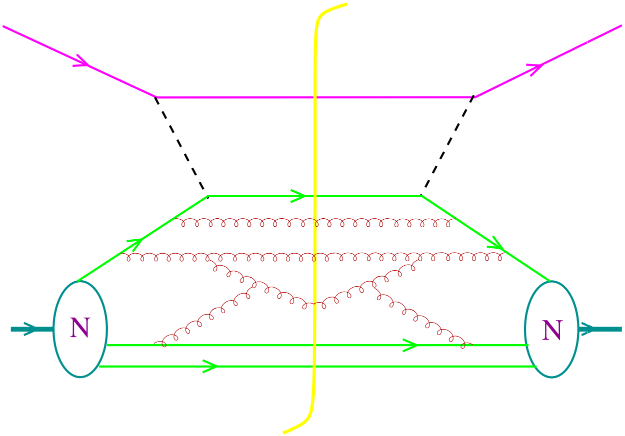


DIS: $ep \to e + X$

Asymptotic freedom: $\alpha_s(Q^2) \to 0$ as $Q^2 \to \infty$



Optical theorem: $\sigma_{\text{tot}} \;=\; \sum_X A^{\dagger}_{ep\to p+X} A_{ep\to p+X} \;=\; \Im A_{\text{forward}}$



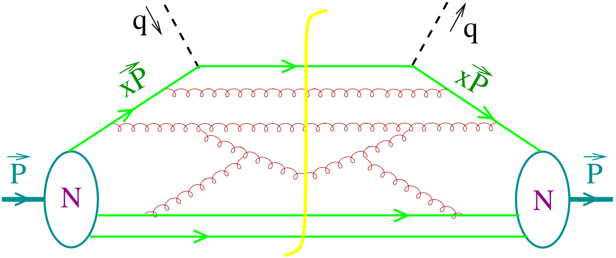
Parton model

$$\sigma_{\rm tot} \sim \int d^4x \, e^{iq\cdot x} \langle N|j_\mu(x) j_\nu(0)|N\rangle$$

Parton model (leading order of pQCD):

$$\sigma_{\rm tot} \sim \sum_q e_q^2 D_q(x_B), \qquad x_B = \frac{Q^2}{2p\cdot q}, \;\; q^2 = -Q^2$$

$D_q(x)$ = probability to find the quark with fraction x of nucleon's momentum



$D_q(x_B) \to D_q(x_B, Q^2)$ - "scaling violations"

DGLAP evolution (LLA(Q^2))

$$Q\frac{d}{dQ}D_q(x_B, Q^2) = K_{\rm DGLAP}D_q(x_B, Q^2)$$

Dokshitzer, Gribov, Lipatov, Altarelli, Parisi, 1972-77

$$K_{\rm DGLAP} = \alpha_s(Q)K_{\rm LO} + \alpha_s^2(Q)K_{\rm NLO} + \alpha_s^3(Q)K_{\rm NNLO}...$$

The DGLAP equation sums up

$$\sum_n \Big(\alpha_s \ln\frac{Q^2}{m_N^2}\Big)^n\big[a_n + b_n\alpha_s + c_n\alpha^2 s + ...\big]$$

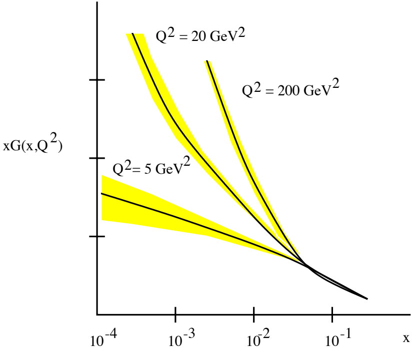
One fit at low $Q_0^2 \sim 1$ GeV^2 describes all the experimental data on DIS!

Regge limit in DIS: $E \gg Q \equiv x_B \ll 1$

DGLAP evolution $\equiv Q^2$ evolution

$$Q\frac{d}{dQ}D_g(x_B, Q^2) = K_{\text{DGLAP}}D_g(x_B, Q^2)$$

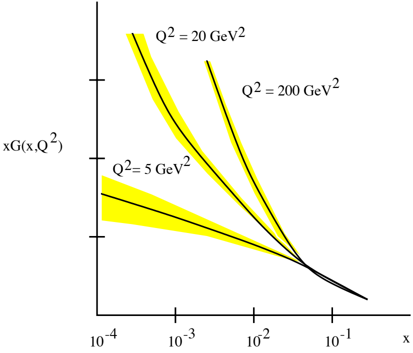
HERA data for $xD_g(x)$



Not really a theory -
needs the x-dependence of the input at $Q_0^2 \sim 1\text{GeV}^2$

Deep inelastic scattering at small x_B

HERA data for $xD_g(x)$



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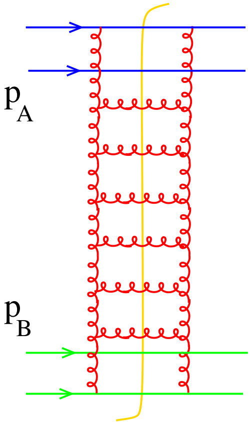
BFKL evolution $\equiv \ x_B$ evolution
(Balitsky, Fadin, Kuraev, Lipatov, 1975-78)

$$\frac{d}{dx_B}D_g(x_B, Q^2) = K_{\text{BFKL}}D_g(x_B, Q^2)$$

Theory, but with problems

In pQCD: Leading Log Approximation ⇒ BFKL pomeron

$$s = (p_A + p_B)^2 \simeq 4E^2$$



Leading Log Approximation (LLA(x)):

$$\alpha_s \ll 1, \quad \alpha_s \ln s \sim 1$$

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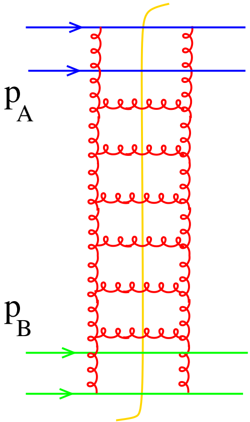

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The sum of gluon ladder diagrams gives

$$\sigma_{\text{tot}} \sim s^{12\frac{\alpha_s}{\pi}\ln 2}$$ BFKL pomeron

Numerically: for DIS at HERA

$$\sigma \sim s^{0.3} = x_B^{-0.3}$$

- qualitatively OK

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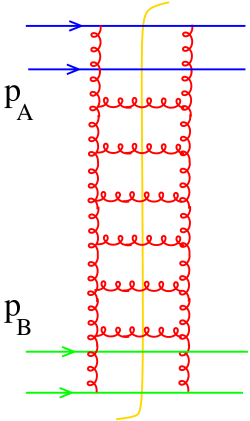

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Numerically: for DIS at HERA

$$\sigma \sim s^{0.3} = x_B^{-0.3}$$

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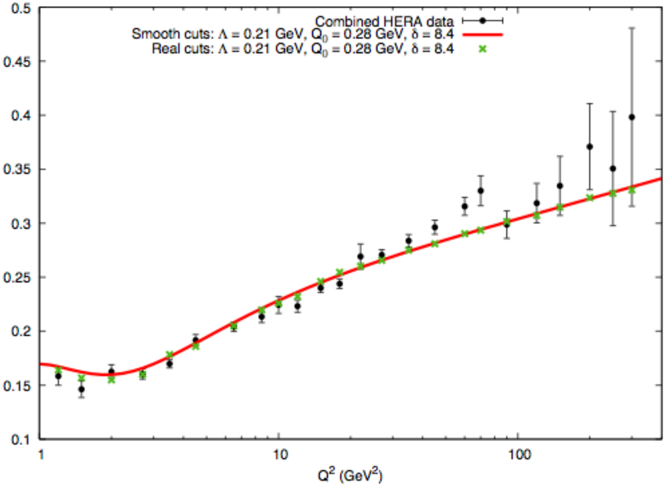
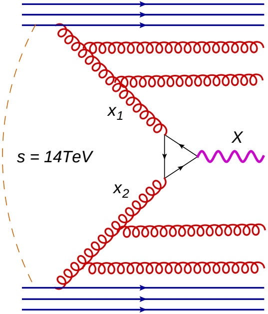
BFKL *vs* HERA data

$$F_2(x_B, Q^2) \;=\; c(Q^2)x_B^{-\lambda(Q^2)}$$



M.Hentschinski, A. Sabio Vera and C. Salas, 2010

DGLAP vs BFKL in particle production



Collinear factorization (LLA(Q^2)):

$$\sigma_H = \int dx_1 dx_2 D_g(x_1, m_H) D_g(x_2, m_H) \sigma_{gg\to H}$$

sum of the logs $\left(\alpha_s \ln \frac{m_X^2}{m_N^2}\right)^n$, $\ln \frac{s}{m_X^2} \sim 1$

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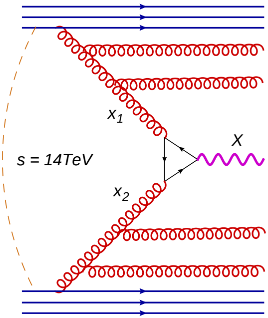

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LLA(x): k_T-factorization

$$\sigma_H = \int dk_1^\perp dk_2^\perp g(k_1^\perp, x_A) g(k_2^\perp, x_B) \sigma_{gg\to H}$$

- sum of the logs $\left(\alpha_s \ln x_i\right)^n$, $\ln \frac{m_X^2}{m_N^2} \sim 1$

Much less understood theoretically.

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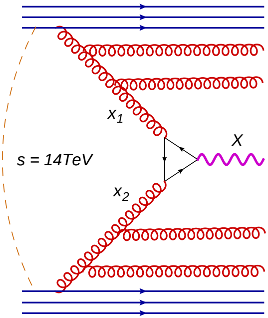

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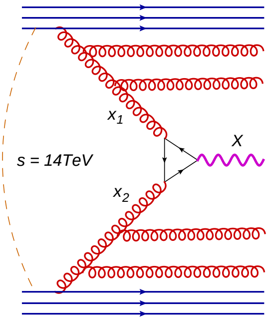
Much less understood theoretically.

For Higgs production in the central rapidity region $x_{1.2} \sim \frac{m_H}{\sqrt{s}} \simeq 0.01$ and we know from DIS experiments that at such x_B the DGLAP formalism works pretty well $\Rightarrow$ no need for BFKL resummation

DGLAP vs BFKL in particle production



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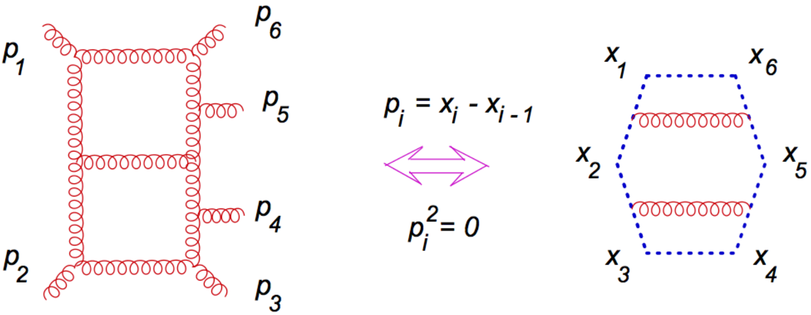
Much less understood theoretically.

For $m_X \sim 10$GeV (like $\bar{b}b$ pair or mini-jet) collinear factorization does not seem to work well $\Rightarrow$ some kind of BFKL resummation is needed.

MHV gluon amplitudes $\Leftrightarrow$ light-like Wilson-loop polygons

Alday, Maldacena (at large $\alpha_s N_c$)



Checked up to 6 gluons/2 loops (Korchemsky et. al).

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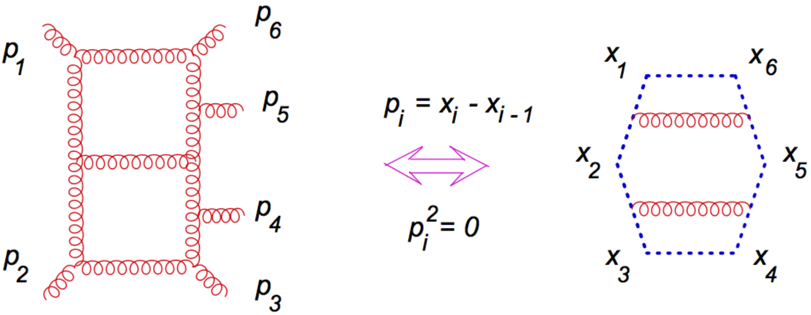

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BDS ansatz: $\ln A^{\rm MHV} =$ IR terms $+F_n, \quad F_n = \Gamma_{\rm cusp}(\text{angles}) + (F_n^{1)} + R_n)$

BFKL in multi-Regge region ⇒ asymptotics of remainder function R_n (Lipatov et a)l

Uses of BFKL: Anomalous dimensions of twist-2 operators

Structure functions of DIS are determined by matrix elements of twist-2 operators

$$\mathcal{O}_G^{(j)} = F_{\mu_1\xi} D_{\mu_2} ... D_{\mu_{j-1}} F_{\mu_j}^{\ \xi}$$

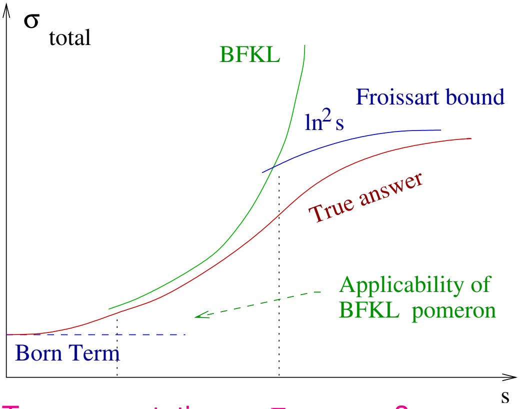
$$\mu^2 \frac{d}{d\mu^2} \mathcal{O}_G^{(j)} = \frac{\gamma_{(j)}(\alpha_s)}{4\pi} \mathcal{O}_G^{(j)}$$

BFKL gives asymptotics of $\gamma_{(j)}$ at $j \to 1$ in all orders in α_s

$$\gamma_{(j)} = \sum_n \Big(\frac{\alpha_s}{j-1}\Big)^n \Big[C^{(n)}_{\text{LO BFKL}} + \alpha_s C^{(n)}_{\text{NLO BFKL}}\Big]$$

Checked by explicit calculation of Feynman diagrams.up to 3 loops in QCD and $\mathcal{N} = 4$ SYM. (Janik et al)

Integrablility of spin chains corresponding to evolution of $\mathcal{N} = 4$ SYM operators $\Rightarrow \gamma_{(j)}$ in 5 loops agrees with BFKL (Janik et al).
For all order of pert. theory: Y-system of equations (Gromov, Kazakov, Viera). Hopefully agrees with BFKL.



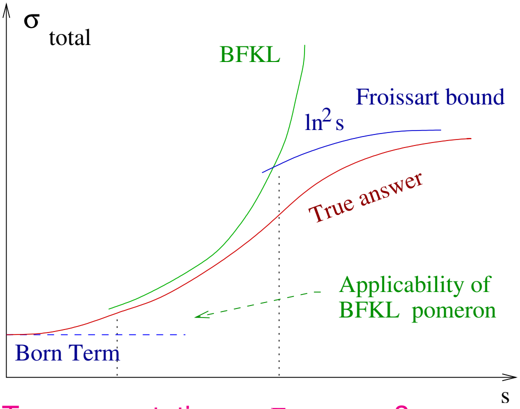
$\sigma_{\text{tot}} \sim s^{12\frac{\alpha_s}{\pi}\ln 2}$ violates Froissart bound $\sigma_{\text{tot}} \leq \ln^2 s$ $\Rightarrow$ pre-asymptotic behavior.

True asymptotics as $E \to \infty$ = ?

Possible approaches:

- Sum all logs $\alpha_s^m \ln^n s$
- Reduce high-energy QCD to $2+1$ effective theory

Towards the high-energy QCD



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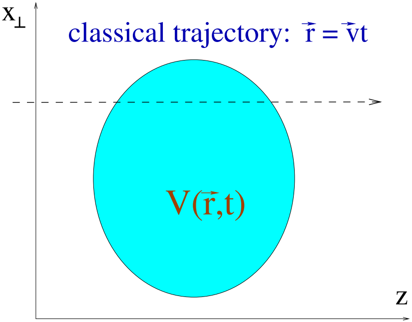
Lecture II: NLO corrections $\alpha_s^{n+1} \ln^n s$

High-energy scattering and "Wilson lines" in quantum mechanics



WKB approximation: $\Psi \sim e^{\frac{i}{\hbar}S}$

$$S = \int (pdz - Edt)$$

$$= -Et + \int^z dz' \sqrt{2m(E - V(z')}$$

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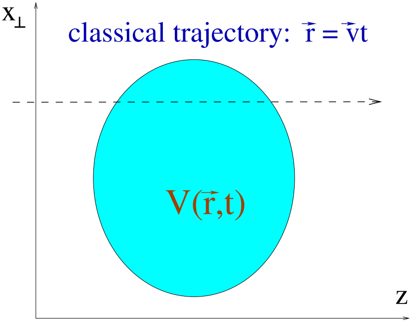

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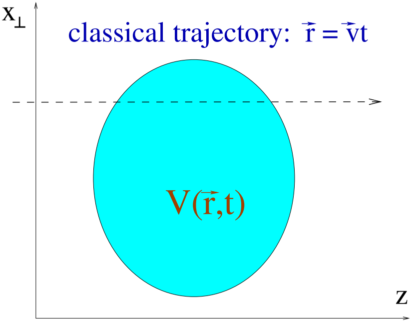
High energy: $E \gg V(x) \Rightarrow$

$$\Psi(\vec{r}, t) = e^{-\frac{i}{\hbar}(Et - kx)} \; e^{-\frac{i}{v\hbar}\int_{-\infty}^{z} dz' V(z')}$$



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Ψ at high energy = free wave $\times$ phase factor ordered along the line $\parallel \vec{v}$.

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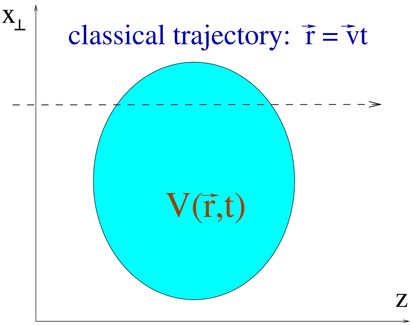

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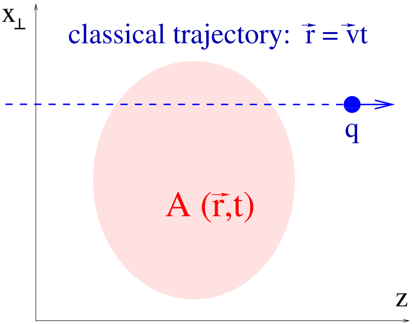
The scattering amplitude is proportional to $\Psi(t = \infty)$ defined by

$$U(x_\perp) \;=\; e^{-\frac{i}{v\hbar}\int_{-\infty}^{\infty} dz' V(z' + x_\perp)}$$

Glauber formula: $\sigma_{\rm tot} = 2\int d^2x_\perp \; [1 - \Re U(x_\perp)]$

High-energy phase factor in QED and QCD



$$S_e = \int dt \left\{ -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A} \right\}$$
$$= S_{\rm free} + \int dt(-e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A})$$

$\Rightarrow$ phase factor for the high-energy scattering is

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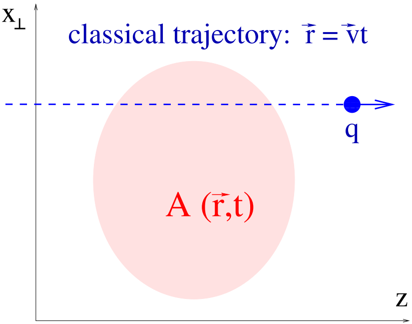
High-energy phase factor in QED and QCD



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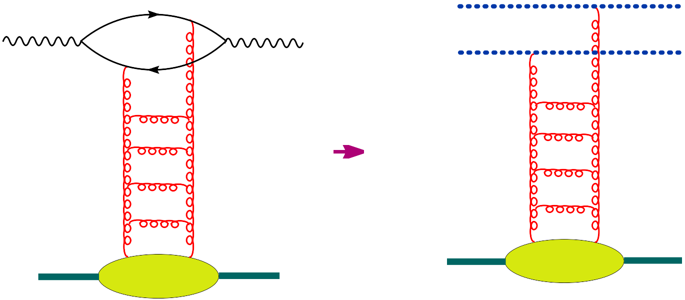
In QCD $e\to -g$, $A_\mu \to A_\mu \equiv A^a_\mu t^a$ t^a - color matrices

$$\Rightarrow\ U(x_\perp, v) = P\exp\{\frac{ig}{\hbar c}\int_{-\infty}^{\infty} dt\ \dot{x}_\mu A^\mu(x(t))\}$$ Wilson – line operator

(Later $\hbar = c = 1$)

DIS at high energy

- At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



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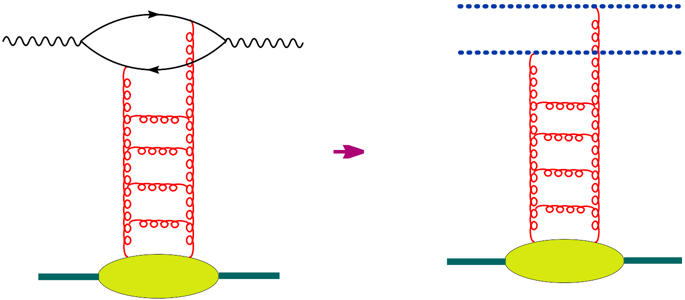
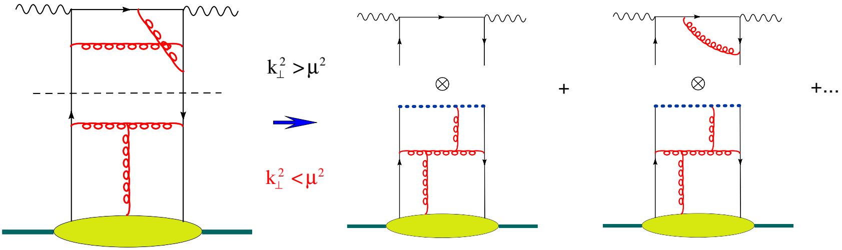


$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \mathrm{Tr}\{U(k_\perp) U^\dagger(-k_\perp)\} | B \rangle$$

Formally, ➔ means the operator expansion in Wilson lines

Light-cone expansion and DGLAP evolution in the NLO

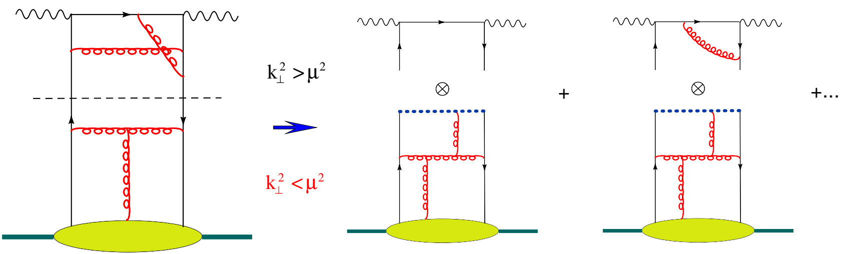


μ^2 - factorization scale (normalization point)

$k_\perp^2 \ > \ \mu^2$ - coefficient functions

$k_\perp^2 \ < \ \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)

Light-cone expansion and DGLAP evolution in the NLO



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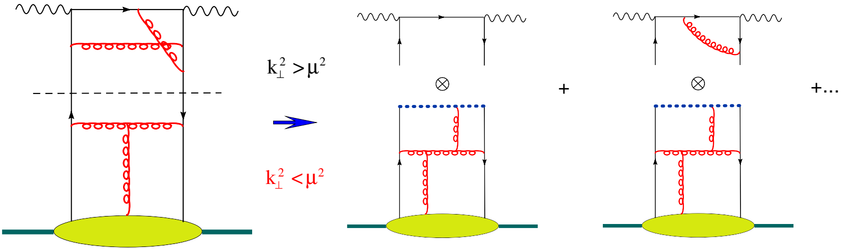
$k_\perp^2 \ < \ \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)

OPE in light-ray operators $(x-y)^2 \to 0$

$$T\{j_\mu(x)j_\nu(y)\} \ = \ \frac{x_\xi}{2\pi^2 x^4}\Big[1 + \frac{\alpha_s}{\pi}(\ln x^2\mu^2 + C)\Big]\bar\psi(x)\gamma_\mu\gamma^\xi\gamma_\nu[x,y]\psi(y) + O(\frac{1}{x^2})$$

$$[x,y] \ \equiv \ Pe^{ig\int_0^1 du\ (x-y)^\mu A_\mu(ux+(1-u)y)}$$ - gauge link



μ^2 - factorization scale (normalization point)

$k_\perp^2 \ > \ \mu^2$ - coefficient functions
$k_\perp^2 \ < \ \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)

Renorm-group equation for light-ray operators $\Rightarrow$ DGLAP evolution of parton densities $(x-y)^2 = 0$

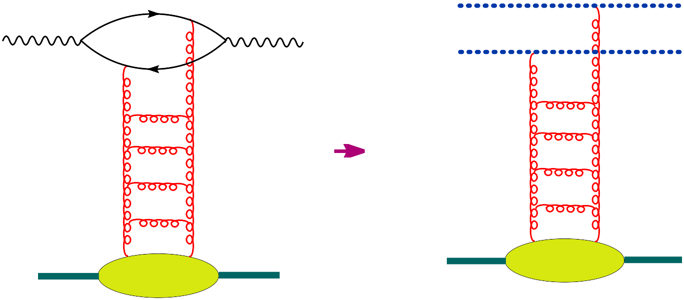
$$\mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x,y]\psi(y) \ = \ K_{\rm LO} \bar{\psi}(x)[x,y]\psi(y) + \alpha_s K_{\rm NLO} \bar{\psi}(x)[x,y]\psi(y)$$

Four steps of an OPE

- Factorize an amplitude into a product of coefficient functions and matrix elements of relevant operators.
- Find the evolution equations of the operators with respect to factorization scale.
- Solve these evolution equations.
- Convolute the solution with the initial conditions for the evolution and get the amplitude

DIS at high energy: relevant operators

- At high energies, particles move along straight lines $\Rightarrow$
  the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp)\langle B|\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\}|B\rangle$$

$$U(x_\perp) = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} du\; n^\mu A_\mu(un + x_\perp)\Big] \qquad \text{Wilson line}$$

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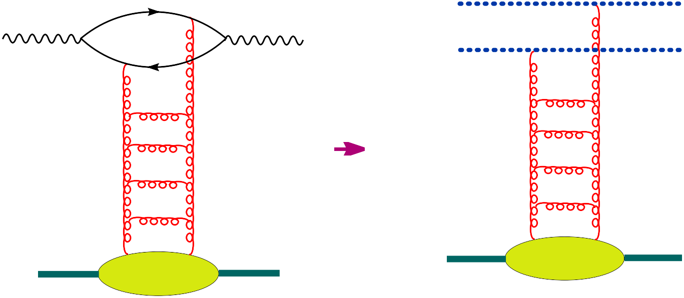

$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \mathrm{Tr}\{U(k_\perp) U^\dagger(-k_\perp)\} | B \rangle$$

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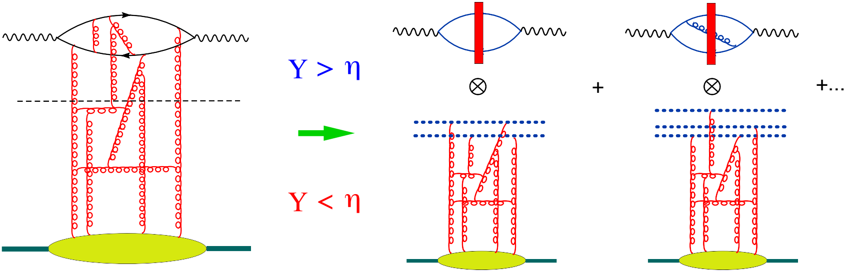
Formally, ➔ means the operator expansion in Wilson lines

Rapidity factorization



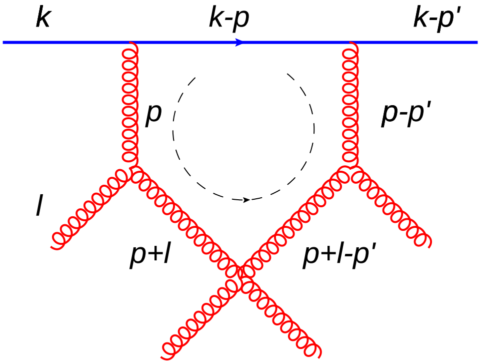
η - rapidity factorization scale

Rapidity Y > η - coefficient function ("impact factor")

Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U^\eta_x = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} dx^+ A^\eta_+(x_+, x_\perp)\Big]$$

$$A^\eta_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$



$p = \alpha p_1 + \beta p_2 + p_\perp$ - Sudakov variables.

$$g_{\mu\nu} \to \frac{2}{s} p_{1\mu} p_{2\nu}$$

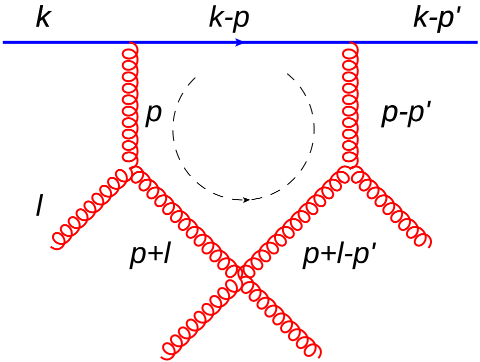
$$\not{p}_2 \frac{\not{k} - \not{p}}{(k-p)^2 + i\epsilon} \not{p}_2 \simeq \frac{\not{p}_2}{\beta_k - \beta_p - \frac{(\vec{k}-\vec{p})_\perp^2}{\alpha_k s} + i\epsilon\alpha_k}.$$

I will prove now that if I replace this by the "eikonal propagator"

$$\frac{\not{p}_2}{-\beta_p + i\epsilon\alpha_k},$$

the value of the loop integral over β_p remains unchanged.

Wilson lines from Feynman diagrams



$\alpha_k \gg \alpha_p, \alpha_l$

$p = \alpha p_1 + \beta p_2 + p_\perp$
- Sudakov variables.

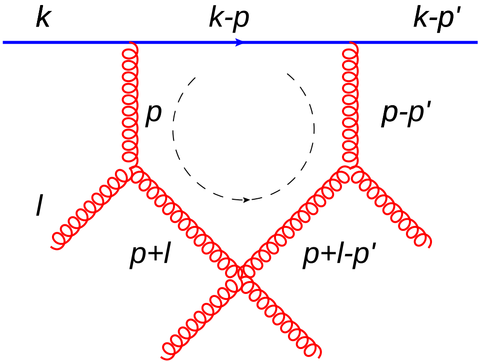
$$g_{\mu\nu} \to \frac{2}{s} p_{1\mu} p_{2\nu}$$

1. Residue at the pole of quark propagator $\beta_p = \beta_k - \frac{(\vec{k}-\vec{p})_\perp^2}{\alpha_k s}$

$$\text{Gluon}: \quad (\alpha_p + \alpha_l)\beta_l s - (p + \tilde{p})_\perp^2 + (\alpha_p + \alpha_l)\beta_k s - \frac{\alpha_p + \alpha_l}{\alpha_k}(\vec{k} - \vec{p})_\perp^2.$$

Fiirst two terms $\sim m^2$ while the second two $\sim \frac{\alpha_p}{\alpha_k} m^2$ ($\beta_k \sim \frac{m^2}{\alpha_k s}$)

$\Rightarrow$ same result as from the pole at $\beta_p = 0$.



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- Sudakov variables.

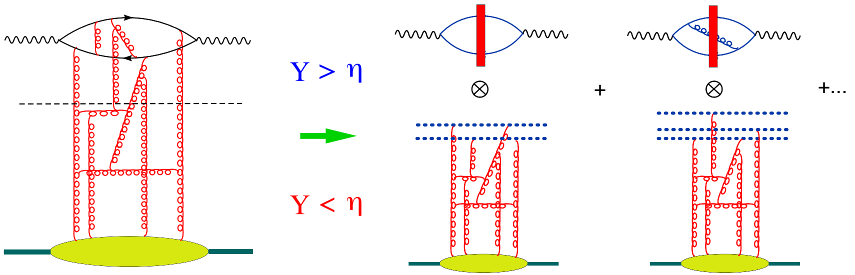
$$g_{\mu\nu} \to \frac{2}{s} p_{1\mu} p_{2\nu}$$

2. Residue at the pole of a gluon propagator $\beta_p = -\beta_l + \frac{(p+l)_\perp^2}{(\alpha_p+\alpha_l)s} \Rightarrow$

Quark prop : $$\frac{\not{p}_2}{\beta_l - \frac{(p+l)_\perp^2}{(\alpha_p+\alpha_l)s} + \beta_k - \frac{(\vec{k}-\vec{p})_\perp^2}{\alpha_k s} + i\epsilon\alpha_k} \to \frac{\not{p}_2}{\beta_l - \frac{(p+l)_\perp^2}{(\alpha_p+\alpha_l)s}}$$

(first two terms $\sim \frac{m^2}{\alpha_p s} \gg$ second two $\sim \frac{m^2}{\alpha_k s}$) $\Leftrightarrow$ quark pole $\frac{\not{p}_2}{-\beta_p + i\epsilon\alpha_k}$

Rapidity factorization



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Rapidity Y > η - coefficient function (“impact factor”)

Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \text{Pexp}\Big[ig\int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp)\Big]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

Spectator frame: propagation in the shock-wave background.

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

[$x \to z$: free propagation]$\times$
[$U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave]$\times$
[$z \to y$: free propagation]

Rescaling in the Regge limit

Amplitude = correlation function of 4 scalar currents

$$
\begin{aligned}
A(s,t) &= -i\int d^2z_\perp e^{-i(r,z)_\perp}\ \mathcal{N}^{-1}\int \mathcal{D}A\, e^{iS(A)}\det(i\nabla) \\
&\times \left\{\int dz_+ \int d^4x\, e^{-ip_A\cdot x}\,\langle j(x_-,x_+ + z_+,x_\perp + z_\perp)j(0,z_+,z_\perp)\rangle_A\right\} \\
&\times \left\{\int dz_- \int d^4y\, e^{-ip_B\cdot y}\,\langle j(y_- + z_-,y_+,y_\perp)j(z_-,0,0_\perp)\rangle_A\right\},
\end{aligned}
$$

Regge limit: $s=(p_A+p_B)^2\to\infty$, $p_A^2, p_B^2, t=-r_\perp^2$ - fixed

$$
\begin{aligned}
p_A &= \lambda\kappa e_- + \frac{p_A^2}{s}\kappa e_+, \quad e_+\cdot e_- = 1, \quad \kappa\equiv\sqrt{\frac{s}{2}} \\
p_A &= \lambda\kappa e_+ + \frac{p_b^2}{s}\kappa e_-,
\end{aligned}
$$

We "freeze" the gluon field, consider the "upper part"

$$\int dz_+ \int d^4x\, e^{-ip_A\cdot x} \langle j(x_-, x_+ + z_+, x_\perp + z_\perp) j(0, z_+, z_\perp)\rangle_A$$

and rescale $z_+ \to \lambda z_+$, $z_- \to \frac{z_-}{\lambda}$ $\quad$ $(p_A^{(0)} = \kappa e_+ + \frac{p_A^2}{s}\kappa e_-)$

$$\int d^4x\, d^4z\, \delta(z_-) e^{-ip_A x - i(r,z)_\perp} \langle j(x+z) j(z)\rangle_A$$

$$= \lambda \int d^4x\, d^4z\, \delta(z_-) e^{-ip_A^{(0)} x - i(r,z)_\perp} \langle j(x+z) j_\nu(z)\rangle_B,$$

The boosted field B_μ has the form

$$B_-(x_-, x_+, x_\perp) = \lambda A_-(\frac{x_-}{\lambda}, \lambda x_+, x_\perp),$$

$$B_*(x_-, x_+, x_\perp) = \frac{1}{\lambda} A_+(\frac{x_-}{\lambda}, \lambda x_+, x_\perp),$$

$$B_\perp(x_-, x_+, x_\perp) = A_\perp(\frac{x_-}{\lambda}, \lambda x_+, x_\perp),$$

We "freeze" the gluon field, consider the "upper part"

$$\int dz_+ \int d^4x\, e^{-ip_A\cdot x}\,\langle j(x_-,x_+ + z_+,x_\perp + z_\perp)j(0,z_+,z_\perp)\rangle_A$$

and rescale $z_+ \to \lambda z_+$, $z_- \to \frac{z_-}{\lambda}$ $\quad (p_A^{(0)} = \kappa e_+ + \frac{p_A^2}{s}\kappa e_-\ )$

$$\int d^4x\, d^4z\, \delta(z_-)e^{-ip_Ax-i(r,z)_\perp}\langle j(x+z)j(z)\rangle_A$$

$$= \quad \lambda\int d^4x\, d^4z\, \delta(z_-)e^{-ip_A^{(0)}x-i(r,z)_\perp}\langle j(x+z)j_\nu(z)\rangle_B,$$

If $F_{\mu\nu}(A) \to 0$ as $x_+ \to \infty$ we get a "pancake" field for $G_{\mu\nu}(B)$

$$\begin{aligned}
G_{-i}(x_-,x_+,x_\perp) &= \lambda F_{-i}(\frac{x_-}{\lambda},\lambda x_+,x_\perp) \ \to\ \delta(x_+)G_i(x_\perp),\\
G_{+i}(x_-,x_+,x_\perp) &= \frac{1}{\lambda}F_{+i}(\frac{x_-}{\lambda},\lambda x_+,x_\perp) \ \to\ 0,\\
G_{+-,ik}(x_-,x_+,x_\perp) &= \frac{1}{\lambda}F_{+-,ik}(\frac{x_-}{\lambda},\lambda x_+,x_\perp) \ \to\ 0,
\end{aligned}$$

We "freeze" the gluon field, consider the "upper part"

$$\int dz_+ \int d^4x\, e^{-ip_A\cdot x}\, \langle j(x_-, x_+ + z_+, x_\perp + z_\perp) j(0, z_+, z_\perp)\rangle_A$$

and rescale $z_+ \to \lambda z_+$, $z_- \to \frac{z_-}{\lambda}$ $\qquad (p_A^{(0)} = \kappa e_+ + \frac{p_A^2}{s}\kappa e_-\ )$

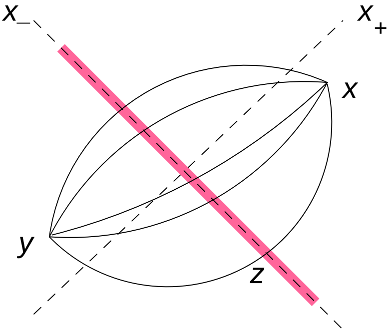
$$\int d^4x\, d^4z\, \delta(z_-) e^{-ip_A x - i(r,z)_\perp} \langle j(x+z) j(z)\rangle_A$$

$$= \quad \lambda \int d^4x\, d^4z\, \delta(z_-) e^{-ip_A^{(0)} x - i(r,z)_\perp} \langle j(x+z) j_\nu(z)\rangle_B,$$

The only component which survives the infinite boost is $F_{-\perp}$ and it exists only within the thin "pancake" near $x_+ = 0$. In the rest of the space the field B_μ is a pure gauge. Let us denote by Ω the corresponding gauge matrix and by B^Ω the rotated gauge field which vanishes everywhere except the pancake:

$$B_-^\Omega = \lim_{\lambda\to\infty} \frac{\partial^i}{\vec{\partial}_\perp^2} G_{i-}^\Omega(0, \lambda x_*, x_\perp) \to \delta(x_*) \frac{\partial^i}{\vec{\partial}_\perp^2} G_i^\Omega(x_\perp),\ B_+^\Omega = B_\perp^\Omega = 0.$$

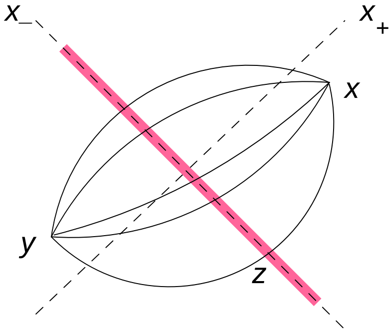
Propagators in the shock-wave background



"Pancake" is very thin ($l_+ \sim \frac{1}{\lambda}$) $\Rightarrow$ path inside the shock wave can be approximated by a segment of the straight line in x_+ direction

$$(x|\frac{1}{\mathcal{P}^2}|y) = \frac{-i}{\mathcal{N}}\int_0^\infty d\tau \int_{x(0)=y}^{x(\tau)=x} \mathcal{D}x(t) e^{-i\int_0^\tau dt \frac{\dot{x}^2}{4}} \text{ Pexp}\Big\{ig\int_0^\tau dt (B_\mu^\Omega(x(t))\dot{x}^\mu(t)\Big\}$$

$$= \int \frac{d^4z}{4\pi^4}\,\delta(z_+)\frac{1}{(x-z)^2}\,\frac{\overleftrightarrow{\partial}}{\partial z_-}\,\frac{1}{(z-y)^2}\ \text{Pexp}\Big\{ig\int dz_+ B_-^\Omega(z_+, z_\perp)\Big\}$$



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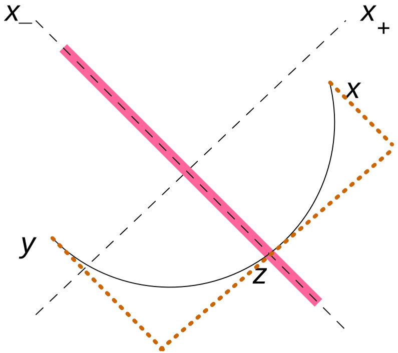
$$= \int \frac{d^4z}{4\pi^4}\,\delta(z_+)\frac{1}{(x-z)^2}\,\frac{\overleftrightarrow{\partial}}{\partial z_-}\,\frac{1}{(z-y)^2}\,U^\Omega(z_\perp)$$

Propagators in the shock-wave background

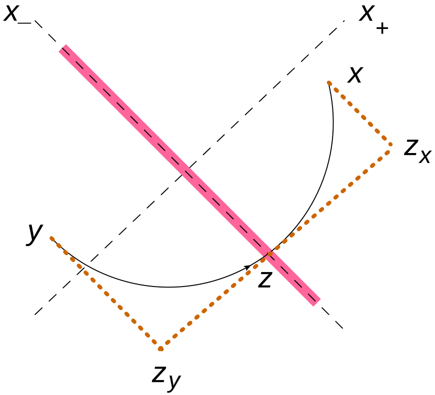


Rotating back $B^{\omega} \rightarrow B$ we get

$$(x|\frac{1}{\mathcal{P}^2}|y) = \int \frac{d^4z}{4\pi^4}\, \delta(z_+)\frac{1}{(x-z)^2}\, \frac{\overleftrightarrow{\partial}}{\partial z_-}\, \frac{1}{(z-y)^2}\, U(z_\perp;x,y),$$

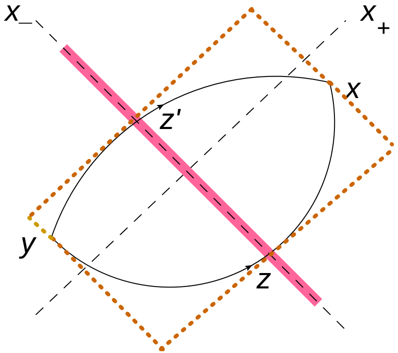
$$U(z_\perp;x,y) = [x,z_x][z_x,z_y][z_y,y], \qquad z_x = x_+e_- + z_-e_+ + z_\perp$$

Propagators in the shock-wave background



Quark propagator

$$(x|\frac{1}{\not{\mathcal{P}}}|y) = i\int dz\delta(z_+)\frac{\not{x}-\not{z}}{2\pi^2(x-z)^4}\ \not{e}_+ U(z_\perp;x,y)\frac{\not{z}-\not{y}}{2\pi^2(z-y)^4}.$$

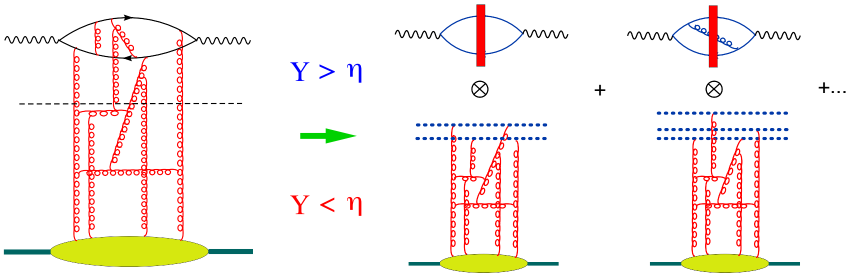


Quark-antiquark pair in a shock-wave background

$$\mathrm{Tr}\{\gamma_\mu(x|\frac{1}{\mathcal{P}}|y)\gamma_\nu(y|\frac{1}{\mathcal{P}}|x)\} \;=\; -\int dz dz'\,\delta(z_+)\delta(z'_+)$$
$$\times\ \mathrm{tr}\{\gamma_\mu\frac{\not x-\not z}{2\pi^2(x-z)^4}\not e_+\frac{\not z-\not y}{2\pi^2(z-y)^4}\gamma_\nu\frac{\not y-\not z'}{2\pi^2(y-z')^4}\not e_+\frac{\not z'-\not x}{2\pi^2(z'-x)^4}\}\mathrm{U}(z_\perp;z'_\perp)$$

$$U(z_\perp,z'_\perp) \;=\; \mathrm{Tr}[z_x,z_y][z_y,z'_y][z'_y,z'_x][z'_x,z_x]$$

High-energy expansion in color dipoles



The high-energy operator expansion is

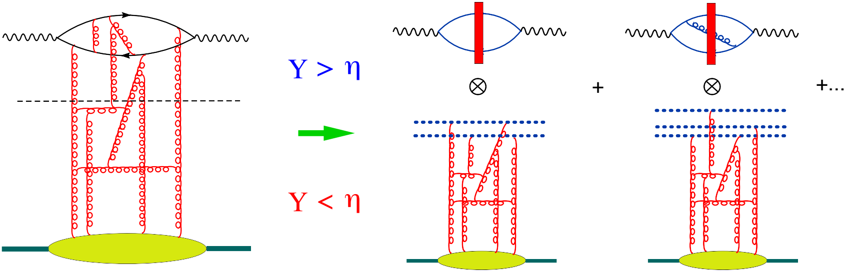
$$(x-y)^4 T\{\bar{\hat{\psi}}(x)\gamma^\mu\hat{\psi}(x)\bar{\hat{\psi}}(y)\gamma^\nu\hat{\psi}(y)\} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4} I^{\rm LO}_{\mu\nu}(z_1,z_2){\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$I^{\rm LO}_{\mu\nu}(z_1,z_2) = \frac{\mathcal{R}^2}{\pi^6(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2) - \frac{1}{2}\kappa^2(\zeta_1\cdot\zeta_2)\Big].$$

$$\kappa \equiv \frac{1}{\sqrt{s}x^+}(\frac{p_1}{s} - x^2p_2 + x_\perp) - \frac{1}{\sqrt{s}y^+}(\frac{p_1}{s} - y^2p_2 + y_\perp)$$

$$\zeta_i \equiv (\frac{p_1}{s} + z_{i\perp}^2p_2 + z_{i\perp}), \qquad \mathcal{R} \equiv \frac{\kappa^2(\zeta_1\cdot\zeta_2)}{2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}$$

High-energy expansion in color dipoles



η - rapidity factorization scale

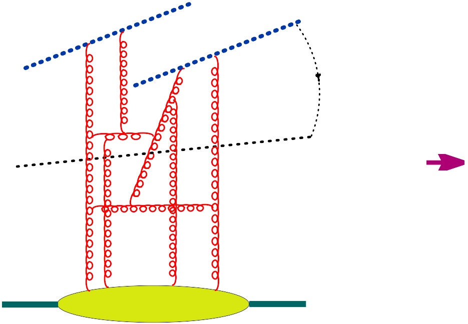
Step II - Evolution equation for color dipoles

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} &= \frac{\alpha_s}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2}[\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} \\
&- N_c\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}] + \alpha_s K_{\mathrm{NLO}}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} + O(\alpha_s^2)
\end{aligned}
$$

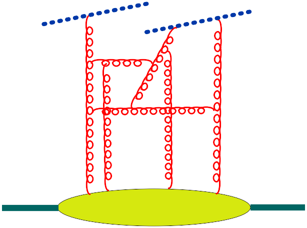
(Linear part of K_{NLO} = $K_{\mathrm{NLO\ BFKL}}$)

Evolution equation for color dipoles

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).



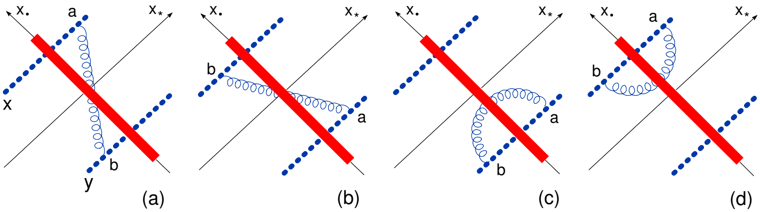
$\alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \quad \otimes$



Evolution equation in the leading order

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

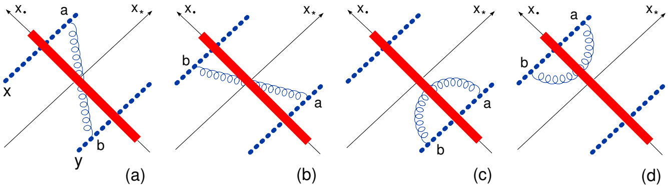
$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}}$$



$$U^{ab}_z = \mathrm{Tr}\{t^a U_z t^b U^\dagger_z\} \quad \Rightarrow (U_x U^\dagger_y)^{\eta_1} \to (U_x U^\dagger_y)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U^\dagger_z U_z U^\dagger_y)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear

Derivation of the non-linear equation



The gluon propagator in a shock-wave external field in the $A_+ = 0$ gauge

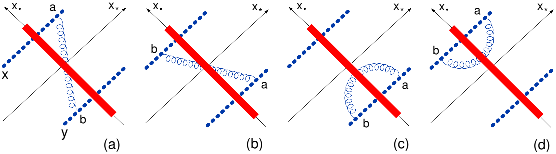
$$\langle \hat{A}^a_\mu(x)\hat{A}^b_\nu(y)\rangle$$

$$\stackrel{x_+>0>y_+}{=} \; -\frac{i}{2}\int d^4z\;\delta(z_+)\;\frac{x_+g^\perp_{\mu\xi} - e^+_\mu(x-z)^\perp_\xi}{\pi^2[(x-z)^2+i\epsilon]^2}\;U^{ab}_{z_\perp}\frac{1}{\partial^{(z)}_+}\;\frac{y_+\delta^{\perp\xi}_\nu - e^+_\nu(y-z)^\xi_\perp}{\pi^2[(z-y)^2+i\epsilon]^2}$$

$$\text{Diagram (a)} \;=\; g^2\int_0^\infty dx_+\int_{-\infty}^0 dy_+\;\langle \hat{A}^{a,Y_1}_\bullet(x_+,x_\perp)\hat{A}^{b,Y_1}_\bullet(y_+,y_\perp)\rangle_{\text{Fig.(a)}}$$

$$=\; -4\alpha_s\int_0^{e^{Y_1}}\frac{d\alpha}{\alpha}(x_\perp|\frac{p_i}{p_\perp^2-i\epsilon}U^{ab}\frac{p_i}{p_\perp^2-i\epsilon}|y_\perp)$$

$(x_\perp|F(p_\perp)|y_\perp) \equiv \int đp\; e^{i(p,x-y)_\perp}F(p_\perp)$ - Schwinger's notations



Formally, the integral over α diverges at the lower limit, but since we integrate over the rapidities $Y > Y_2$ in the leading log approximation, we get ($\Delta Y \equiv Y_1 - Y_2$)

$$g^2\int_0^\infty dx_+ \int_{-\infty}^0 dy_+ \ \langle \hat{A}^{a,Y_1}_\bullet(x_+,x_\perp)\hat{A}^{b,Y_1}_\bullet(y_+,y_\perp)\rangle_{\text{Fig.(a)}} \ = \ -4\alpha_s\Delta Y(x_\perp|\frac{p_i}{p_\perp^2}U^{ab}\frac{p_i}{p_\perp^2}|y_\perp)$$

$$\Rightarrow \ \langle \hat{U}^Y_{z_1}\otimes\hat{U}^{\dagger Y}_{z_2}\rangle^{Y_1}_{\text{Fig.(a)}} \ = \ -\frac{\alpha_s}{\pi^2}\Delta Y\ (t^aU_{z_1}\otimes t^bU^\dagger_{z_2})\int d^2z_3\frac{(z_{13},z_{23})}{z_{13}^2z_{23}^2}U^{ab}_{z_3}$$

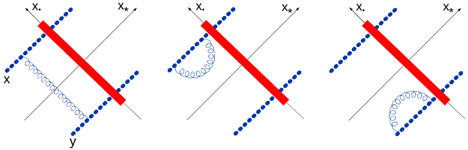
The contribution of the diagram in Fig. (b) is obtained by the replacement $t^aU_{z_1}\otimes t^bU^\dagger_{z_2}\to U_{z_1}t^b\otimes U^\dagger_{z_2}t^a$, $z_2\leftrightarrow z_1$. The two remaining diagrams(c) and (d) are obtained by $z_2\to z_1$ for Fig.(c) and $z_1\to z_2$ for Fig.(d).

Result:

$$\langle \text{Tr}\{\hat{U}^{Y_1}_{z_1}\hat{U}^{\dagger Y_1}_{z_2}\}\rangle_{\text{Figs(a)}-\text{(d)}} \ = \ \frac{\alpha_s\Delta Y}{\pi^2}\int d^2z_3\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\text{Tr}\{t^aU_{z_1}U^\dagger_{z_3}t^aU_{z_3}U^\dagger_{z_2}\} - \frac{1}{N_c}\text{Tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

Derivation of the non-linear equation

Diagrams without the gluon-shockwave intersection:



These diagrams are proportional to the original dipole $\mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\}$ ⇒ corresponding term can be derived from the contribution of Fig. (a)-(d) graphs using the requirement that the r.h.s. of the evolution equation should vanish in the absence of the shock wave ($U \to 1$).

$$\langle \mathrm{Tr}\{\hat U^{Y_1}_{z_1}\hat U^{\dagger Y_1}_{z_2}\}\rangle \;=\; \frac{\alpha_s \Delta Y}{\pi^2}\int d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{Tr}\{t^a U_{z_1}U^\dagger_{z_3}t^a U_{z_3}U^\dagger_{z_2}\} - N_c \mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

⇒ non-linear equation for the evolution of the color dipole

$$\frac{d}{dY}\mathrm{Tr}\{\hat U^{Y}_{z_1}\hat U^{\dagger Y}_{z_2}\} \;=\; \frac{\alpha_s}{2\pi^2}\int d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{Tr}\{\hat U^{Y}_{z_1}\hat U^{\dagger Y}_{z_3}\}\mathrm{Tr}\{\hat U^{Y}_{z_1}\hat U^{\dagger Y}_{z_2}\} - N_c \mathrm{Tr}\{\hat U^{Y}_{z_1}\hat U^{\dagger Y}_{z_2}\}]$$

Non linear evolution equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) \;=\; \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

I. B. (1996), Yu. Kovchegov (1999)
Alternative approach: JIMWLK (1997-2000)

Non-linear evolution equation

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I. B. (1996), Yu. Kovchegov (1999)
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LLA for DIS in pQCD $\Rightarrow$ BFKL

(LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

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I. B. (1996), Yu. Kovchegov (1999)
Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1,\ \alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

Argument of coupling constant

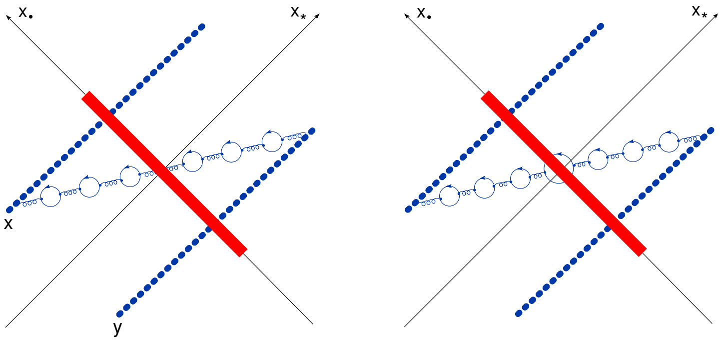
$$\frac{d}{d\eta}\hat{\mathcal{U}}(z_1,z_2) =$$

$$\frac{\alpha_s(?_\perp)N_c}{2\pi^2}\int dz_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big\{\hat{\mathcal{U}}(z_1,z_3)+\hat{\mathcal{U}}(z_3,z_2)-\hat{\mathcal{U}}(z_1,z_2)-\hat{\mathcal{U}}(z_1,z_3)\hat{\mathcal{U}}(z_3,z_2)\Big\}$$

Argument of coupling constant

$$\frac{d}{d\eta}\hat{\mathcal{U}}(z_1,z_2) =$$
$$\frac{\alpha_s(?_\perp)N_c}{2\pi^2}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big\{\hat{\mathcal{U}}(z_1,z_3)+\hat{\mathcal{U}}(z_3,z_2)-\hat{\mathcal{U}}(z_1,z_2)-\hat{\mathcal{U}}(z_1,z_3)\hat{\mathcal{U}}(z_3,z_2)\Big\}$$

Renormalon-based approach: summation of quark bubbles



$$-\tfrac{2}{3}n_f \to b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

Argument of coupling constant (rcBK)

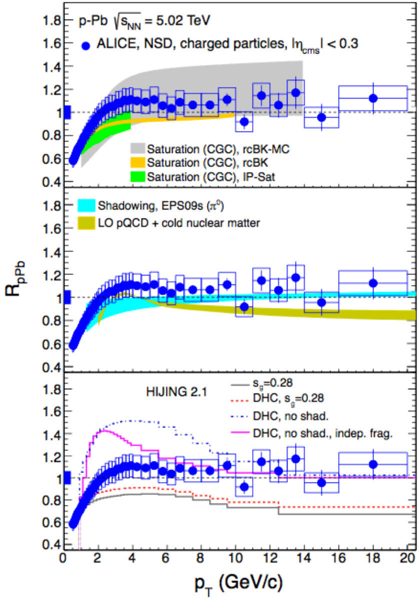
$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\} = \frac{\alpha_s(z_{12}^2)}{2\pi^2}\int d^2z\ [\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{Tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]$$
$$\times\Big[\frac{z_{12}^2}{z_{13}^2 z_{23}^2} + \frac{1}{z_{13}^2}\Big(\frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1\Big) + \frac{1}{z_{23}^2}\Big(\frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1\Big)\Big] + \ldots$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

$$\frac{\alpha_s(z_{12}^2)}{2\pi^2}\frac{z_{12}^2}{z_{13}^2 z_{23}^2} \qquad |z_{12}| \ll |z_{13}|, |z_{23}|$$
$$\frac{\alpha_s(z_{13}^2)}{2\pi^2 z_{13}^2} \qquad |z_{13}| \ll |z_{12}|, |z_{23}|$$
$$\frac{\alpha_s(z_{23}^2)}{2\pi^2 z_{23}^2} \qquad |z_{23}| \ll |z_{12}|, |z_{13}|$$

$\Rightarrow$ the argument of the coupling constant is given by the size of the smallest dipole.



ALICE arXiv:1210.4520

Nuclear modification factor

$$R^{pPb}(p_T) = \frac{d^2N^{pPb}_{\rm ch}/d\eta dp_T}{\langle T_{pPb}\rangle d^2\sigma^{\rm pp}_{\rm ch}/d\eta dp_T}$$

$N^{pPb} \equiv$ charged particle yield in p-Pb collisions.