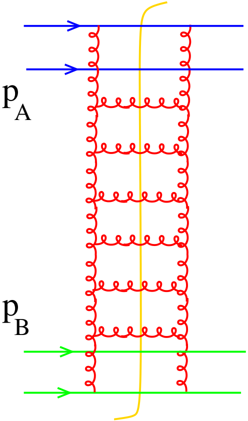
Outline

1. Lecture I highlights:
   - Rapidity factorization and High-energy operator expansion in Wilson lines.
   - Evolution equation for color dipoles.
   - Propagators in a shock-wave background.
   - Leading order: BK equation.
2. NLO high-energy amplitudes in $\mathcal{N}=4$ SYM
   - Conformal (Möbius) invariance of the LO BK kernel
   - Conformal composite dipoles and NLO BK kernel in $\mathcal{N}=4$.
   - Regge limit in the coordinate space.
   - NLO amplitude in $\mathcal{N}=4$ SYM
3. NLO high-energy amplitudes in QCD
   - Photon impact factor.
   - NLO BK kernel in QCD.
   - rcBK.
   - Conclusions

In pQCD: Leading Log Approximation ⇒ BFKL pomeron

$$s = (p_A + p_B)^2 \simeq 4E^2$$



Leading Log Approximation (LLA(x)):

$$\alpha_s \ll 1, \quad \alpha_s \ln s \sim 1$$

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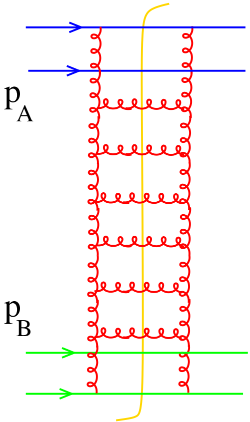

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The sum of gluon ladder diagrams gives

$$\sigma_{\text{tot}} \sim s^{12\frac{\alpha_s}{\pi}\ln 2} \qquad \text{BFKL pomeron}$$

Numerically: for DIS at HERA

$$\sigma \sim s^{0.3} = x_B^{-0.3}$$

- qualitatively OK

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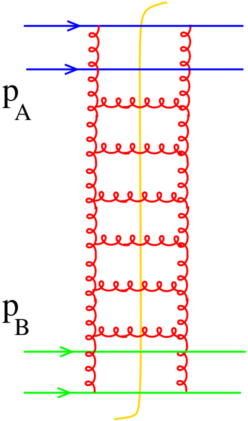

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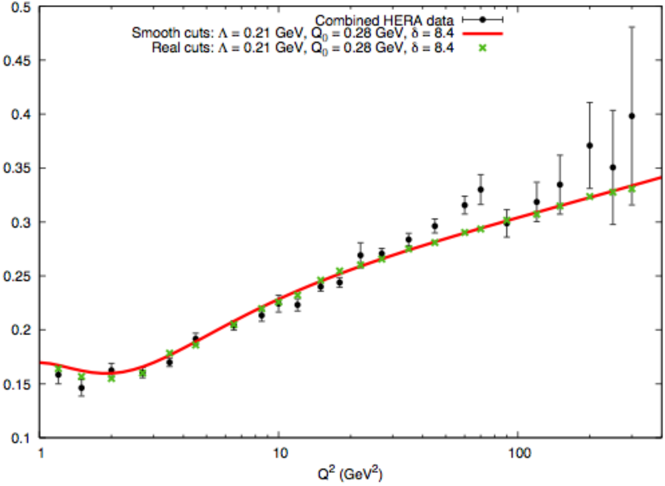
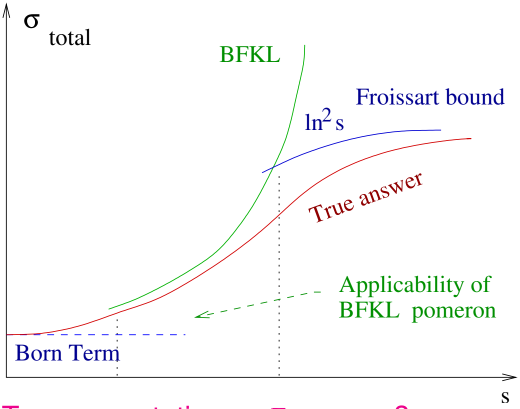
BFKL *vs* HERA data

$$F_2(x_B, Q^2) \; = \; c(Q^2) x_B^{-\lambda(Q^2)}$$



M.Hentschinski, A. Sabio Vera and C. Salas, 2010



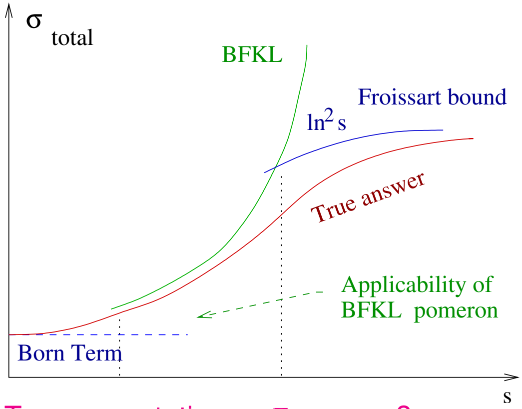
$\sigma_{\rm tot} \sim s^{12\frac{\alpha_s}{\pi}\ln 2}$ violates Froissart bound $\sigma_{\rm tot} \leq \ln^2 s$ $\Rightarrow$ pre-asymptotic behavior.

True asymptotics as $E \to \infty$ = ?

Possible approaches:

- Sum all logs $\alpha_s^m \ln^n s$
- Reduce high-energy QCD to $2+1$ effective theory

Towards the high-energy QCD



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- Reduce high-energy QCD to $2+1$ effective theory

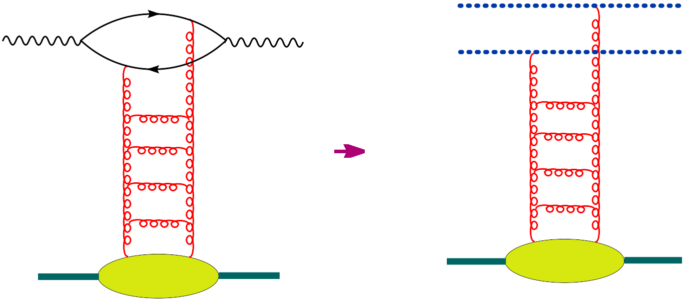
Lecture II: NLO corrections $\alpha_s^{n+1} \ln^n s$

Four steps of an OPE

- Factorize an amplitude into a product of coefficient functions and matrix elements of relevant operators.
- Find the evolution equations of the operators with respect to factorization scale.
- Solve these evolution equations.
- Convolute the solution with the initial conditions for the evolution and get the amplitude

DIS at high energy: relevant operators

- At high energies, particles move along straight lines ⇒ the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp) \langle B|\mathrm{Tr}\{U(k_\perp)U^\dagger(-k_\perp)\}|B\rangle$$

$$U(x_\perp) = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} du\ n^\mu A_\mu(un + x_\perp)\Big] \qquad \text{Wilson line}$$

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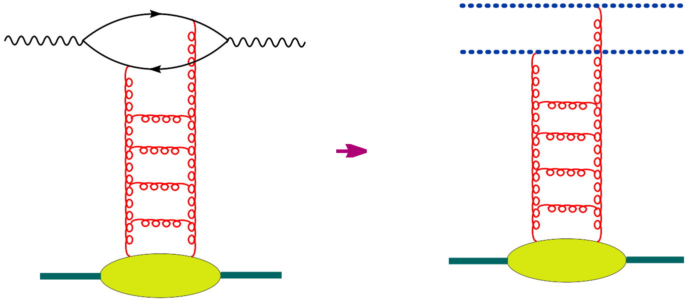

$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \mathrm{Tr}\{U(k_\perp) U^\dagger(-k_\perp)\} | B \rangle$$

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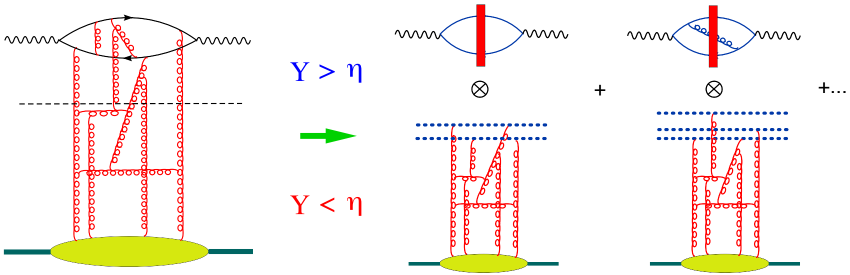
Formally, ➔ means the operator expansion in Wilson lines

Rapidity factorization



η - rapidity factorization scale

Rapidity Y > η - coefficient function ("impact factor")

Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp)\Big]$$

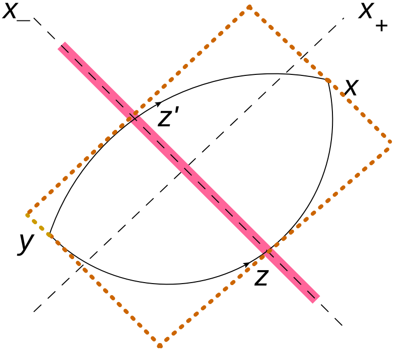
$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

Spectator frame: propagation in the shock-wave background.

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space ⇒ we can replace the gauge factor along the actual path with the one along the straight-line path.

[$x \to z$: free propagation]×
$[U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave$]\times$
[$z \to y$: free propagation]

Propagators in the shock-wave background

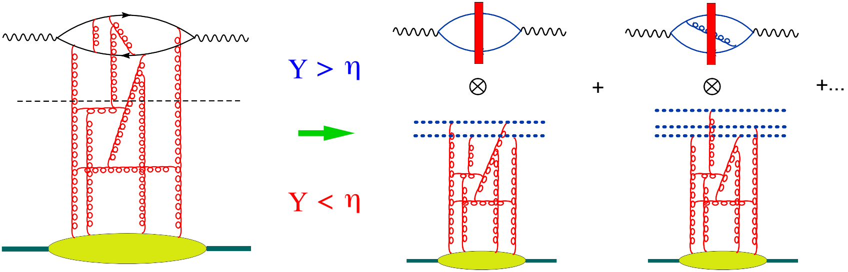


Quark-antiquark pair in a shock-wave background

$$\mathrm{Tr}\{\gamma_\mu(x|\frac{1}{\mathcal{P}}|y)\gamma_\nu(y|\frac{1}{\mathcal{P}}|x)\} \;=\; -\int dz dz'\,\delta(z_+)\delta(z'_+)$$

$$\times\ \mathrm{tr}\{\gamma_\mu\frac{\not x-\not z}{2\pi^2(x-z)^4}\not e_+\frac{\not z-\not y}{2\pi^2(z-y)^4}\gamma_\nu\frac{\not y-\not z'}{2\pi^2(y-z')^4}\not e_+\frac{\not z'-\not x}{2\pi^2(z'-x)^4}\}\mathrm{U}(z_\perp;z'_\perp)$$

$$U(z_\perp,z'_\perp) \;=\; \mathrm{Tr}[z_x,z_y][z_y,z'_y][z'_y,z'_x][z'_x,z_x]$$



The high-energy operator expansion is

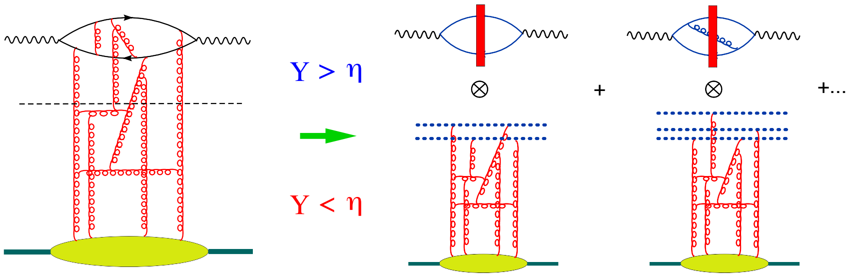
$$(x-y)^4 T\{\bar{\hat{\psi}}(x)\gamma^\mu\hat{\psi}(x)\bar{\hat{\psi}}(y)\gamma^\nu\hat{\psi}(y)\} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(z_1,z_2){\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$I^{\rm LO}_{\mu\nu}(z_1,z_2) = \frac{\mathcal{R}^2}{\pi^6(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)-\frac{1}{2}\kappa^2(\zeta_1\cdot\zeta_2)\Big].$$

$$\kappa \equiv \frac{1}{\sqrt{s}x^+}\Big(\frac{p_1}{s}-x^2p_2+x_\perp\Big)-\frac{1}{\sqrt{s}y^+}\Big(\frac{p_1}{s}-y^2p_2+y_\perp\Big)$$

$$\zeta_i \equiv \Big(\frac{p_1}{s}+z_{i\perp}^2p_2+z_{i\perp}\Big), \qquad \mathcal{R} \equiv \frac{\kappa^2(\zeta_1\cdot\zeta_2)}{2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}$$

High-energy expansion in color dipoles



η - rapidity factorization scale

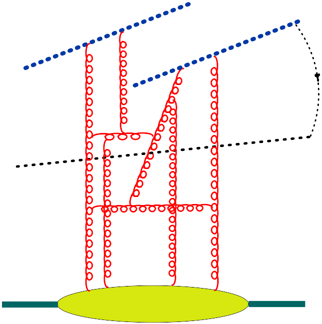
Step II - Evolution equation for color dipoles

$$\frac{d}{d\eta}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} = \frac{\alpha_s}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2}[\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} - N_c\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}] + \alpha_s K_{\mathrm{NLO}}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} + O(\alpha_s^2)$$

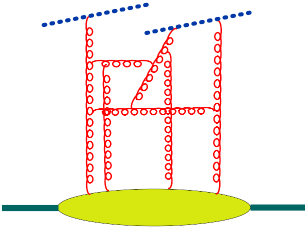
(Linear part of K_{NLO} = $K_{\mathrm{NLO\ BFKL}}$)

Evolution equation for color dipoles

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).

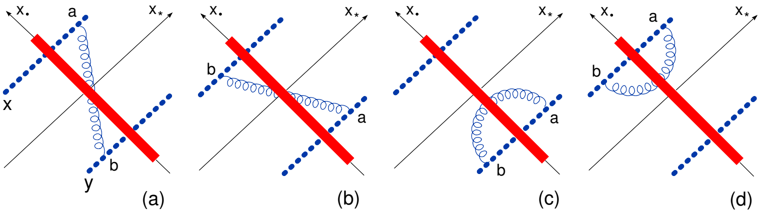


$$\alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \quad \otimes$$



$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

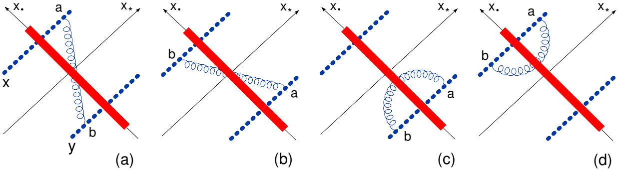
$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}}$$



$$U^{ab}_z = \mathrm{Tr}\{t^a U_z t^b U^\dagger_z\} \quad \Rightarrow (U_x U^\dagger_y)^{\eta_1} \to (U_x U^\dagger_y)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U^\dagger_z U_z U^\dagger_y)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear

Derivation of the non-linear equation



The gluon propagator in a shock-wave external field in the $A_+ = 0$ gauge

$$\langle \hat{A}^a_\mu(x)\hat{A}^b_\nu(y)\rangle = -\int_0^\infty đ\alpha \, \frac{e^{-i\alpha(x-y)_\bullet}}{2\alpha}$$

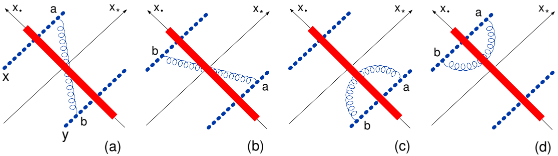
$$\times (x_\perp| e^{-i\frac{p_\perp^2}{\alpha\sqrt{s}}x_-}\Big[g^\perp_{\mu\xi} - \frac{2}{\alpha s}(p^\perp_\mu p_{2\xi} + p_{2\mu}p^\perp_\xi)\Big]U^{ab}\Big[g^{\perp\xi}_\nu - \frac{2}{\alpha s}(+^\xi_2 p^\perp_\nu + p_{2\nu}p^\xi_\perp)\Big]e^{i\frac{p_\perp^2}{\alpha\sqrt{s}}y_+}|y_\perp)$$

$$\text{Diagram (a)} = g^2\int_0^\infty dx_+\int_{-\infty}^0 dy_+ \, \langle \hat{A}^{a,Y_1}_\bullet(x_+,x_\perp)\hat{A}^{b,Y_1}_\bullet(y_+,y_\perp)\rangle_{\text{Fig.(a)}}$$

$$= -4\alpha_s\int_0^{e^{Y_1}}\frac{d\alpha}{\alpha}(x_\perp|\frac{p_i}{p_\perp^2 - i\epsilon}U^{ab}\frac{p_i}{p_\perp^2 - i\epsilon}|y_\perp)$$

$(x_\perp|F(p_\perp)|y_\perp) \equiv \int đp \; e^{i(p,x-y)_\perp}F(p_\perp)$ - Schwinger's notations

Derivation of the non-linear equation



Formally, the integral over α diverges at the lower limit, but since we integrate over the rapidities $Y > Y_2$ in the leading log approximation, we get ($\Delta Y \equiv Y_1 - Y_2$)

$$g^2\int_0^\infty dx_+ \int_{-\infty}^0 dy_+ \, \langle \hat{A}_-^{a,Y_1}(x_+,x_\perp)\hat{A}_-^{b,Y_1}(y_+,y_\perp)\rangle_{\text{Fig.(a)}} \;=\; -4\alpha_s\Delta Y(x_\perp|\frac{p_i}{p_\perp^2}U^{ab}\frac{p_i}{p_\perp^2}|y_\perp)$$

$$\Rightarrow \;\langle \hat{U}^Y_{z_1}\otimes\hat{U}^{\dagger Y}_{z_2}\rangle^{Y_1}_{\text{Fig.(a)}} \;=\; -\frac{\alpha_s}{\pi^2}\Delta Y\,(t^aU_{z_1}\otimes t^bU^\dagger_{z_2})\int d^2z_3\frac{(z_{13},z_{23})}{z_{13}^2z_{23}^2}U^{ab}_{z_3}$$

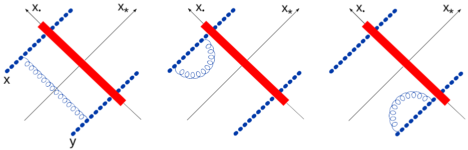
The contribution of the diagram in Fig. (b) is obtained by the replacement $t^aU_{z_1}\otimes t^bU^\dagger_{z_2}\to U_{z_1}t^b\otimes U^\dagger_{z_2}t^a$, $z_2\leftrightarrow z_1$. The two remaining diagrams(c) and (d) are obtained by $z_2\to z_1$ for Fig.(c) and $z_1\to z_2$ for Fig.(d).

Result:

$$\langle \text{Tr}\{\hat{U}^{Y_1}_{z_1}\hat{U}^{\dagger Y_1}_{z_2}\}\rangle_{\text{Figs(a)}-\text{(d)}} \;=\; \frac{\alpha_s\Delta Y}{\pi^2}\int d^2z_3\frac{z_{12}^2}{z_{13}^2z_{23}^2}[\text{Tr}\{t^aU_{z_1}U^\dagger_{z_3}t^aU_{z_3}U^\dagger_{z_2}\} - \frac{1}{N_c}\text{Tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

Derivation of the non-linear equation

Diagrams without the gluon-shockwave intersection:



These diagrams are proportional to the original dipole $\mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\}$ ⇒ corresponding term can be derived from the contribution of Fig. (a)-(d) graphs using the requirement that the r.h.s. of the evolution equation should vanish in the absence of the shock wave ($U \to 1$).

$$\langle\mathrm{Tr}\{\hat{U}^{Y_1}_{z_1}\hat{U}^{\dagger Y_1}_{z_2}\}\rangle \;=\; \frac{\alpha_s \Delta Y}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{Tr}\{t^a U_{z_1}U^\dagger_{z_3}t^a U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{Tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

⇒ non-linear equation for the evolution of the color dipole

$$\frac{d}{dY}\mathrm{Tr}\{\hat{U}^{Y}_{z_1}\hat{U}^{\dagger Y}_{z_2}\} \;=\; \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{Tr}\{\hat{U}^{Y}_{z_1}\hat{U}^{\dagger Y}_{z_3}\}\mathrm{Tr}\{\hat{U}^{Y}_{z_1}\hat{U}^{\dagger Y}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^{Y}_{z_1}\hat{U}^{\dagger Y}_{z_2}\}]$$

Non linear evolution equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) \;=\; \frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

I. B. (1996), Yu. Kovchegov (1999)
Alternative approach: JIMWLK (1997-2000)

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LLA for DIS in pQCD ⇒ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

Non-linear evolution equation

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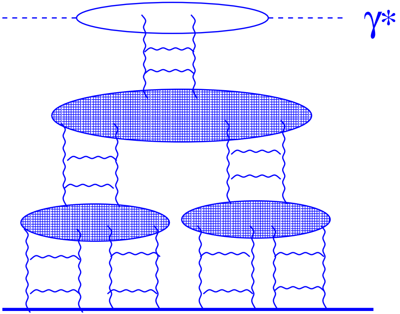
I. B. (1996), Yu. Kovchegov (1999)
Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

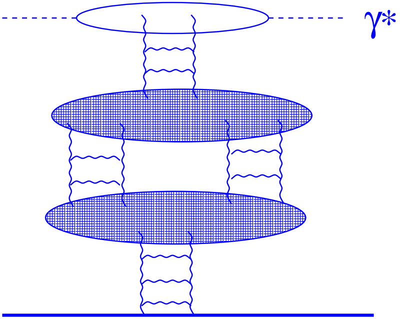
LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$, $\alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

Non-linear equation sums up the “fan” diagrams

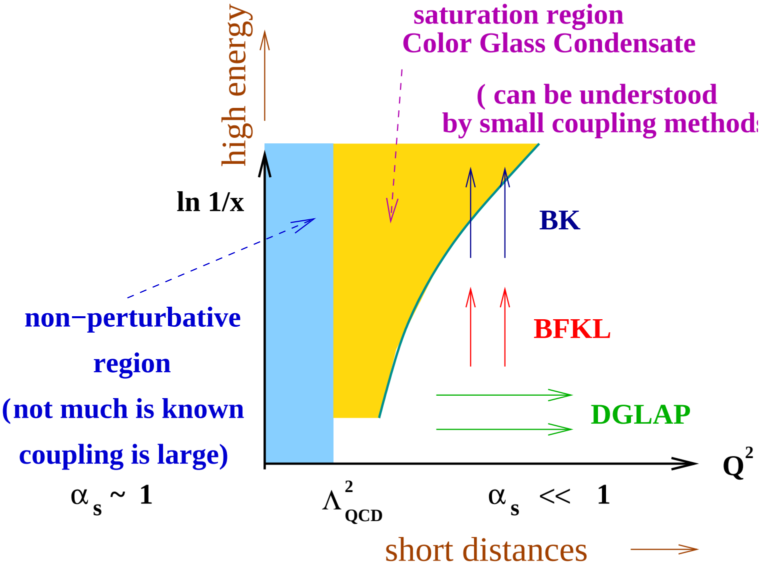


Example of the diagrams left behind by the NL eqn: pomeron loops



Timeline for the NL evolution at high energies

- Gribov, Levin, Ryskin (1983) - GLR eqn suggested
- Mueller, Qui (1986) - DLA limit of GLR eqn proved
- I.B. (1996) - the NL equation derived
- Kovchegov (1999) - the NL eqn rederived (in the dipole model) and used for DIS from large nuclei
- Braun (M.A.) (2000) - NL = GLR + 3-pomeron vertex from Bartels *et. al.*
- JIMWLK(2000) - obtained from the RG eqn for Color Glass Condensate


saturation region
Color Glass Condensate
(can be understood
by small coupling method
high energy
ln 1/x
BK
BFKL
DGLAP
non−perturbative
region
(not much is known
coupling is large)
$\alpha_s \sim 1$
Λ^2_{QCD}
$\alpha_s \ll 1$
Q^2
short distances

Why NLO correction?

- To check that high-energy OPE works at the NLO level.
- To determine the argument of the coupling constant.
- To get the region of application of the leading order evolution equation.
- To check conformal invariance (in $\mathcal{N}$=4 SYM)

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;=\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+\, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

Conformal invariance of the BK equation

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Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp / x_\perp^2$ and $x^+ \to x^+ / x_\perp^2$

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$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;\to\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2}\, A_+\left(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2}\right)\right\} \;=\; [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;=\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+\, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp/x_\perp^2$ and $x^+ \to x^+/x_\perp^2 \Rightarrow$

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;\to\; \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2}\, A_+\Big(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2}\Big)\right\} \;=\; [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

$\Rightarrow$The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta}\mathrm{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2 z}{\cancel{z^4}}\,\frac{(x-y)^2\ \cancel{z^4}}{(x-z)^2(z-y)^2}\left[\mathrm{Tr}\{V_x V_z^\dagger\}\mathrm{Tr}\{V_z V_y^\dagger\} - N_c \mathrm{Tr}\{V_x V_y^\dagger\}\right]$$

Conformal invariance of the BK equation

SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm\hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z,\bar{z})] = z^2\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_0, \hat{U}(z,\bar{z})] = z\partial_z\hat{U}(z,\bar{z}), \quad [\hat{S}_+, \hat{U}(z,\bar{z})] = -\partial_z\hat{U}(z,\bar{z})$$

$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z,\bar{z})$

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$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 + iz^2, \quad U(z_\perp) = U(z,\bar{z})$

Conformal invariance of the evolution kernel

$$\frac{d}{d\eta}[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}] = \frac{\alpha_s N_c}{2\pi^2}\int dz\, K(x,y,z)[\hat{S}_-, \mathrm{Tr}\{U_x U_y^\dagger\}\mathrm{Tr}\{U_x U_y^\dagger\}]$$

$$\Rightarrow \left[x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}\right]K(x,y,z) = 0$$

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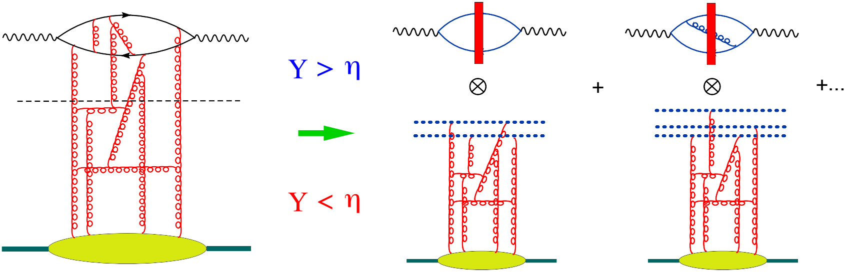
In the leading order - OK. In the NLO - ?

Expansion of the amplitude in color dipoles in the NLO



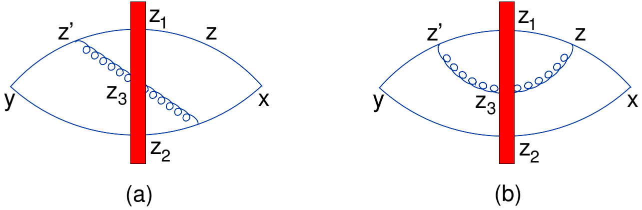
The high-energy operator expansion is

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}(z_1,z_2)\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}(z_1,z_2,z_3)[\frac{1}{N_c}\mathrm{Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - \mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

In the leading order - conf. invariant impact factor

$$I_{\rm LO} = \frac{x_+^{-2}y_+^{-2}}{\pi^2\mathcal{Z}_1^2\mathcal{Z}_2^2}, \qquad \mathcal{Z}_i \equiv \frac{(x-z_i)_\perp^2}{x_+} - \frac{(y-z_i)_\perp^2}{y_+} \qquad CCP, 2007$$



$$I^{\rm NLO}(x,y;z_1,z_2,z_3;\eta) \ = \ -I^{\rm LO}\times\frac{\lambda}{\pi^2}\frac{z_{13}^2}{z_{12}^2z_{23}^2}\Big[\ln\frac{\sigma s}{4}\mathcal{Z}_3-\frac{i\pi}{2}+C\Big]$$

The NLO impact factor is not Möbius invariant ⇒ the color dipole with the cutoff η is not invariant

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$[{\rm Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf} \ = \ {\rm Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$+\frac{\lambda}{2\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}[{\rm Tr}\{T^n\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^n\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c{\rm Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{13}^2z_{23}^2}\ +\ O(\lambda^2)$$

the impact factor becomes conformal in the NLO.

Operator expansion in conformal dipoles

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2\, I^{\rm LO}(z_1,z_2){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$$
$$+\int d^2z_1 d^2z_2 d^2z_3\, I^{\rm NLO}(z_1,z_2,z_3)[\frac{1}{N_c}{\rm Tr}\{T^n\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^n\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO} = -I^{\rm LO}\frac{\lambda}{2\pi^2}\int dz_3\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[\ln\frac{z_{12}^2 e^{2\eta}as^2}{z_{13}^2z_{23}^2}\mathcal{Z}_3^2 - i\pi + 2C\Big]$$

The new NLO impact factor is conformally invariant
$\Rightarrow$ ${\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}^{\rm conf}$ is Möbius invariant

We think that one can construct the composite conformal dipole operator order by order in perturbation theory.

Analogy: when the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator in must be corrected by finite counterterms order by order in perturbaton theory.

Non-linear evolution equation in the NLO

$$\frac{d}{d\eta}Tr\{U_x U_y^\dagger\} =$$

$$\int \frac{d^2z}{2\pi^2}\left(\alpha_s \frac{(x-y)^2}{(x-z)^2(z-y)^2} + \alpha_s^2 K_{NLO}(x,y,z)\right)[Tr\{U_x U_z^\dagger\}Tr\{U_z U_y^\dagger\} - N_c Tr\{U_x U_y^\dagger\}] +$$

$$\alpha_s^2 \int d^2z d^2z' \left(K_4(x,y,z,z')\{U_x, U^\dagger_{z'}, U_z, U^\dagger_y\} + K_6(x,y,z,z')\{U_x, U^\dagger_{z'}, U_{z'}, U_z, U^\dagger_z, U^\dagger_y\}\right)$$

K_{NLO} is the next-to-leading order correction to the dipole kernel and K_4 and K_6 are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

Definition of the NLO kernel

In general

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

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$$\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} - \alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + O(\alpha_s^3)$$

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We calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle\alpha_s^2 K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle = \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle - \langle\alpha_s K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle + O(\alpha_s^3)$$

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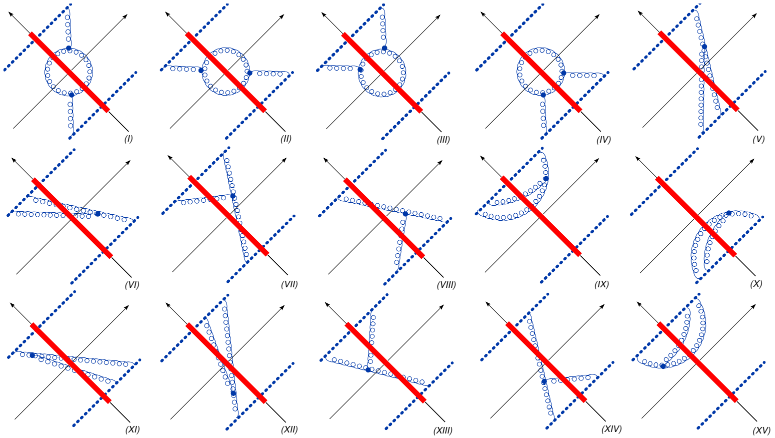
Subtraction of the (LO) contribution (with the rigid rapidity cutoff)
$\Rightarrow$ $\left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

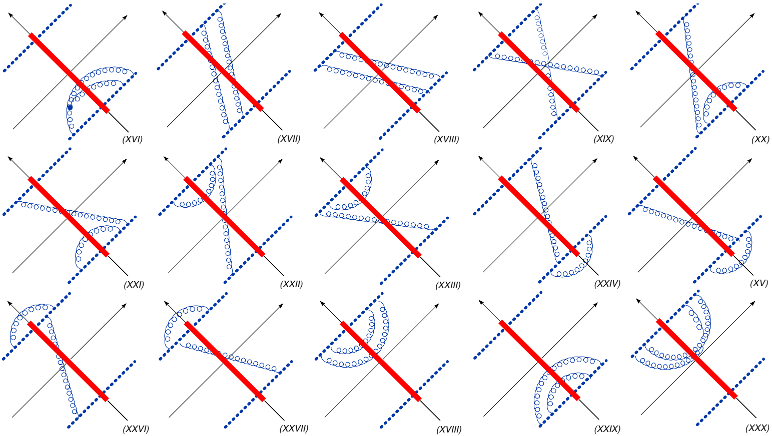
Typical integral

$$\int_0^1 dv\,\frac{1}{(k-p)_\perp^2 v + p_\perp^2(1-v)}\left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2}\ln\frac{(k-p)_\perp^2}{p_\perp^2}$$

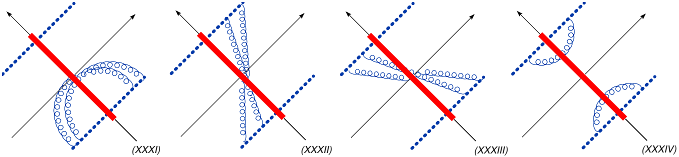
Gluon part of the NLO BK kernel: diagrams



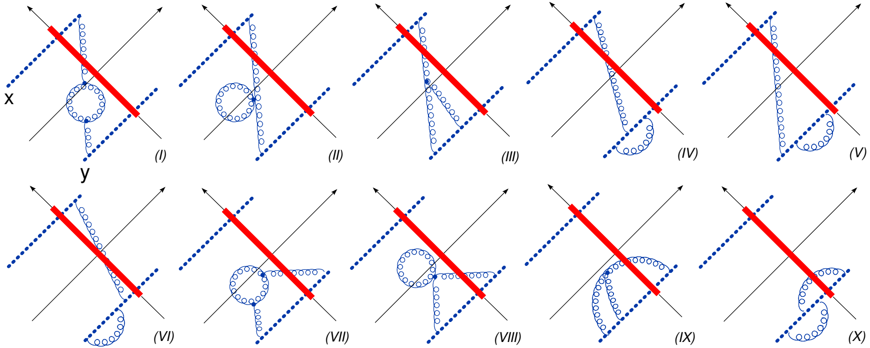
Diagrams for 1→3 dipoles transition



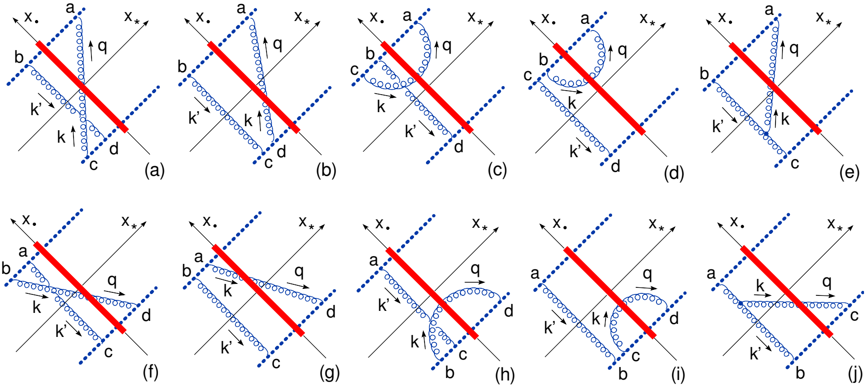
Diagrams for 1→3 dipoles transition



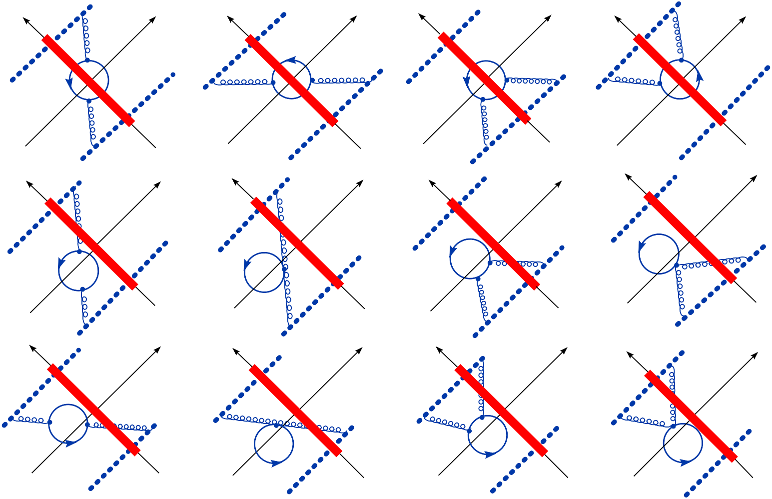
"Running coupling" diagrams



1 → 2 dipole transition diagrams



Gluino and scalar loops



$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$= \frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}$$
$$\times\ [\mathrm{Tr}\{T^a\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$
$$-\ \frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\ \frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}$$
$$\times\ \mathrm{Tr}\{[T^a,T^b]\hat{U}^{\eta}_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2}+T^bT^a\hat{U}^{\eta}_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}(\hat{U}^{\eta}_{z_3})^{aa'}(\hat{U}^{\eta}_{z_4}-\hat{U}^{\eta}_{z_3})^{bb'}$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

Evolution equation for color dipole in $\mathcal{N}=4$ (I.B. and G. Chirilli)

$$
\begin{aligned}
&\frac{d}{d\eta}\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\\
&= \frac{\alpha_s}{\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{1-\frac{\alpha_sN_c}{4\pi}\Big[\frac{\pi^2}{3}+2\ln\frac{z_{13}^2}{z_{12}^2}\ln\frac{z_{23}^2}{z_{12}^2}\Big]\Big\}\\
&\times\ [\mathrm{Tr}\{T^a\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}T^a\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\\
&-\frac{\alpha_s^2}{4\pi^4}\int\frac{d^2z_3d^2z_4}{z_{34}^4}\ \frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\\
&\times\ \mathrm{Tr}\{[T^a,T^b]\hat U^\eta_{z_1}T^{a'}T^{b'}\hat U^{\dagger\eta}_{z_2}+T^bT^a\hat U^\eta_{z_1}[T^{b'},T^{a'}]\hat U^{\dagger\eta}_{z_2}\}(\hat U^\eta_{z_3})^{aa'}(\hat U^\eta_{z_4}-\hat U^\eta_{z_3})^{bb'}
\end{aligned}
$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha<\sigma=e^{2\eta}$ in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant

Evolution equation for composite conformal dipoles in $\mathcal{N} = 4$

$$\frac{d}{d\eta}\big[\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\big]^{\rm conf}$$

$$= \frac{\alpha_s}{\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1-\frac{\alpha_sN_c}{4\pi}\frac{\pi^2}{3}\Big]\big[\mathrm{Tr}\{T^a\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}T^a\hat{U}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\big]^{\rm conf}$$

$$-\frac{\alpha_s^2}{4\pi^4}\int d^2z_3d^2z_4\frac{z_{12}^2}{z_{13}^2z_{24}^2z_{34}^2}\Big\{2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big[1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big]\ln\frac{z_{13}^2z_{24}^2}{z_{14}^2z_{23}^2}\Big\}$$

$$\times\,\mathrm{Tr}\{[T^a,T^b]\hat{U}^\eta_{z_1}T^{a'}T^{b'}\hat{U}^{\dagger\eta}_{z_2}+T^bT^a\hat{U}^\eta_{z_1}[T^{b'},T^{a'}]\hat{U}^{\dagger\eta}_{z_2}\}[(\hat{U}^\eta_{z_3})^{aa'}(\hat{U}^\eta_{z_4})^{bb'}-(z_4\to z_3)]$$

Now Möbius invariant!

NLO BFKL equation in $\mathcal{N}=4$ SYM

To find $A(x,y;x',y')$ we need the linearized (NLO BFKL) equation. With two-gluon accuracy

$$\hat{\mathcal{U}}^\eta(x,y) = 1 - \frac{1}{N_c^2-1}\mathrm{Tr}\{\hat{U}_x^\eta \hat{U}_y^{\dagger\eta}\}$$

Conformal dipole operator in the BFKL approximation

$$\hat{\mathcal{U}}^\eta_{\mathrm{conf}}(z_1,z_2) = \hat{\mathcal{U}}^\eta(z_1,z_2)+\frac{\alpha_s N_c}{4\pi^2}\int d^2z \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\ln\frac{a z_{12}^2}{z_{13}^2 z_{23}^2}[\hat{\mathcal{U}}^\eta(z_1,z_3)+\hat{\mathcal{U}}^\eta(z_2,z_3)-\hat{\mathcal{U}}^\eta(z_1,z_2)]$$

NLO BFKL equation in $\mathcal{N} = 4$ SYM

To find $A(x, y; x', y')$ we need the linearized (NLO BFKL) equation. With two-gluon accuracy

$$\hat{\mathcal{U}}^\eta(x, y) \;=\; 1 - \frac{1}{N_c^2 - 1}\mathrm{Tr}\{\hat{U}_x^\eta \hat{U}_y^{\dagger\eta}\}$$

Conformal dipole operator in the BFKL approximation

$$\hat{\mathcal{U}}^\eta_{\mathrm{conf}}(z_1, z_2) \;=\; \hat{\mathcal{U}}^\eta(z_1, z_2) + \frac{\alpha_s N_c}{4\pi^2}\int d^2z \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln\frac{a z_{12}^2}{z_{13}^2 z_{23}^2}[\hat{\mathcal{U}}^\eta(z_1, z_3) + \hat{\mathcal{U}}^\eta(z_2, z_3) - \hat{\mathcal{U}}^\eta(z_1, z_2)]$$

Define

$$\hat{\mathcal{U}}^a_{\mathrm{conf}}(z_1, z_2) = \hat{\mathcal{U}}^\eta(z_1, z_2) + \frac{\alpha_s N_c}{4\pi^2}\int d^2z \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln\frac{a e^{2\eta} z_{12}^2}{z_{13}^2 z_{23}^2}[\hat{\mathcal{U}}^\eta(z_1, z_3) + \hat{\mathcal{U}}^\eta(z_2, z_3) - \hat{\mathcal{U}}^\eta(z_1, z_2)] + \ldots$$

such that $\frac{d}{d\eta}\hat{\mathcal{U}}^a_{\mathrm{conf}}(z_1, z_2) = 0$.

$\Rightarrow$The evolution can be rewritten in terms of a

NLO BFKL

$$
\begin{aligned}
& a\frac{d}{da}\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2) \\
& = \frac{\alpha_s N_c}{2\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[1-\frac{\alpha_s N_c}{4\pi}\frac{\pi^2}{3}\Big][\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_3)+\hat{\mathcal{U}}^a_{\rm conf}(z_2,z_3)-\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2)] \\
& + \frac{\alpha_s^2 N_c^2}{8\pi^4}\int \frac{d^2z_3 d^2z_4}{z_{34}^4}\ \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2}\Big\{2\ln\frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2}+\Big[1+\frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2-z_{14}^2 z_{23}^2}\Big]\ln\frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2}\Big\}\hat{\mathcal{U}}^a_{\rm conf}(z_3,z_4) \\
& \qquad + \frac{3\alpha_s^2 N_c^2}{2\pi^3}\zeta(3)\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2)
\end{aligned}
$$

Eigenfunctions are determined by conformal invariance

$$
E_{\nu,n}(z_{10},z_{20}) \ = \ \Big[\frac{\tilde{z}_{12}}{\tilde{z}_{10}\tilde{z}_{20}}\Big]^{\frac{1}{2}+i\nu+\frac{n}{2}}\Big[\frac{\bar{z}_{12}}{\bar{z}_{10}\bar{z}_{20}}\Big]^{\frac{1}{2}+i\nu-\frac{n}{2}}
$$

The expansion in eigenfunctions

$$
\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2) \ = \ \sum_{n=0}^{\infty}\int d^2z_0\int d\nu\ E_{\nu,n}(z_{10},z_{20})\hat{\mathcal{U}}^a_{z_0,\nu,n} \quad \Rightarrow \quad a\frac{d}{da}\hat{\mathcal{U}}^a_{z_0,\nu,n} \ = \ \omega(n,\nu)\hat{\mathcal{U}}^a_{z_0,\nu,n}
$$

$\omega(n,\nu)\equiv$ pomeron intercept = eigenvalue of the BFKL equation

Pomeron intercept

Pomeron intercept = the eigenvalue of the BFKL equation

$$\omega(n,\nu) = \frac{\alpha_s}{\pi}N_c\Big[\chi(n,\frac{1}{2}+i\nu) + \frac{\alpha_s N_c}{4\pi}\delta(n,\frac{1}{2}+i\nu)\Big],$$

$$\delta(n,\gamma) = 6\zeta(3) - \frac{\pi^2}{3}\chi(n,\gamma) - \chi"(n,\gamma) - 2\Phi(n,\gamma) - 2\Phi(n,1-\gamma)$$

where $\gamma = \frac{1}{2} + i\nu$ and

$$\chi(n,\gamma) = 2\psi(1) - \psi(\gamma+\frac{n}{2}) - \psi(1-\gamma+\frac{n}{2})$$

$$\Phi(n,\gamma) = \int_0^1 \frac{dt}{1+t}\, t^{\gamma-1+\frac{n}{2}}\Big\{\frac{\pi^2}{12} - \frac{1}{2}\psi'\Big(\frac{n+1}{2}\Big) - \mathrm{Li}_2(t) - \mathrm{Li}_2(-t)$$

$$- \Big(\psi(n+1) - \psi(1) + \ln(1+t) + \sum_{k=1}^{\infty}\frac{(-t)^k}{k+n}\Big)\ln t - \sum_{k=1}^{\infty}\frac{t^k}{(k+n)^2}[1-(-1)^k]\Big\}$$

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Coincides with Lipatov & Kotikov

Conformal four-point amplitude

$$A(x,y,x',y') = (x-y)^4(x'-y')^4 N_c^2 \langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle$$

$\mathcal{O} = \mathrm{Tr}\{Z^2\}$ ($Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$) - chiral primary operator
In a conformal theory the amplitude is a function of two conformal ratios

$$A = F(R,R')$$
$$R = \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2}, \qquad R' = \frac{(x-y)^2(x'-y')^2}{(x-y')^2(x'-y)^2}$$

At large N_c

$$A(x,y,x',y') = A(g^2N_c) \qquad\qquad g^2N_c = \lambda \ - \ \text{'t Hooft coupling}$$

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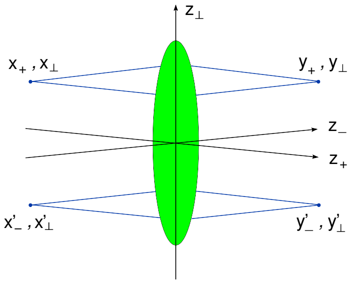
AdS/CFT gives predictions at large $\lambda \to \infty$.

Our goal is perturbative expansion and resummation of $(\lambda \ln s)^n$ at large energies in the next-to-leading approximation

$$(\lambda \ln s)^n (c_n^{\mathrm{LO}} + c_n^{\mathrm{NLO}}\lambda)$$

Regge limit in the coordinate space

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+, y_- \to \rho' y_-,\ y'_- \to \rho' y-_- \qquad \rho, \rho' \to \infty$



Full 4-dim conformal group: $A \ =\ F(R, r)$

$$R \ =\ \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2} \ \to\ \frac{\rho^2\rho'^2 x_+x'_+y_-y'_-}{(x-x')^2_\perp(y-y')^2_\perp} \ \to\ \infty$$

$$r \ =\ \frac{[(x-y)^2(x'-y')^2 - (x'-y)^2(x-y')^2]^2}{(x-x')^2(y-y')^2(x-y)^2(x'-y')^2}$$

$$\to\ \frac{[(x'-y')^2_\perp x_+y_- + x'_+y'_-(x-y)^2_\perp + x_+y'_-(x'-y)^2_\perp + x'_+y_-(x-y')^2_\perp]^2}{(x-x')^2_\perp(y-y')^2_\perp x_+x'_+y_-y'_-}$$

Regge limit: $x_+ \to \rho x_+,\ x'_+ \to \rho x'_+, y_- \to \rho' y_-,\ y'_- \to \rho' y-_-$
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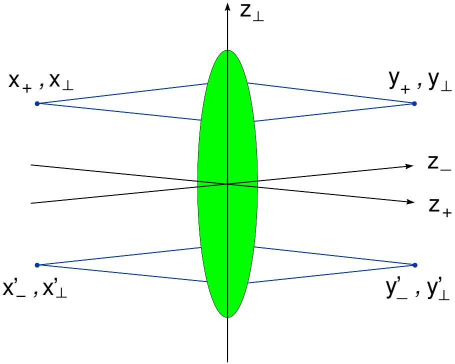

Regge limit symmetry: 2-dim conformal group $SL(2,C)$ formed from P_1, P_2, M^{12}, D, K_1 and K_2 which leave the plane $(0,0,z_\perp)$ invariant.

$$A(x,y;x',y') \overset{s\to\infty}{=} \frac{i}{2}\int d\nu\, f_+(\omega(\lambda,\nu))F(\lambda,\nu)\Omega(r,\nu)R^{\omega(\lambda,\nu)/2}$$

L. Cornalba (2007)

$f_+(\omega) = \frac{e^{i\pi\omega}-1}{\sin\pi\omega}$ - signature factor

$\Omega(r,\nu)$ - solution of the eqn $(\Box_{H_3} + \nu^2 + 1)\Omega(r,\nu) = 0$.
Explicit form:

$$\Omega(r,\nu) = \frac{\nu^2}{\pi^3}\int d^2z\Big(\frac{\kappa^2}{(2\kappa\cdot\zeta)^2}\Big)^{\frac{1}{2}+i\nu}\Big(\frac{\kappa'^2}{(2\kappa'\cdot\zeta)^2}\Big)^{\frac{1}{2}-i\nu} = \frac{\sin\nu\rho}{\sinh\rho}, \qquad \cosh\rho = \frac{\sqrt{r}}{2}$$

$$\zeta = p_1 + \frac{z_\perp^2}{s}p_2 + z_\perp, \qquad p_1^2 = p_2^2 = 0,\ \ 2(p_1,p_2) = s$$

$$\kappa = \frac{1}{2x_+}(p_1 - \frac{x^2}{s}p_2 + x_\perp) - \frac{1}{2y_+}(p_1 - \frac{y^2}{s}p_2 + y_\perp), \qquad \kappa^2\kappa'^2 = \frac{1}{R}$$

$$\kappa' = \frac{1}{2x'_-}(p_1 - \frac{x'^2}{s}p_2 + x'_\perp) - \frac{1}{2y'_-}(p_1 - \frac{y'^2}{s}p_2 + y'_\perp, \qquad 4(\kappa\cdot\kappa')^2 = \frac{r}{R}$$

The dynamics is described by $\omega(\lambda,\nu)$ and $F(\lambda,\nu)$.

Pomeron in the conformal theory

$$A(x,y;x',y') \overset{s\to\infty}{=} \frac{i}{2}\int d\nu\, f_+(\omega(\lambda,\nu))F(\lambda,\nu)\Omega(r,\nu)R^{\omega(\lambda,\nu)/2}$$

Pomeron intercept $\omega(\nu,\lambda)$ is known in two limits:

$$1. \qquad \lambda\to 0: \qquad \omega(\nu,\lambda) = \frac{\lambda}{\pi}\chi(\nu) + \lambda^2\omega_1(\nu) + ...$$

$\chi(\nu) = 2\psi(1) - \psi(\frac{1}{2}+i\nu) - \psi(\frac{1}{2}-i\nu)$ - BFKL intercept,

$\omega_1(\nu)$ - NLO BFKL intercept Lipatov, Kotikov (2000)

$$2. \qquad \lambda\to\infty: \qquad AdS/CFT \quad \Rightarrow \quad \omega(\nu,\lambda) = 2 - \frac{\nu^2+4}{2\sqrt{\lambda}} + ...$$

2 = gravition spin , next term - Brower, Polchinski, Strassler, Tan (2006)

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$$A(x,y;x',y') \stackrel{s\to\infty}{=} \frac{i}{2}\int d\nu\, f_+(\omega(\lambda,\nu))F(\lambda,\nu)\Omega(r,\nu)R^{\omega(\lambda,\nu)/2}$$

The function $F(\nu,\lambda)$ in two limits:

$$1. \qquad \lambda\to 0: \qquad F(\nu,\lambda) = \lambda^2 F_0(\nu) + \lambda^3 F_1(\nu) + ...$$

$F_0(\nu) = \frac{\pi \sinh \pi\nu}{4\nu \cosh^3 \pi\nu}$ Cornalba, Costa, Penedones (2007)

$F_1(\nu)$ = see below G. Chirilli and I.B. (2009)

$$2. \qquad \lambda\to\infty: \qquad AdS/CFT \quad \Rightarrow \quad \omega(\nu,\lambda) = \pi^3\nu^2\frac{1+\nu^2}{\sinh^2\pi\nu} + ...$$

L.Cornalba (2007)

Pomeron in the conformal theory

$$A(x,y;x',y') \stackrel{s\to\infty}{=} \frac{i}{2}\int d\nu\, f_+(\omega(\lambda,\nu))F(\lambda,\nu)\Omega(r,\nu)R^{\omega(\lambda,\nu)/2}$$

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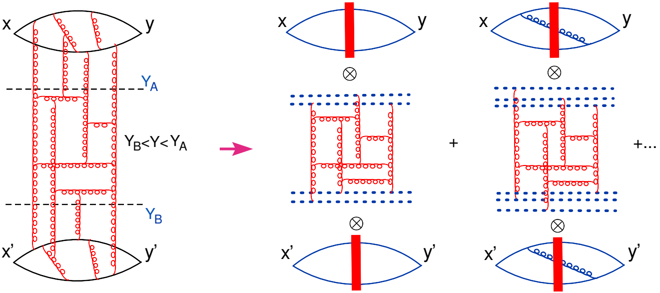
L.Cornalba (2007)

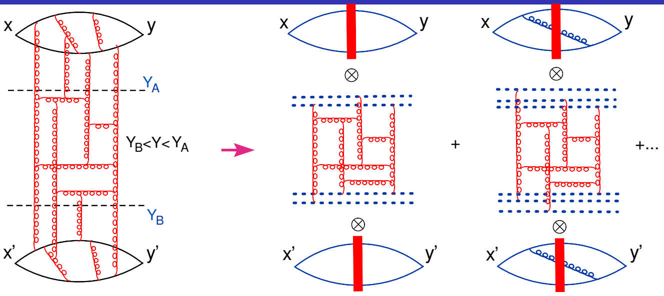
We calculate $F_1(\nu)$ (and confirm $\omega_1(\nu)$) using the expansion of high-energy amplitudes in Wilson lines (color dipoles)

NLO Amplitude in $\mathcal{N}$=4 SYM theory: factorization in rapidity



$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

$a_0 = \frac{x_+y_+}{(x-y)^2}$, $b_0 = \frac{x'_-y'_-}{(x'-y')^2}$ $\Leftrightarrow$ impact factors do not scale with energy
$\Rightarrow$ all energy dependence is contained in $[\mathrm{DD}]^{a_0,b_0}$ $(a_0b_0 = R)$



$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\,\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

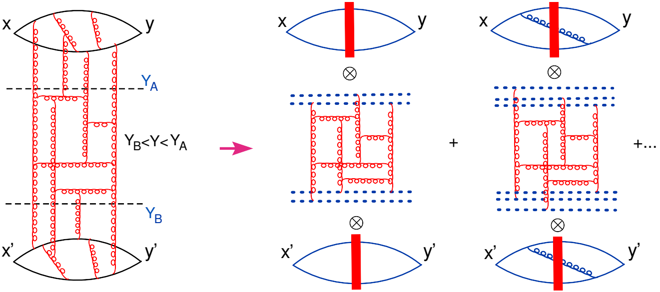
Dipole-dipole scattering

$$\chi(\gamma)\equiv 2C-\psi(\gamma)-\psi(1-\gamma)$$

$$[DD] = \int d\nu\int dz_0\,\Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^{\frac{1}{2}+i\nu}\Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^{\frac{1}{2}-i\nu}D(\frac{1}{2}+i\nu;\lambda)R^{\omega(\nu)/2}$$

$$D(\gamma;\lambda) = \frac{\Gamma(-\gamma)\Gamma(\gamma-1)}{\Gamma(1+\gamma)\Gamma(2-\gamma)}\Big\{1-\frac{\lambda}{4\pi^2}\Big[\frac{\chi(\gamma)}{\gamma(1-\gamma)}-\frac{\pi^2}{3}\Big]+O(\lambda^2)\Big\}$$

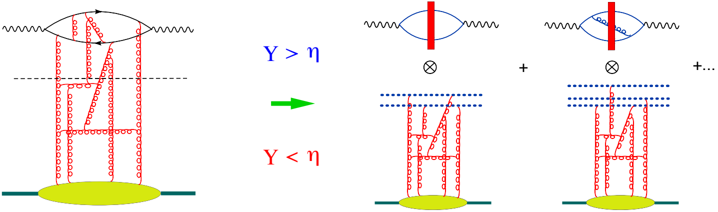
NLO Amplitude in $\mathcal{N}$=4 SYM theory: factorization in rapidity



$$(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle$$
$$= \int d^2z_{1\perp}d^2z_{2\perp}d^2z'_{1\perp}d^2z'_{2\perp}\mathrm{IF}^{a_0}(x,y;z_1,z_2)[\mathrm{DD}]^{a_0,b_0}(z_1,z_2;z'_1,z'_2)\mathrm{IF}^{b_0}(x',y';z'_1,z'_2)$$

Result : (G.A. Chirilli and I.B.)

$$F(\nu) = \frac{N_c^2}{N_c^2-1}\frac{4\pi^4\alpha_s^2}{\cosh^2\pi\nu}\left\{1+\frac{\alpha_s N_c}{\pi}\left[\frac{\pi^2}{2}-\frac{2\pi^2}{\cosh^2\pi\nu}-\frac{8}{1+4\nu^2}\right]+O(\alpha_s^2)\right\}$$



DIS structure function $F_2(x)$: photon impact factor + evolution of color dipoles+ initial conditions for the small-x evolution

Photon impact factor in the LO

$$(x-y)^4 T\{\bar{\hat\psi}(x)\gamma^\mu\hat\psi(x)\bar{\hat\psi}(y)\gamma^\nu\hat\psi(y)\} = \int\frac{d^2z_1d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(z_1,z_2){\rm tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$

$$I^{\rm LO}_{\mu\nu}(z_1,z_2) = \frac{\mathcal{R}^2}{\pi^6(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)-\frac{1}{2}\kappa^2(\zeta_1\cdot\zeta_2)\Big].$$

$$\kappa \equiv \frac{1}{\sqrt{s}x^+}\Big(\frac{p_1}{s}-x^2p_2+x_\perp\Big)-\frac{1}{\sqrt{s}y^+}\Big(\frac{p_1}{s}-y^2p_2+y_\perp\Big)$$

$$\zeta_i \equiv \Big(\frac{p_1}{s}+z_{i\perp}^2p_2+z_{i\perp}\Big), \qquad \mathcal{R}\equiv\frac{\kappa^2(\zeta_1\cdot\zeta_2)}{2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}$$

Composite "conformal" dipole $[\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}$ - same as in $\mathcal{N}=4$ case.

$$(x-y)^4 T\{\bar{\hat{\psi}}(x)\gamma^\mu\hat{\psi}(x)\bar{\hat{\psi}}(y)\gamma^\nu\hat{\psi}(y)\}$$
$$= \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\ \Big\{I^{\mu\nu}_{\mathrm{LO}}(z_1,z_2)\big[1+\frac{\alpha_s}{\pi}\big][\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}$$
$$+\int d^2z_3\Big[\frac{\alpha_s}{4\pi^2}\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big(\ln\frac{\kappa^2(\zeta_1\cdot\zeta_3)(\zeta_1\cdot\zeta_3)}{2(\kappa\cdot\zeta_3)^2(\zeta_1\cdot\zeta_2)}-2C\Big)I^{\mu\nu}_{\mathrm{LO}}+\ I_2^{\mu\nu}\Big]$$
$$\times\ [\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\}-N_c\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_{a_0}\Big\}$$

$$(I_2)_{\mu\nu}(z_1,z_2,z_3)\ =\ \frac{\alpha_s}{16\pi^8}\frac{\mathcal{R}^2}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big\{\frac{(\kappa\cdot\zeta_2)}{(\kappa\cdot\zeta_3)}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[-\frac{(\kappa\cdot\zeta_1)^2}{(\zeta_1\cdot\zeta_3)}$$
$$+\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}{(\zeta_2\cdot\zeta_3)}+\ \frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_3)(\zeta_1\cdot\zeta_2)}{(\zeta_1\cdot\zeta_3)(\zeta_2\cdot\zeta_3)}-\frac{\kappa^2(\zeta_1\cdot\zeta_2)}{(\zeta_2\cdot\zeta_3)}\Big]$$
$$+\frac{(\kappa\cdot\zeta_2)^2}{(\kappa\cdot\zeta_3)^2}\frac{\partial^2}{\partial x^\mu\partial y^\nu}\Big[\frac{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_3)}{(\zeta_2\cdot\zeta_3)}-\frac{\kappa^2(\zeta_1\cdot\zeta_3)}{2(\zeta_2\cdot\zeta_3)}\Big]+(\zeta_1\leftrightarrow\zeta_2)\Big\}$$

With two-gluon (NLO BFKL) accuracy

$$\frac{1}{N_c}(x-y)^4 T\{\bar{\hat\psi}(x)\gamma^\mu\hat\psi(x)\bar{\hat\psi}(y)\gamma^\nu\hat\psi(y)\} = \frac{\partial\kappa^\alpha}{\partial x^\mu}\frac{\partial\kappa^\beta}{\partial y^\nu}\int\frac{dz_1dz_2}{z_{12}^4}\,\hat{\mathcal{U}}_{a_0}(z_1,z_2)\Big[\mathcal{I}^{\rm LO}_{\alpha\beta}\big(1+\frac{\alpha_s}{\pi}\big)+\mathcal{I}^{\rm NLO}_{\alpha\beta}\Big]$$

$$\mathcal{I}^{\alpha\beta}_{\rm LO}(x,y;z_1,z_2) = \mathcal{R}^2\frac{g^{\alpha\beta}(\zeta_1\cdot\zeta_2)-\zeta_1^\alpha\zeta_2^\beta-\zeta_2^\alpha\zeta_1^\beta}{\pi^6(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}$$

$$\begin{aligned}\mathcal{I}^{\alpha\beta}_{\rm NLO}(x,y;z_1,z_2) = \frac{\alpha_sN_c}{4\pi^7}\mathcal{R}^2\Big\{&\frac{\zeta_1^\alpha\zeta_2^\beta+\zeta_1\leftrightarrow\zeta_2}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}\Big[4{\rm Li}_2(1-\mathcal{R})-\frac{2\pi^2}{3}+\frac{2\ln\mathcal{R}}{1-\mathcal{R}}+\frac{\ln\mathcal{R}}{\mathcal{R}}\\ &-4\ln\mathcal{R}+\frac{1}{2\mathcal{R}}-2+2\big(\ln\frac{1}{\mathcal{R}}+\frac{1}{\mathcal{R}}-2\big)\big(\ln\frac{1}{\mathcal{R}}+2C\big)-4C-\frac{2C}{\mathcal{R}}\Big]\\ +\Big(\frac{\zeta_1^\alpha\zeta_1^\beta}{(\kappa\cdot\zeta_1)^2}+\zeta_1\leftrightarrow\zeta_2\Big)&\Big[\frac{\ln\mathcal{R}}{\mathcal{R}}-\frac{2C}{\mathcal{R}}+2\frac{\ln\mathcal{R}}{1-\mathcal{R}}-\frac{1}{2\mathcal{R}}\Big]-\frac{2}{\kappa^2}\Big(g^{\alpha\beta}-2\frac{\kappa^\alpha\kappa^\beta}{\kappa^2}\Big)\\ +\Big[\frac{\zeta_1^\alpha\kappa^\beta+\zeta_1^\beta\kappa^\alpha}{(\kappa\cdot\zeta_1)\kappa^2}+\zeta_1\leftrightarrow\zeta_2\Big]&\Big[-2\frac{\ln\mathcal{R}}{1-\mathcal{R}}-\frac{\ln\mathcal{R}}{\mathcal{R}}+\ln\mathcal{R}-\frac{3}{2\mathcal{R}}+\frac{5}{2}+2C+\frac{2C}{\mathcal{R}}\Big]\\ +\frac{g^{\alpha\beta}(\zeta_1\cdot\zeta_2)}{(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)}&\Big[\frac{2\pi^2}{3}-4{\rm Li}_2(1-\mathcal{R})\\ &-2\big(\ln\frac{1}{\mathcal{R}}+\frac{1}{\mathcal{R}}+\frac{1}{2\mathcal{R}^2}-3\big)\big(\ln\frac{1}{\mathcal{R}}+2C\big)+6\ln\mathcal{R}-\frac{2}{\mathcal{R}}+2+\frac{3}{2\mathcal{R}^2}\Big]\Big\}\end{aligned}$$

5 tensor structures (CCP, 2009)

Photon Impact Factor at NLO

Reminder

$$\kappa^\mu = \frac{1}{\sqrt{s}x^+}\left(\frac{p_1^\mu}{s} - x^2 p_2^\mu + x_\perp^\mu\right) - \frac{1}{\sqrt{s}y^+}\left(\frac{p_1^\mu}{s} - y^2 p_2^\mu + y_\perp^\mu\right)$$

$$\zeta_1^\mu = \left(\frac{p_1^\mu}{s} + z_{1\perp}^2 p_2^\mu + z_{1\perp}^\mu\right), \qquad \zeta_2^\mu = \left(\frac{p_1^\mu}{s} + z_{2\perp}^2 p_2^\mu + z_{2\perp}^\mu\right)$$

DIS photon impact factor is a linear combination of the following tensor basis

$$\mathcal{I}_1^{\mu\nu} = g^{\mu\nu} \qquad \mathcal{I}_2^{\mu\nu} = \frac{\kappa^\mu\kappa^\nu}{\kappa^2}$$

$$\mathcal{I}_3^{\mu\nu} = \frac{\kappa^\mu\zeta_1^\nu + \kappa^\nu\zeta_1^\mu}{\kappa\cdot\zeta_1} + \frac{\kappa^\mu\zeta_2^\nu + \kappa^\nu\zeta_2^\mu}{\kappa\cdot\zeta_2}$$

$$\mathcal{I}_4^{\mu\nu} = \frac{\kappa^2\zeta_1^\mu\zeta_1^\nu}{(\kappa\cdot\zeta_1)^2} + \frac{\kappa^2\zeta_2^\mu\zeta_2^\nu}{(\kappa\cdot\zeta_2)^2} \qquad \mathcal{I}_5^{\mu\nu} = \frac{\zeta_1^\mu\zeta_2^\nu + \zeta_2^\mu\zeta_1^\nu}{\zeta_1\cdot\zeta_2}$$

Cornalba, Costa, Penedones (2010)

Mellin representation of the LO impact factor

$$\int \frac{d^2z_1 d^2z_2}{z_{12}^2} I_{LO}^{\mu\nu}(z_1,z_2)\Big(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\Big)^\gamma = \frac{1}{\pi^4} B(1-\gamma,1-\gamma)\Gamma(\gamma+2)\Gamma(3-\gamma)$$
$$\times \Big\{\frac{\gamma(1-\gamma)D_1}{12(1+\gamma)(2-\gamma)} + \frac{D_2}{2(1+\gamma)(2-\gamma)} - \frac{D_3^{\mu\nu}}{8(1+\gamma)(2-\gamma)}$$
$$-\frac{\gamma(1-\gamma)D_4^{\mu\nu}}{16(1+2\gamma)(3-2\gamma)(1+\gamma)(2-\gamma)} - \frac{D_1^{\mu\nu}+D_2^{\mu\nu}}{8}\Big\}_{\mu\nu}\Big(\frac{\kappa^2}{(\kappa\cdot\zeta_0)^2}\Big)^\gamma$$

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$$\times\Big\{\frac{\gamma(1-\gamma)D_1^{\mu\nu}}{12(1+\gamma)(2-\gamma)} + \frac{D_2^{\mu\nu}}{2(1+\gamma)(2-\gamma)} - \frac{D_3^{\mu\nu}}{8(1+\gamma)(2-\gamma)}$$
$$-\frac{\gamma(1-\gamma)D_4}{16(1+2\gamma)(3-2\gamma)(1+\gamma)(2-\gamma)} - \frac{D_1^{\mu\nu}+D_2^{\mu\nu}}{8}\Big\}_{\mu\nu}\Big(\frac{\kappa^2}{(\kappa\cdot\zeta_0)^2}\Big)^\gamma$$

where

$$(D_1+D_2)^{\mu\nu} = -2\Delta^2 x^+ y^+ \kappa^{-2}\partial_x^\mu \partial_y^\nu \kappa^2$$
$$D_2^{\mu\nu} = -\Delta^2 x^+ y^+ \partial_x^\mu(\ln\kappa^2)\partial_y^\nu \ln\kappa^2$$
$$D_3^{\mu\nu} = 4\gamma\Delta^2 x^+ y^+\big[(\partial_x^\mu \ln\kappa^2)\partial_\nu^y \ln(\kappa\cdot\zeta_0) + (\partial_y^\nu \ln\kappa^2)\partial_\mu^x \ln(\kappa\cdot\zeta_0) - (\partial_x^\mu \ln\kappa^2)\partial_y^\nu \ln\kappa^2\big]$$
$$D_4^{\mu\nu} = 4\gamma(1+2\gamma)\Delta^2 x^+ y^+\big[-\frac{1}{3}\partial_x^\mu\partial_y^\nu \ln\kappa^2 - \partial_x^\mu(\ln\kappa^2)\partial_y^\nu \ln\kappa^2$$
$$+(\partial_x^\mu \ln\kappa^2)\partial_\nu^y \ln(\kappa\cdot\zeta_0) + (\partial_y^\nu \ln\kappa^2)\partial_\mu^x \ln(\kappa\cdot\zeta_0) - 2\partial_\mu^x \ln(\kappa\cdot\zeta_0)\partial_\nu^y \ln(\kappa\cdot\zeta_0)\big]$$

$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $C = -\psi(1)$ is the Euler constant, and $\psi'(a) = \frac{d}{da}\ln\Gamma(a)$

$$\int \frac{d^2z_1 d^2z_2}{z_{12}^4}[I^{\mu\nu}_{LO}(z_1,z_2) +^{\mu\nu}_{NLO}(z_1,z_2)]\Big(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\Big)^\gamma = \frac{N_c}{4\pi^6\Delta^4}\frac{\Gamma(\gamma+1)\Gamma(2-\gamma)}{\Delta^2 x^+ y^+}$$

$$\times\Big[\frac{\bar\gamma\gamma D_1}{3}\Big\{1+\frac{\alpha_s N_c}{4\pi}\Big[\frac{\pi^2}{3}-\frac{\pi^2}{\sin^2\pi\gamma}-C\chi_\gamma-\frac{1}{\gamma\bar\gamma}+\frac{1}{2}--\frac{\chi_\gamma}{\gamma\bar\gamma}\Big]\Big\}$$

$$+2D_2\Big\{1+\frac{\alpha_s N_c}{4\pi}\Big[\frac{\pi^2}{3}-\frac{\pi^2}{\sin^2\pi\gamma}C\chi_\gamma-\frac{3}{4\gamma\bar\gamma}+\frac{1}{2}\chi_\gamma+\frac{\chi_\gamma}{2\gamma\bar\gamma}\Big]\Big\}$$

$$-\frac{D_3}{2}\Big\{1+\frac{\alpha_s N_c}{4\pi}\Big[\frac{\pi^2}{3}-\frac{\pi^2}{\sin^2\pi\gamma}-C\chi_\gamma+\frac{1}{2}-\frac{1}{\gamma\bar\gamma}+\frac{\chi_\gamma}{4}+\frac{\chi_\gamma}{2\gamma\bar\gamma}\Big]\Big\}$$

$$+\frac{\bar\gamma\gamma D_4}{4(3+4\bar\gamma\gamma)}\Big\{1+\frac{\alpha_s N_c}{4\pi}\Big[\frac{\pi^2}{3}-\frac{\pi^2}{\sin^2\pi\gamma}-C\chi_\gamma+\frac{1}{2}-\frac{4}{\gamma\bar\gamma}+\frac{3}{2\gamma^2\bar\gamma^2}-\frac{\chi_\gamma}{2\gamma\bar\gamma}\Big]\Big\}$$

$$-\frac{D_1+D_2}{2}(2+\bar\gamma\gamma)\Big\{1+\frac{\alpha_s N_c}{4\pi}\Big[\frac{\pi^2}{3}-\frac{\pi^2}{\sin^2\pi\gamma}-C\chi_\gamma+\frac{1}{2}$$

$$-\frac{4\gamma\bar\gamma+3}{2\gamma\bar\gamma(2+\bar\gamma\gamma)}+\frac{1+2\gamma\bar\gamma}{\gamma\bar\gamma(2+\bar\gamma\gamma)}\chi_\gamma\Big]\Big\}\Big]^{\mu\nu}\Big(\frac{\kappa^2}{(2\kappa\cdot\zeta_0)^2}\Big)^\gamma\frac{\Gamma^2(\bar\gamma)}{\Gamma(2\bar\gamma)} \qquad \bar\gamma\equiv 1-\gamma$$

NLO evolution of composite “conformal” dipoles in QCD

I. B. and G. Chirilli

$$
\begin{aligned}
a\frac{d}{da}[\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\rm conf}_a \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\Big([\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\rm conf}_a \\
& \times\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big] \\
& +\frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2+\frac{z_{23}{}^2z_{23}^2+z_{24}^2z_{13}^2-4z_{12}^2z_{34}^2}{2(z_{23}{}^2z_{23}^2-z_{24}^2z_{13}^2)}\ln\frac{z_{23}{}^2z_{23}^2}{z_{24}^2z_{13}^2}\Big] \\
& \times[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_3}U_{z_4}U^\dagger_{z_2}U_{z_3}U^\dagger_{z_4}\} - (z_4\to z_3)] \\
& +\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{z_{23}{}^2z_{23}^2}+\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}{}^2z_{23}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{z_{23}{}^2z_{23}^2}\Big] \\
& \times[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\mathrm{tr}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_4}U_{z_3}U^\dagger_{z_2}U_{z_4}U^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}
\end{aligned}
$$

$$b=\frac{11}{3}N_c-\frac{2}{3}n_f$$

$K_{\rm NLO\ BK}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N}=4$) part

Linearized $K_{\rm NLO\ BK}$ reproduces the known result for the forward NLO BFKL kernel.

Evolution equation for color dipole in momentum representation

$\mathcal{V}_a(z) \equiv z^{-2}\mathcal{U}_a(z)$
$\mathcal{V}_a(k) \equiv \int dz\, e^{-i(k,z)_\perp}\mathcal{V}_a(z)$ - “unintegrated gluon TMD”

$$2a\frac{d}{da}\mathcal{V}_a(k) = \frac{\alpha_s N_c}{2\pi^2}\int\frac{d^2k'}{(k-k')^2}\Big\{\Big(2\mathcal{V}(k') - \frac{k^2}{k'^2}\mathcal{V}_a(k)\Big)$$
$$+ \frac{\alpha_s b}{4\pi}\Big[\Big(2\mathcal{V}(k') - \frac{k^2}{k'^2}\mathcal{V}_a(k)\Big)\Big(\ln\frac{\mu^2}{k^2} + \big(\frac{67}{9} - \frac{\pi^2}{3}\big) - \frac{10n_f}{9N_c}\Big)$$
$$- 2\Big(\mathcal{V}_a(k')\ln\frac{(k-k')^2}{k'^2} - \mathcal{V}_a(k)\frac{k^2}{k'^2}\ln\frac{(k-k')^2}{k^2}\Big) + \mathcal{V}_a(k')\frac{4(k',k-k')}{k'^2}\ln\frac{(k-k')^2}{k^2}\Big]\Big\}$$
$$+ \frac{\alpha_s^2 N_c^2}{4\pi^3}\int d^2k'\,\Big[-\frac{1}{(k-k')^2}\ln^2\frac{k^2}{k'^2} + F(k,k') + \Phi(k,k')\Big]\,\mathcal{V}_a(k') + 3\frac{\alpha_s^2 N_c^2}{2\pi^2}\zeta(3)\mathcal{V}_a(k)$$

$$F(k,k') = \Big(1+\frac{n_f}{N_c^3}\Big)\frac{3(k,k')^2 - 2k^2k'^2}{16k^2k'^2}\Big(\frac{2}{k^2} + \frac{2}{k'^2} + \frac{k^2-k'^2}{k^2k'^2}\ln\frac{k^2}{k'^2}\Big)$$
$$- \Big[3 + \Big(1+\frac{n_f}{N_c^3}\Big)\Big(1 - \frac{(k^2+k'^2)^2}{8k^2k'^2} + \frac{3k^4+3k'^4-2k^2k'^2}{16k^4k'^4}(k,k')^2\Big)\Big]\int_0^\infty\frac{dt}{k^2+t^2k'^2}\ln\frac{1+t}{|1-t|},$$
$$\Phi(k,k') = \frac{(k^2-k'^2)}{(k-k')^2(k+k')^2}\Big[\ln\frac{k^2}{k'^2}\ln\frac{k^2k'^2(k-k')^4}{(k^2+k'^2)^4}$$
$$+ 2\mathrm{Li}_2\Big(-\frac{k'^2}{k^2}\Big) - 2\mathrm{Li}_2\Big(-\frac{k^2}{k'^2}\Big)\Big] - \Big(1 - \frac{(k^2-k'^2)^2}{(k-k')^2(k+k')^2}\Big)\Big[\int_0^1 - \int_1^\infty\Big]\frac{du}{(k-k'u)^2}\ln\frac{u^2k'^2}{k^2}$$

coincidess with NLO BFKL

Argument of coupling constant

$$\frac{d}{d\eta}\hat{\mathcal{U}}(z_1,z_2) =$$

$$\frac{\alpha_s(?_\perp)N_c}{2\pi^2}\int dz_3\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big\{\hat{\mathcal{U}}(z_1,z_3)+\hat{\mathcal{U}}(z_3,z_2)-\hat{\mathcal{U}}(z_1,z_2)-\hat{\mathcal{U}}(z_1,z_3)\hat{\mathcal{U}}(z_3,z_2)\Big\}$$

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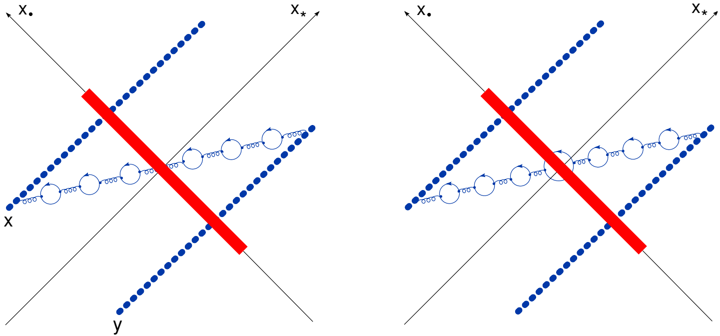
Renormalon-based approach: summation of quark bubbles



$$-\tfrac{2}{3}n_f \rightarrow b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

Argument of coupling constant

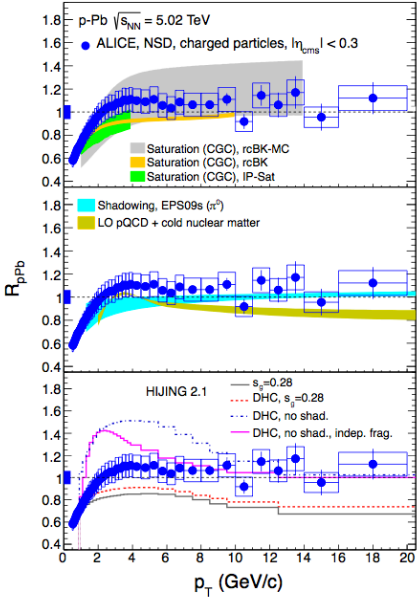
$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\} = \frac{\alpha_s(z_{12}^2)}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{Tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]$$
$$\times\Big[\frac{z_{12}^2}{z_{13}^2 z_{23}^2} + \frac{1}{z_{13}^2}\Big(\frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1\Big) + \frac{1}{z_{23}^2}\Big(\frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1\Big)\Big] + \dots$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

$$\frac{\alpha_s(z_{12}^2)}{2\pi^2}\frac{z_{12}^2}{z_{13}^2 z_{23}^2} \qquad |z_{12}| \ll |z_{13}|, |z_{23}|$$
$$\frac{\alpha_s(z_{13}^2)}{2\pi^2 z_{13}^2} \qquad |z_{13}| \ll |z_{12}|, |z_{23}|$$
$$\frac{\alpha_s(z_{23}^2)}{2\pi^2 z_{23}^2} \qquad |z_{23}| \ll |z_{12}|, |z_{13}|$$

$\Rightarrow$ the argument of the coupling constant is given by the size of the smallest dipole.



ALICE arXiv:1210.4520

Nuclear modification factor

$$R^{pPb}(p_T) = \frac{d^2N^{pPb}_{\rm ch}/d\eta dp_T}{\langle T_{pPb}\rangle d^2\sigma^{\rm pp}_{\rm ch}/d\eta dp_T}$$

N^{pPb} $\equiv$ charged particle yield in p-Pb collisions.

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- The NLO BK kernel agrees with NLO BFKL equation.
- The correlation function of four Z^2 operators is calculated at the NLO order.
- NLO photon impact factor is calculated.

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Outlook: relation to conformal light-ray operators

Gluon parton density $\mathcal{D}(x_B,\mu^2)$ is proportional to matrix element of the light-ray operator

$$\mathcal{O}(x_B,\mu^2) \ = \ \int d\lambda\ e^{i\lambda x_B}\ \mathrm{Tr}\{G_{+i}(\lambda e^+)[\lambda e^+,0]G_{+i}(0)[0,\lambda e^+]\}^\mu$$

Conformal light-ray operator $\mathcal{O}_j$ (j - I spin in $SL(2,R)$ group)

$$\mathcal{O}_j^\mu \ = \ \int d\lambda\ \lambda^{1-j}\ \mathrm{Tr}\{G_{+i}(\lambda e^+)[\lambda e^+,0]G_{+i}(0)[0,\lambda e^+]\}^\mu$$

Anomalous dimension

$$\mu\frac{d}{d\mu}\mathcal{O}_j \ = \ \gamma_j(\alpha_s)O_j$$

At $j=n$ γ_n is an anomalous dimension of the local twist-2 operator

$$G^{+i}(D^+)^{n-2}G_i^+$$

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Expansion of conformal dipoles in conformal light-ray operators - ?

In the leading order relation this expansion is trivial: $x_\perp^2$ is the normalization point of gluon light-ray operator and $x_B = e^{-\eta}$:

$$\mathrm{Tr}\{\partial_i U_x \partial^i U_0\}^\eta \quad = \quad \mathcal{D}^{\mu^2 = x_\perp^{-2}}_{x_B = e^{-\eta}} + O(x_\perp^2) \; = \; \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{dj}{2\pi i} \frac{\Gamma(j-1)}{x_B^{j-1}} (x_\perp^2 \mu^2)^{-\gamma_j} \mathcal{O}_j^{\mu^2}$$

$$= \quad \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{d\omega}{2\pi i} \Gamma(\omega) e^{\omega\eta} (x_\perp^2 \mu^2)^{-\gamma_\omega} \mathcal{O}_\omega^{\mu^2}$$

This should be compared to LO rapidity evolution of color dipole $\omega_{\gamma=\frac{1}{2}+i\nu} = \omega(\nu)$ - pomeron intercept)

$$\mathrm{Tr}\{\partial_i U_x \partial^i U_0\}^\eta \quad = \quad = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\gamma}{2\pi i} e^{\omega_\gamma(\eta-\eta_0)} (x_\perp^2 \mu^2)^{-\gamma} \int d^2 z \; (z_\perp^2)^{1-\gamma} \mathcal{U}(z_\perp)^{\eta_0}$$

In the leading order relation this expansion is trivial: $x_\perp^2$ is the normalization point of gluon light-ray operator and $x_B = e^{-\eta}$:

$$\mathrm{Tr}\{\partial_i U_x \partial^i U_0\}^\eta = \mathcal{D}^{\mu^2=x_\perp^{-2}}_{x_B=e^{-\eta}} + O(x_\perp^2) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{dj}{2\pi i} \frac{\Gamma(j-1)}{x_B^{j-1}} (x_\perp^2\mu^2)^{-\gamma_j} \mathcal{O}_j^{\mu^2}$$
$$= \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{d\omega}{2\pi i} \Gamma(\omega) e^{\omega\eta} (x_\perp^2\mu^2)^{-\gamma_\omega} \mathcal{O}_\omega^{\mu^2}$$

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$$\mathrm{Tr}\{\partial_i U_x \partial^i U_0\}^\eta = = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\gamma}{2\pi i} e^{\omega_\gamma(\eta-\eta_0)} (x_\perp^2\mu^2)^{-\gamma} \int d^2z\, (z_\perp^2)^{1-\gamma} \mathcal{U}(z_\perp)^{\eta_0}$$

$\Rightarrow$

$$\omega = \omega(\gamma, \alpha_s) \Leftrightarrow \gamma = \gamma(\omega, \alpha_s) \simeq \sum \frac{\alpha_s^n}{\omega^n} = \frac{\alpha_s}{\omega} + \frac{\alpha_s^3}{\omega^3} + \ldots$$

BFKL gives the anomalous dimensions in all orders as $\omega \to 0$ which corresponds to the the non-physical point $j = n = 1$ for γ_n of local operators

In the NLO the expansion of conformal dipoles in conformal light-ray operators is not straightforward due to mismatch of UV and rapidity regularizations.

$$\tilde{\omega}(\alpha_s, \gamma) = \omega(\alpha_s, \gamma + \frac{1}{2}\omega) \quad \Rightarrow \gamma = \gamma(\tilde{\omega}, \alpha_s)$$

$\omega(\alpha_s, \gamma)$ is the pomeron intercept which stands in the formula for the amplitude in terms of conformal ratios.
$\tilde{\omega}(\alpha_s, \gamma)$ determines anomalous dimensions of conformal light-ray operators.

The difficulty is probably due to the fact that conformal dipoles are invariant under $SL(2, C)$ and light-ray operators under $SL(2, R)$