1 Rutherford Formula in Classical Mechanics

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1.1 Motion in a (x,y) plane in a central potential $V(r)$

It is convenient to use polar coordinates

$$x = r\cos\phi, \qquad y = r\sin\phi \tag{1.1}$$

In general, both r and ϕ change as particle moves:

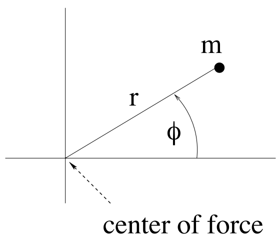

Figure 1. Polar coordinates in x, y plane.

$$\begin{aligned} v_x &= \dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi \\ v_y &= \dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi \end{aligned} \tag{1.2}$$

The kinetic energy in polar coordinates takes the form

$$T = \frac{m}{2}(v_x^2+v_y^2) = \frac{m}{2}\big[(\dot{r}\cos\phi - r\dot{\phi}\sin\phi)^2 + (\dot{r}\sin\phi + r\dot{\phi}\cos\phi)^2\big] = \frac{m}{2}\big[\dot{r}^2 + r^2\dot{\phi}^2\big] \tag{1.3}$$

The total energy is conserved

$$E = \frac{m}{2}(v_x^2+v_y^2) + V(r) = \frac{m}{2}\big(\dot{r}^2 + r^2\dot{\phi}^2\big) + V(r) = \text{const} \tag{1.4}$$

Similarly, the angular momentum $\vec{L} = L\hat{e}_z$ is conserved

$$\begin{aligned} L_z &= (\vec{r}\times\vec{p})_z = xp_y - yp_x = m(x\dot{y} - y\dot{x}) \\ &= m\big[r\cos\phi(\dot{r}\sin\phi + r\dot{\phi}\cos\phi) - r\sin\phi(\dot{r}\cos\phi - r\dot{\phi}\sin\phi)\big] = mr^2\dot{\phi} = \text{const} \\ &\Rightarrow \quad L = mr^2\dot{\phi} = \text{const} \end{aligned} \tag{1.5}$$

1.2 Effective potential

Due to the conservation of angular momentum the problem of motion of a particle in a central potential $V(r)$ can be reduced to 1-dimensional problem with an "effective potential":

$$\begin{aligned} E &= \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\phi}^2 + V(r) = \frac{m}{2}\dot{r}^2 + V(r) + \frac{m}{2}r^2\frac{L^2}{m^2r^4} \\ &= \frac{m}{2}\dot{r}^2 + V(r) + \frac{L^2}{2mr^2} = \frac{m}{2}\dot{r}^2 + V_{\text{eff}}(r) \end{aligned} \tag{1.6}$$

Thus, the energy of the particle in central potential is equal to the energy of the particle moving in one dimension (at $r > 0$) in the effective potential

$$V_{\text{eff}}(r) \;=\; V(r) + \frac{L^2}{2mr^2} \tag{1.7}$$

Since $E - V_{\text{eff}}(r) \;=\; \frac{m}{2}\dot{r}^2 \geq 0$, the equation

$$V(r) + \frac{L^2}{2mr^2} \;\leq\; E \tag{1.8}$$

determines the region of space where the motion can occur.

Also, we can determine the form of the trajectory $r(\phi)$ from the following consideration. From Eq. (1.6) we get

$$\dot{r} \;=\; \frac{dr}{dt} \;=\; \pm\sqrt{\frac{2}{m}}\sqrt{E - V_{\text{eff}}(r)} \tag{1.9}$$

where the sign depends on whether $r(t)$ is increasing (sign "+") or decreasing (sign "-") at time t, In other words, the sign depends on the direction of radial motion (sign "+" for the motion out and sign "-" for the motion in). We will consider motion "in" and take "-" sign.

From Eqs. (1.5) and (1.9) we see that

$$\begin{aligned}
\frac{d\phi}{dt} \;&=\; \frac{L}{mr^2} \quad\Rightarrow\quad dt \;=\; \frac{mr^2}{L}d\phi \\
\frac{dr}{dt} \;&=\; -\sqrt{\frac{2}{m}}\sqrt{E - V_{\text{eff}}(r)} \quad\Rightarrow\quad dt \;=\; -\frac{dr}{\sqrt{\frac{2}{m}}\sqrt{E - V_{\text{eff}}(r)}}
\end{aligned} \tag{1.10}$$

Comparing the two expressions for dt we see that

$$\begin{aligned}
&-\sqrt{\frac{m}{2}}\frac{dr}{\sqrt{E - V_{\text{eff}}(r)}} \;=\; \frac{mr^2}{L}d\phi \\
&\Rightarrow\; d\phi \;=\; -\frac{L}{\sqrt{2m}}\frac{dr}{r^2\sqrt{E - V_{\text{eff}}(r)}} \\
&\Rightarrow\; \phi_2 - \phi_1 \;=\; \int_{\phi_1}^{\phi_2} d\phi' \;=\; -\frac{L}{\sqrt{2m}}\int_{r_1}^{r_2} dr'\frac{1}{r'^2\sqrt{E - V_{\text{eff}}(r')}}
\end{aligned} \tag{1.11}$$

1.3 Scattering

Consider motion of a particle in central potential $V(r)$ which we assume to vanish at infinity $V(r) \overset{r\to\infty}{\to} 0$. The energy of a free motion at $t \to -\infty$ is $E = \frac{m}{2}v_\infty^2$ and the angular momentum is $L = mv_\infty b$ where b is called an impact factor. The typical picture of the scattering from a repulsive potential is shown in Fig. 1

One can have in mind Coulomb potential $V(r) = \frac{qQ}{4\pi r}$ as a typical example.

The point at the minimal distance r_0 is the inversion point for given energy E and angular momentum L. Since $\dot{r}(r_0) \;=\; 0$ from Eq. (1.6) we see that r_0 is a solution of the equation

$$V_{\text{eff}}(r_0) \;=\; E \quad\Leftrightarrow\quad V(r_0) + \frac{L^2}{2mr_0^2} \;=\; E \tag{1.12}$$

If we know r_0, the angle ϕ_0 can be obtained from the formula (1.11). Taking $\phi_1 = 0$ at $r = \infty$ and $\phi_2 = \phi_0$ at $r = r_0$ we get

$$\phi_0 \;=\; -\frac{L}{\sqrt{2m}}\int_{\infty}^{r_0} dr' \frac{1}{r'^2\sqrt{E - V(r') - \frac{L^2}{2mr'^2}}} \tag{1.13}$$

(The minus sign is due to the fact that $\dot{r} < 0$ if the particle is approaching the scattering center).

After reaching r_0 the particle moves again to infinity and the change of angle between r_0 and infinity is

$$\phi_0' \;=\; \frac{L}{\sqrt{2m}}\int_{r_0}^{\infty} dr' \frac{1}{r'^2\sqrt{E - V(r') - \frac{L^2}{2mr'^2}}} \tag{1.14}$$

Note that $\phi_0 = \phi_0'$ and the trajectory is symmetric with respect to line parallel to vector $\vec{r}_0$ (see Fig. 2)

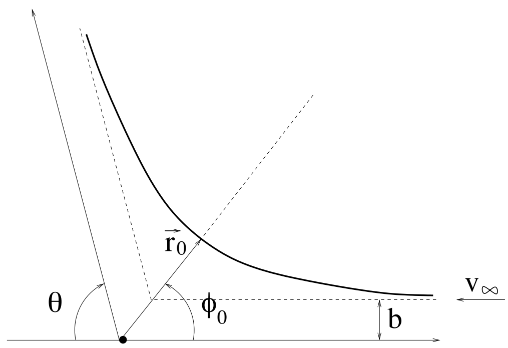

Figure 2. Scattering from a repulsive potential

For future use, it is convenient to represent ϕ_0 in terms of b and v_∞ as

$$\phi_0 \;=\; \int_{r_0}^{\infty} dr' \frac{b}{r'^2\sqrt{1 - \frac{b}{r'^2} - \frac{V(r')}{mv_\infty^2/2}}} \tag{1.15}$$

The deflection angle (the angle between velocities at plus and minus infinity) is

$$\theta \;=\; \pi - 2\phi_0 \tag{1.16}$$

1.4 Cross section

Consider a uniform beam of particles incident on a central potential

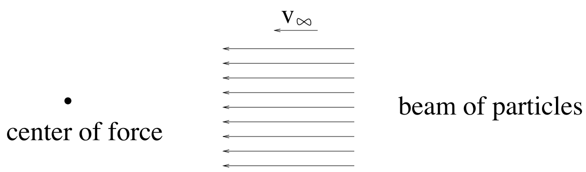

Figure 3. A beam of particles incident on a central potential

Flux Φ is a number of particles per unit area per unit time

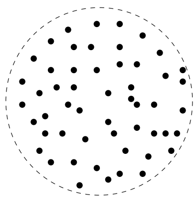

Figure 4. Transverse view of a beam of particles

Each particle has a definite b abd v_∞ and will be deflected by angle $\theta = |\pi - 2\phi_0|$. Let us consider now particles in a ring between b and $b + \Delta b$. The number of particles crossing area of a ring $b < r < b + \Delta b$ per unit time is

$$dn \;=\; 2\pi b \Delta b \; \Phi \tag{1.17}$$

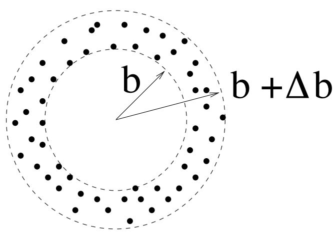

Figure 5. Particles in a ring between b and $b + \Delta b$

These particles will be deflected by angle between θ and $\theta + \Delta\theta$, see Fig. 6. (Due to azimuthal symmetry, the deflection angle $\Delta\theta$ does not depend on ϕ).

Cross section $d\sigma$ is defined as

$$dn(\theta) \;=\; \Phi d\sigma(\theta) \tag{1.18}$$

Note that $d\sigma$ has the dimension of an area since dn has a dimension of $\frac{1}{\text{time}}$ (from Eq. (1.17) $dn \;=\; \frac{\text{number of particles}}{\text{time}}$).

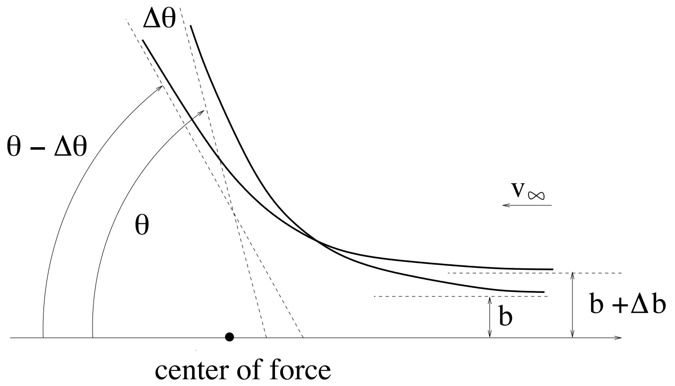

Figure 6. Scattering of particles with impact parameter between b and $b+\Delta b$

Since $\theta = |\pi - 2\phi_0(b)|$ one may think of b as a function of θ and get from Eqs. (1.17) and (1.18)

$$\cancel{\Phi} d\sigma(\theta) \;=\; \cancel{\Phi} 2\pi b db \quad\Rightarrow\quad d\sigma(\theta) \;=\; 2\pi b\Big|\frac{db(\theta)}{d\theta}\Big|d\theta \tag{1.19}$$

The reason for modulus $\big|\frac{db(\theta)}{d\theta}\big|$ in the r.h.s. of this equation is that $d\sigma(\theta)$ is a positive definite quantity ($=\frac{\text{number of particles}}{\text{flux}}$) while $b(\theta)$ is generally decreasing function of θ (the greater the impact parameter b, the smaller is the deflection angle θ), see Fig. 6.

It is convenient to write down the derivative of the cross section with respect to solid angle (so-called "differential cross section" $\frac{d\sigma}{d\Omega}$). Recall that $d\Omega \equiv \sin\theta d\theta d\phi \Rightarrow$

$$d\sigma(\theta) \;=\; \frac{b}{\sin\theta}\Big|\frac{db(\theta)}{d\theta}\Big|d\Omega \quad\Rightarrow\quad \frac{d\sigma}{d\Omega} \;=\; \frac{b}{\sin\theta}\Big|\frac{db(\theta)}{d\theta}\Big| \tag{1.20}$$

The total cross section is defined as

$$\sigma_{\rm tot} \;\equiv\; \int d\Omega \frac{d\sigma}{d\Omega} \tag{1.21}$$

so it is a number of particles scattered in a unit time in all directions divided by flux.

Example: scattering from a rigid ball of radius a. The potential is

$$V(r) \;=\; 0 \quad \text{if } r \geq a \quad \text{and} \quad V(r) \;=\; \infty \quad \text{if } r < a \tag{1.22}$$

From Fig. 7 we see that $\sin\phi_0 \;=\; \frac{b}{a}$ (for $b < a$, at $b \geq a$ the particle will not be deflected) so

$$\theta \;=\; \pi - 2\arcsin\frac{b}{a} \quad\Rightarrow\quad \frac{b}{a} \;=\; \sin\frac{\pi-\theta}{2} \;=\; \cos\frac{\theta}{2} \quad\Rightarrow\quad \frac{db}{d\theta} \;=\; -\frac{a}{2}\sin\frac{\theta}{2} \tag{1.23}$$

and therefore

$$\frac{d\sigma}{d\Omega} \;=\; \frac{b}{\sin\theta}\Big|\frac{db(\theta)}{d\theta}\Big| \;=\; \frac{a\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \times \frac{a}{2}\sin\frac{\theta}{2} \;=\; \frac{a^2}{4} \tag{1.24}$$

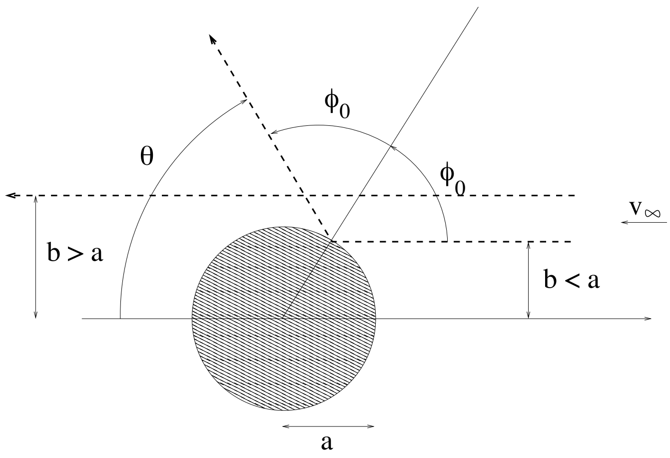

Figure 7. Scattering from the rigid ball

Not that the obtained cross section

$$\frac{d\sigma}{d\Omega} = \frac{a^2}{4} \tag{1.25}$$

is isotropic (does not depend on θ. In other words, regardless of where the detector is placed, it will detect the same number of particles per unit time per unit solid angle (for a given flux Φ).

The total cross section is

$$\sigma_{\rm tot} \equiv \int d\Omega \frac{d\sigma}{d\Omega} = \int d\Omega\, \frac{a^2}{4} = 4\pi \times \frac{a^2}{4} = \pi a^2 \tag{1.26}$$

(which means that we defined the cross section (1.19) in accordance with our everyday intuition).

1.5 Rutherford formula

Consider two particle with masses m and $M \gg m$ and charges ze and Ze. The effective potential is

$$V_{\rm eff}(r) = \frac{Zze^2}{r} + \frac{L^2}{2\mu r^2}, \qquad \mu \equiv \frac{mM}{m+M} \simeq m \tag{1.27}$$

(see Fig. 8)

The inversion point r_0 can be found from the equation

$$E = V_{\rm eff}(r_0) = \frac{Zze^2}{r_0} + \frac{L^2}{2mr_0^2} \tag{1.28}$$

or, in terms of v_∞ and b

$$2\alpha\frac{b}{r_0} + \Big(\frac{b}{r_0}\Big)^2 = 1, \qquad \alpha \equiv \frac{Zze^2}{mv_\infty^2 b} \tag{1.29}$$

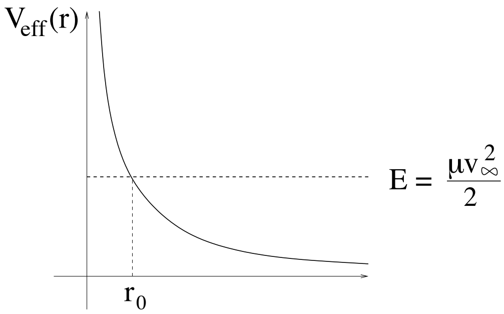

Figure 8. Effective potential for a scattering from a Coulomb center

This is a quadratic equation with a (positive) solution

$$r_0 \;=\; \frac{b}{\sqrt{1+\alpha^2}-\alpha} \tag{1.30}$$

Quick check: for a head-on collision $b \to 0$ so $\alpha \to \infty$ and

$$r_0 \;\stackrel{b\to 0}{=}\; \frac{b}{\alpha\sqrt{1+\alpha^{-2}}-1} \;\simeq\; 2\alpha b \;=\; \frac{2Zze^2}{mv_\infty^2} \tag{1.31}$$

Now we can find the angle ϕ_0. Since we are considering repulsive force ($Zz > 0$) the trajectory looks like Fig. 9

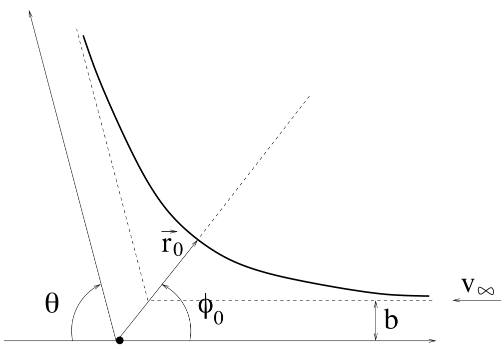

Figure 9. Scattering of particles from a Coulomb center

and therefore $\theta \;=\; \pi - 2\phi_0$ where ϕ_0 is given by Eq. (1.15)

$$\phi_0 \;=\; \int_{r_0}^{\infty} dr' \frac{b}{r'^2\sqrt{1-\frac{b}{r'^2}-\frac{2Zze^2}{mv_\infty^2 r'}}} \;\stackrel{u'=1/r'}{=}\; \int_0^{\frac{1}{r_0}} \frac{du'}{\sqrt{1-b^2u'^2-2\alpha bu'}} \tag{1.32}$$

$$\stackrel{x=u'b}{=}\; \int_0^{\frac{b}{r_0}} \frac{dx}{\sqrt{1-x^2-2\alpha x}} \;=\; \arcsin\frac{x+\alpha}{\sqrt{1+\alpha^2}}\Bigg|_0^{b/r_0} \;=\; \frac{\pi}{2} - \arcsin\frac{\alpha}{\sqrt{1+\alpha^2}}$$

because $(\frac{b}{r_0}+\alpha)^2 \;=\; 1+\alpha^2$, see Eq. (1.30). The deflection angle takes the form

$$\theta \;=\; \pi - 2\phi_0 \;\Rightarrow\; 2\arcsin\frac{\alpha}{\sqrt{1+\alpha^2}} \;\Rightarrow\; \sin\frac{\theta}{2} \;=\; \frac{\alpha}{\sqrt{1+\alpha^2}} \tag{1.33}$$

and therefore

$$\frac{1}{\sin^2\frac{\theta}{2}} \;=\; 1+\frac{1}{\alpha^2} \;=\; 1+b^2\Big(\frac{mv_\infty^2}{Zze^2}\Big)^2 \tag{1.34}$$

To find differential cross section from Eq. (1.20) we need to rewrite the impact parameter b as a function of deviation angle θ which is easily done inverting the above equation:

$$b(\theta) \;=\; \Big|\frac{Zze^2}{mv_\infty^2}\Big|\cot\frac{\theta}{2} \tag{1.35}$$

The differential cross section (1.20) takes the form

$$\frac{d\sigma}{d\Omega} \;=\; \frac{b}{\sin\theta}\Big|\frac{db(\theta)}{d\theta}\Big| \;=\; \Big|\frac{zZe^2}{2mv_\infty^2}\Big|^2\frac{1}{\sin^4\frac{\theta}{2}} \tag{1.36}$$

This is the famous Rutherford's formula.

Properties:

- $\frac{d\sigma}{d\Omega}$ is independent of the sign of charges ze and Ze ($\equiv$ cross section is the same for attractive and repulsive Coulomb potential).
- $\frac{d\sigma}{d\Omega} \;\sim\; \frac{1}{\theta^4}$ for small angles (large impact parameters) $\Rightarrow$
- The integral for the total cross section (1.21) $\sigma_{\rm tot} \;=\; \int d\Omega\frac{d\sigma}{d\Omega}$ diverges at small θ

The last property means that the total cross section $\sigma_{\rm tot}$ is poorly defined for Coulomb potential since all particles are deflected regardless of how large is b. This behavior (divergence of $\sigma_{\rm tot}$) is a characteristic of potentials falling as $\frac{1}{r}$ at large separations.

Remarkable fact about the Rutherford formula is that for Coulomb scattering Classical Mechanics, Quantum Mechanics, and Quantum Field Theory all lead to the same result! (Otherwise, physicists in the beginning of 20th century would be extremely confused...)