Problem 1. True (**T**) or false (**F**)?

1. If two events are separated by space-like interval, there is a frame where they occur simultaneously. (**T**)
2. The energy of even a massless particle like photon can be arbitrary large. (**T**)
3. If I know the wave function of some quantum-mechanical system, I can predict the outcome of any future observation of that system (e.g. position) with certainty. (**F**)
4. If I measure the energy of a quantum-mechanical system, I will get an answer that is an eigenvalue of the Hamiltonian of that system. (**T**)
5. An atom with 12 electrons must have some electrons in $n = 3$ state. (**T**)

Problem 2.

Two spaceships approach Earth with speeds $0.8c$. One of them goes along x axis, another along y axis. What is the magnitude and direction of the velocity of one of the ships in another ship's frame.

Solution

Let us find the velocity of the ship moving along y axis with respect to frame (second ship) moving along x axis. Lorentz transformation for velocities from Earth's frame to ship's frame reads

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}, \qquad u'_y = \frac{u_y/\gamma}{1 - \frac{u_x v}{c^2}} \qquad u'_z = \frac{u_z/\gamma}{1 - \frac{u_x v}{c^2}}$$

(see Eq. 1-23 from 5th edition). In our case $u_x = u_z = 0$ and $u_y = v = 0.8C$ so we get

$$u'_x = -v, \qquad u'_y = \frac{v}{\gamma} \qquad u'_z = 0 \quad \Rightarrow \quad |u'| = v\sqrt{2 - \frac{v^2}{c^2}} \simeq 0.93c$$

The angle with respect to x axis is obtained from

$$\tan\theta = \frac{u'_y}{|u'_x|} = \frac{1}{\gamma} = 0.6 \quad \Rightarrow \quad \theta \simeq 31^\circ$$

Problem 3.

Photons from a helium-neon laser λ=632.82 nm collide head on with incident electrons of energy E_1=100 MeV. Some of the photons are scattered back in the direction from which they came. What is the wavelength of the back-scattered light?

Solution

First, one cannot use Eq. (3.25) since the electron with energy 100 MeV is hardly at rest. Thus, back to conservation laws.

The momentum of the photon with λ_1=632.82 nm is $k_1 = \frac{h}{\lambda} \simeq 2\frac{\text{eV}}{c}$. Conservation of momentum and energy for a head-on collision

$$\begin{aligned} k_2 - p_2 &= p_1 - k_1 = p, && \text{conservation of momentum} \\ k_2 c + \sqrt{m^2c^4 + p_2^2c^2} &= k_1 c + E_1 = E, && \text{conservation of energy} \\ \Rightarrow k_2 &= \frac{E^2 - p^2c^2 - m^2c^4}{2c(E - pc)} = k_1 \frac{E_1 + p_1 c}{E_1 - p_1 c + 2k_1 c} \end{aligned}$$

Since $E_1 \gg mc^2$ we can use approximation $E_1 = \sqrt{m^2c^4 + p_1^2c^2} \simeq p_1 c + \frac{m^2c^3}{2p_1}$ and get

$$k_2 \simeq k_1 \frac{p_1}{k_1 + \frac{m^2c^2}{4p_1}} \simeq 3.2 \times 10^5 \frac{\text{eV}}{c} \simeq 3.2 \times 10^5 \frac{\text{eV}}{c}$$

which corresponds to $\lambda_2 \simeq 3.87{\times}10^{-12}\text{m} = 38.7\text{fm}$.

Problem 4.

The average energy of a proton in a certain nucleus is 20 MeV. Using Heisenberg uncertainty relation, estimate the size of the nucleus. (10 fm)

Solution

The momentum of the proton with average energy 20 MeV is $p = \frac{1}{c}\sqrt{E^2 - m^2c^4} \simeq \frac{20\text{MeV}}{c}$ and therefore

$$R_{\text{nucleus}} \sim \Delta x \sim \frac{\hbar}{p} \simeq 10\text{fm}$$

The estimation $\Delta x \geq \frac{\hbar}{2p} \simeq 5\text{fm}$ is OK also.

Problem 5.

At $t = 0$ the one-dimensional harmonic oscillator with Hamiltonian $\frac{\hat{p}^2}{2m} + \frac{m\omega^2x^2}{2}$ is in the state

$$\frac{1}{\sqrt{2}}[\psi_0(x) + \psi_1(x)]$$

a) Show that a later time t the oscillator is still in the $\frac{1}{\sqrt{2}}[\psi_0(x,t) + \psi_1(x,t)]$ state.

b) Find the average momentum $\langle \hat{p} \rangle$ at time t.

You may need Gaussian integrals

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \qquad \int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

Solution

(a): Wave function is correct if it satisfies Schrödinger equation and initial condition. Check of Schrödinger equation

$$
\begin{aligned}
i\hbar\frac{d}{dt}\frac{1}{\sqrt{2}}[\psi_0(x,t)+\psi_1(x,t)] &= \frac{1}{\sqrt{2}}[i\hbar\frac{d}{dt}\psi_0(x,t)+i\hbar\frac{d}{dt}\psi_1(x,t)] \\
= \frac{1}{\sqrt{2}}\Big[\Big(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{m\omega^2x^2}{2}\Big)\psi_0(x,t)+\Big(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{m\omega^2x^2}{2}\Big)\psi_1(x,t)\Big] \\
= \Big(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{m\omega^2x^2}{2}\Big)\frac{1}{\sqrt{2}}[\psi_0(x,t)+\psi_1(x,t)]
\end{aligned}
$$

Thus, $\psi(x) = \frac{1}{\sqrt{2}}[\psi_0(x,t)+\psi_1(x,t)]$ satisfies both Schrödinger equation and initial condition $\psi(x,0) = \frac{1}{\sqrt{2}}[\psi_0(x)+\psi_1(x)]$ so it is a correct expression for wave function at any t.

(b): Average momentum

$$
\begin{aligned}
\langle\hat{p}\rangle &= \int dx\ \frac{1}{\sqrt{2}}[\psi_0^*(x,t)+\psi_1^*(x,t)](-i\hbar\frac{\partial}{\partial x})\frac{1}{\sqrt{2}}[\psi_0(x)+\psi_1(x)] \\
&= \int dx\ \frac{1}{\sqrt{2}}\Big[e^{i\frac{E_0}{\hbar}t}\psi_0^*(x)+\psi_1^*(x)e^{i\frac{E_1}{\hbar}t}\Big](-i\hbar\frac{\partial}{\partial x})\frac{1}{\sqrt{2}}\Big[e^{-i\frac{E_0}{\hbar}t}\psi_0^*(x)+\psi_1^*(x)e^{-i\frac{E_1}{\hbar}t}\Big] \\
&= \frac{1}{2}\int dx\ [\psi_0^*(x)(-i\hbar\frac{\partial}{\partial x})\psi_0(x)+\psi_1^*(x)(-i\hbar\frac{\partial}{\partial x})\psi_1(x)] \\
&+ \frac{1}{2}e^{\frac{i}{\hbar}(E_1-E_0)t}\int dx\ \psi_1^*(x)(-i\hbar\frac{\partial}{\partial x})\psi_0(x)+\ \frac{1}{2}e^{-\frac{i}{\hbar}(E_1-E_0)t}\int dx\ \psi_0^*(x)(-i\hbar\frac{\partial}{\partial x})\psi_1(x) \\
&= 0\ +\ 0\ +\ \frac{1}{2}e^{\frac{i}{\hbar}(E_1-E_0)t}\int dx\ \psi_1^*(x)(-i\hbar\frac{\partial}{\partial x})\psi_0(x)-\frac{1}{2}e^{-\frac{i}{\hbar}(E_1-E_0)t}\int dx\ \psi_1(x)(-i\hbar\frac{\partial}{\partial x})\psi_0^*(x) \\
&= \hbar\sin\frac{E_1-E_0}{\hbar}t\int dx\ \psi_1(x)\frac{\partial}{\partial x}\psi_0(x)
\end{aligned}
$$

where we used the property that ψ_0 and ψ_1 are real. Using the explicit form of ψ_0 and ψ_1 we get

$$
\begin{aligned}
\psi_0(x) &= \Big(\frac{m\omega}{\pi\hbar}\Big)^{\frac{1}{4}}e^{-\frac{m\omega}{2\hbar}x^2}, \qquad \psi_1(x) = 2\sqrt{\frac{m\omega}{\hbar}}\Big(\frac{m\omega}{\pi\hbar}\Big)^{\frac{1}{4}}xe^{-\frac{m\omega}{2\hbar}x^2} \\
\Rightarrow \int dx\ \psi_1^*(x)\frac{d}{dx}\psi_0(x) &= \frac{1}{\sqrt{\pi}}(\frac{m\omega}{\hbar})^2\int dx\ x^2e^{-\frac{m\omega}{\hbar}x^2} = \frac{1}{2}\sqrt{\frac{m\omega}{\hbar}}
\end{aligned}
$$

and therefore

$$
\langle\hat{p}\rangle = \hbar\int dx\ \psi_1^*(x)\frac{d}{dx}\psi_0(x) = \frac{1}{2}\sqrt{m\omega\hbar}\sin\omega t
$$

Problem 6.

The electron in a hydrogen atom is in the state described by wave function

$$
\frac{1}{\sqrt{2}}(\psi_{100}e^{-i\frac{E_1}{\hbar}t}+\psi_{210}e^{-i\frac{E_2}{\hbar}t}), \qquad\qquad \psi_{nlm} = R_{nl}(r)Y_{lm}(\theta,\phi)
$$

(here we disregard spin of the electron). What is the expectation value of $\hat{L}^2$ in this state at time t?

Solution

$$\langle \hat{L}^2 \rangle \;=\; \frac{1}{2}\int d^3x (\psi_{100}^* e^{i\frac{E_1}{\hbar}t} + \psi_{210}^* e^{i\frac{E_2}{\hbar}t})\hat{L}^2(\psi_{100}e^{-i\frac{E_1}{\hbar}t} + \psi_{210}e^{-i\frac{E_2}{\hbar}t})$$

The functions ψ_{nlm} are the eigenfunctions of $\hat{L}^2$ operator

$$\hat{L}^2\psi_{nlm} \;=\; \hbar^2 l(l+1)\psi_{nlm}$$

so we get

$$\hat{L}^2(\psi_{100}e^{-i\frac{E_1}{\hbar}t} + \psi_{210}e^{-i\frac{E_2}{\hbar}t}) \;=\; 0 \;+\; 2\hbar^2\psi_{210}e^{-i\frac{E_2}{\hbar}t}$$

and therefore

$$\langle \hat{L}^2 \rangle \;=\; \hbar^2\int d^3x \; (\psi_{100}^* e^{i\frac{E_1}{\hbar}t} + \psi_{210}^* e^{i\frac{E_2}{\hbar}t})\psi_{210}e^{-i\frac{E_2}{\hbar}t} \;=\; \hbar^2\int d^3x \; \psi_{210}^*\psi_{210} \;=\; \hbar^2$$

Problem 7.

What are possible values of total angular momentum for the system of three particles with spin $\frac{1}{2}$? (Assume there is no orbital angular momentum)

Solution

Let us do the addition in two steps: (spin $\frac{1}{2}$+spin $\frac{1}{2}$) +spin $\frac{1}{2}$. When we add two $\frac{1}{2}$ spins we can get spin 0 and spin 1 systems. At the second step, we have a problem of addition either angular momentum 0 and angular momentum $\frac{1}{2}$ or angular momentum 1 andangular momentum $\frac{1}{2}$. In the first case we can get only angular momentum $\frac{1}{2}$ whereas in the second case we can get $\frac{3}{2}$ and $\frac{1}{2}$. Thus, the possible values of total angular momentum are $\frac{1}{2}$ or $\frac{3}{2}$.

Problem 8.

Question #1 (p. 322) from *Tipler & Llewellyn, 5th ed.*

Solution

From Fig. 8-3

$$v_{\text{rms}} \;=\; \sqrt{\frac{3kT}{m}} \quad\Rightarrow\quad \frac{v_{\text{rms}}^{H_2}}{v_{\text{rms}}^{O_2}} \;=\; \sqrt{\frac{m_{O_2}}{m_{H_2}}} \;=\; \sqrt{16} \;=\; 4$$