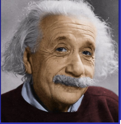
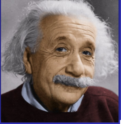


PHYSICS 453

Electromagnetism II
Lecture 16

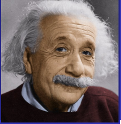
Physics Department
Old Dominion University

March 27, 2025



Outline

1. Special Theory of Relativity and Covariant Electrodynamics
   - Galilean Transformations
   - Maxwellian Mechanics under Galilean Transformations
   - Wave Equation
   - Einstein's postulates
   - Kinematic Results of Special Relativity
   - Lorentz transformations
   - Problem 12.19



Galilean Transformations

- In Newtonian Mechanics: **inertial frame**; in which a body, acted on by no external forces, moves with a constant velocity
- A transformation between two inertial frames is a **Galilean Transformation**
- Practical definition of an inertial frame is one moving with constant velocity relative to the distant stars (Mach's principle)
- Consider two inertial frames K, K', moving with a relative velocity $\mathbf{v}$
- The coordinates in the two frames are related by

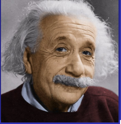
$$t' = t \quad , \quad \mathbf{x}' = \mathbf{x} - \mathbf{v}t$$

- Consider the interactions of N particles at positions $\mathbf{x}_i; i = 1, \ldots, N$, acting solely under the influence of a central potential $V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|)$
- Then the equation of motion of particle i in K is

$$m_i \frac{d\mathbf{v}_i}{dt} = -\sum_j \nabla_{x_i} V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|)$$

- Suppose that we look at the equation of motion in K'
- Then we should have

$$m_i \frac{d\mathbf{v}'_i}{dt} = -\sum_j \nabla_{x'_i} V_{ij}(|\mathbf{x}'_i - \mathbf{x}'_j|)$$



Wave equation

$$m_i \frac{d\mathbf{v}'_i}{dt} = -\sum_j \nabla_{x'_i} V_{ij}(|\mathbf{x}'_i - \mathbf{x}'_j|)$$

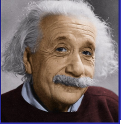
- It is evident that $\mathbf{v}'_i = \mathbf{v}_i - \mathbf{v}$, and under the transformation, $\partial/\partial x'_i = \partial/\partial x_i$
- We also have $d\mathbf{v}'_i/dt = d\mathbf{v}_i/dt$ and $|\mathbf{x}'_i - \mathbf{x}'_j| = |\mathbf{x}_i - \mathbf{x}_j|$
- Thus, we see that the equation of motion in K' is of exactly the same form as that in K
- Classical Newtonian mechanics transforms **covariantly** under Galilean Transformations
- Electric and magnetic propagation in a vacuum satisfies the wave equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(x, y, z; t) = 0$$

- Consider its transformation under $t' = t$, $\mathbf{x}' = \mathbf{x} - \mathbf{v}t$. We have

$$\frac{\partial}{\partial x_i} = \frac{\partial x'_j}{\partial x_i}\frac{\partial}{\partial x'_j} + \frac{\partial t'}{\partial x_i}\frac{\partial}{\partial t'} = \delta_{ij}\frac{\partial}{\partial x'_j} + 0 = \frac{\partial}{\partial x'_i}$$

$$\frac{\partial}{\partial t} = \frac{\partial x'_j}{\partial t}\frac{\partial}{\partial x'_j} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = -v_j\frac{\partial}{\partial x'_j} + \frac{\partial}{\partial t'} = \frac{\partial}{\partial t'} - \mathbf{v}\cdot\nabla'$$



Wave Equation

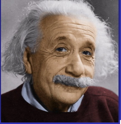
$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(x,y,z;t) = 0$$

- Thus, under $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x'_i}$, $\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \mathbf{v}\cdot\nabla'$, the wave equation becomes

$$\left[\nabla'^2 - \frac{1}{c^2}\left(\frac{\partial}{\partial t'} - \mathbf{v}\cdot\nabla'\right)\left(\frac{\partial}{\partial t'} - \mathbf{v}\cdot\nabla'\right)\right]\psi = 0$$

$$\left[\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} + \frac{2}{c^2}\mathbf{v}\cdot\nabla'\frac{\partial}{\partial t'} - \frac{1}{c^2}(\mathbf{v}\cdot\nabla')(\mathbf{v}\cdot\nabla')\right]\psi = 0$$

- This equation is clearly different from the wave equation
- It does not transform covariantly under Galilean Transformations
- For sound waves there is no problem; they propagate in a medium
- It is then natural to write wave equation in medium's rest frame
- The natural question: - *Is there a frame in which the "ether" is at rest"?*
- We all know the answer (Michelson-Morley): there is no ether
- The velocity of light is the same in all frames
- The resolution of this nasty transformation property is the **Special Theory of Relativity**



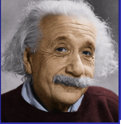
Einstein's postulates

1. Postulate 1: Same laws of nature hold in all inertial systems $\Rightarrow$
2. Postulate 2: Velocity of light is the same in all systems moving uniformly with respect to each other.

- Derive the relationship between coordinates in two frames K, K' moving with constant relative velocity $\mathbf{v}$
- Choose that the origins of the coordinates coincide at $t = t' = 0$
- Take a flashlight rapidly switched on and off at the origin at $t = t' = 0$
- By postulate 2, observers in both K and K' see a spherical shell of radiation expanding with the velocity of light c. The wavefront satisfies

$$\text{In } K\text{:} \quad c^2t^2 - (x^2 + y^2 + z^2) = 0$$
$$\text{In } K'\text{:} \quad c^2t'^2 - (x'^2 + y'^2 + z'^2) = 0$$

- Thus we need a transformation, under which the quantity $c^2t^2 - (x^2 + y^2 + z^2) = 0$ remains invariant
- The emission of the light, and its subsequent absorption at some later times, are each **events**
- These events are separated by something traveling at the speed of light



Kinematic Results of Special Relativity

- Consider the case where the axes in K, K' are parallel and the frames are moving with a relative velocity $\mathbf{v} = v\mathbf{e_3}$
- The transformation must reduce to the Galilean transformation for small relative velocities.
- First guess: the linear relations

$$t' = a_1 t + b_1 z$$
$$z' = a_2 t + b_2 z \ , \quad x' = x \ , \quad y' = y$$

- The transverse dimensions do not change $\Leftarrow$ the gedanken experiment of Taylor and Wheeler discussed in *Griffiths'* textbook)
- The event $z' = 0$ corresponds to $z = vt$, yielding

$$a_2 = -v b_2$$

- We now impose invariance of Δs^2:

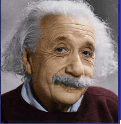
$$c^2 t^2 - (x^2 + y^2 + z^2) = c^2 (a_1 t + b_1 z)^2 - x^2 - y^2 - (a_2 t + b_2 z)^2$$

- Expand it as

$$c^2 t^2 [1 - a_1^2 + a_2^2 / c^2] - z^2 [1 + b_1^2 c^2 - b_2^2] + 2zt[a_2 b_2 - c^2 a_1 b_1] = 0$$

- This is true $\forall x, t$, so equating the coefficients to zero yields

$$a_1^2 - a_2^2 / c^2 = 1 \ , \quad b_2^2 - c^2 b_1^2 = 1 \ , \quad a_2 b_2 = c^2 a_1 b_1$$



Finding coefficients

$$a_2 = -vb_2 \ , \ a_1^2 - a_2^2/c^2 = 1 \ , \ b_2^2 - c^2 b_1^2 = 1 \ , \ a_2 b_2 = c^2 a_1 b_1$$

- Using $a_2 = -vb_2$ converts the system into

$$a_1^2 - b_2^2 v^2/c^2 = 1 \ , \ b_2^2 - c^2 b_1^2 = 1 \ , \ b_2^2 = -c^2 a_1 b_1 / v$$

- Excluding b_2^2 through last equation, we have

$$a_1^2 + a_1 b_1 v = 1 \ , \ -c^2 a_1 b_1/v - c^2 b_1^2 = 1$$

- Substituting $b_1 = (1 - a_1^2)/a_1 v$ into the second equation produces

$$-\frac{c^2}{v^2}(1 - a_1^2) - \frac{c^2}{v^2 a_1^2}(1 - a_1^2)^2 = 1 \ \text{ or } \ (1 - a_1^2) + \frac{1}{a_1^2}(1 - a_1^2)^2 = -\frac{v^2}{c^2}$$

which simplifies into $1/a_1^2 - 1 = -v^2/c^2$. Thus,

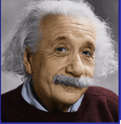
$$a_1^2 = \frac{1}{1 - v^2/c^2} \equiv \gamma^2$$

$a_1 = \gamma$

- The *gamma-factor*

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$$

plays important role in the coordinate transformations of special relativity



Lorentz transformations

- For zero velocity v we have $\gamma^2 = 1$ and hence $a_1^2 = 1$
- Since a_1 relates t' at the origin $z = 0$ to t, choose positive

$$a_1 = +\gamma$$

- Then t' runs in the same direction as t, i.e. there is no *time inversion*
- For the $b_1 = (1/a_1^2 - 1)a_1/v$ coefficient this gives $b_1 = -\gamma\, v/c^2$ and hence

$$ct' = \gamma \left[ct - \frac{v}{c} z\right]$$

- Also, $b_2^2 = -c^2 a_1 b_1/v = \gamma^2$
- The coefficient b_2 relates z' to z at the initial moment of time $t = 0$
- Choosing $b_2 = +\gamma$ means that there is no *z-axis inversion*
- Finally, we have $a_2 = -v b_2$, or $a_2 = -v\gamma$, which gives

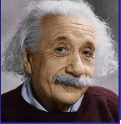
$$z' = \gamma \left[z - \frac{v}{c} ct\right]$$

- Recall also the relations $x' = x \;\; , \;\; y' = y$
- We can write these transformations in an axis-independent form as

$$ct' = \gamma(ct - \beta x_{\parallel}) \;\; , \;\; x'_{\parallel} = \gamma(x_{\parallel} - \beta ct) \;\; , \;\; \mathbf{x'}_{\perp} = \mathbf{x}_{\perp}$$

where

$$\beta = v/c \;\; , \;\; \gamma = (1 - \beta^2)^{-1/2} \;\; , \;\; x_{\parallel} = \frac{\mathbf{x} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{\boldsymbol{\beta} \cdot \mathbf{x}}{\beta}$$



Lorentz transformation, cont.

- It is easy to derive the inverse transformation

$$\left.\begin{aligned} ct &= \gamma(ct' + \beta x'_{\|}) \\ x_{\|} &= \gamma(x'_{\|} + \beta ct) \end{aligned}\right\}$$

$\Delta x^2 \neq x_2^2 - x_1^2$

$(x_2 - x_1)^2$

- It involves $-\beta$, in accordance with the fact that K moves with respect to K' with the opposite velocity $-v$

$\Delta x^2 = (\Delta x)^2$

- For any two events, the combination

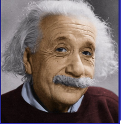
$$\Delta s^2 = c^2 \Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2) ,$$

where $\Delta t = t_2 - t_1$, $\Delta x = x_2 - x_1$, etc. is called the **interval** between two events (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2)

- Lorentz transformations $\Rightarrow$ for any two events, the interval Δs^2 is invariant under transformations between inertial frames
- Check: $c\Delta t' = \gamma(c\Delta t - \beta \Delta x_{\|})$ $\Delta x'_{\|} = \gamma(\Delta x_{\|} - \beta c \Delta t)$

$$c^2(\Delta t')^2 - (\Delta x'_{\|})^2 = \gamma^2[c^2(\Delta t)^2 - (\Delta x_{\|})^2)] + \gamma^2\beta^2[(\Delta x_{\|})^2 - c^2(\Delta t)^2]$$

$$= \gamma^2(1-\beta^2)(c^2(\Delta t)^2 - (\Delta x_{\|})^2) = c^2(\Delta t)^2 - (\Delta x_{\|})^2$$



Problem 12.19: rapidity

- Let us introduce a parameter ζ, called *rapidity*, defined by

$$\beta \equiv \tanh\zeta = \frac{\sinh\zeta}{\cosh\zeta}$$

- When ζ changes from 0 to ∞, β changes from 0 to 1
- An inverse transformation may be found from

$$\beta = \frac{e^{\zeta} - e^{-\zeta}}{e^{\zeta} + e^{-\zeta}} \quad \Rightarrow \quad e^{2\zeta} = \frac{1+\beta}{1-\beta} \quad \text{or} \quad \zeta = \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)$$

- We also have

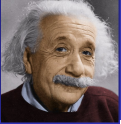
$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{\cosh\zeta}{\sqrt{\cosh^2\zeta - \sinh^2\zeta}} = \cosh\zeta$$

and

$$\beta\gamma = \tanh\zeta\cosh\zeta = \sinh\zeta$$

- Then, for frames moving parallel to the z axis, we have

$$ct' = ct\cosh\zeta - z\sinh\zeta$$
$$z' = z\cosh\zeta - ct\sinh\zeta$$



Rapidity, cont.

$$ct' = ct\cosh\zeta - z\sinh\zeta$$
$$z' = z\cosh\zeta - ct\sinh\zeta$$

- Transformation has the form of a "rotation" by a complex angle $\phi = i\zeta$

$$(ict') = (ict)\cos\phi - z\sin\phi$$
$$z' = z\cos\phi + (ict)\sin\phi\ ,$$

or

$$x_4' = x_4\cos\phi - z\sin\phi$$
$$z' = x_4\sin\phi + z\cos\phi\ ,$$

- $x_4 \equiv ict$ is the imaginary "fourth" coordinate
- The "Euclidean" rotation in the (x_4, z) plane does not change the value of $x_4^2 + z^2 = -(c^2t^2 - z^2)$, i.e. the interval between the event (x_4, t) and the $t = 0$ event at the origin $z = 0$
- Moreover, such a rotation does not change the interval

$$\Delta s^2 = -(x_4^{(2)} - x_4^{(1)})^2 - (z^{(2)} - z^{(1)})^2 = c^2\Delta t^2 - \Delta z^2$$

between any two events $(x_4^{(1)}, z^{(1)})$ and $(x_4^{(2)}, z^{(2)})$ on the (x_4, z) plane