PHYSICS 453

Electromagnetism I
Lecture 17

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April 3, 2025

Outline

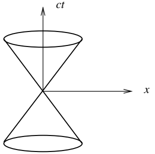
Light Cone

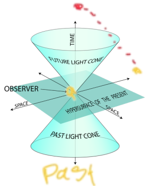
- Given two events $(ct_1, \mathbf{x}_1)$ and $(ct_2, \mathbf{x}_2)$, Lorentz transformations leave the **interval**

$$\Delta s^2 = c^2(t_2 - t_1)^2 - (\mathbf{x}_2 - \mathbf{x}_1)^2$$

invariant. Thus we can classify the interval by the **sign** of Δs^2, as follows

1. $\Delta s^2 > 0$. This is **timelike** separation. We have $c|t_2 - t_1| > |\mathbf{x}_2 - \mathbf{x}_1|$: the two points can communicate by a signal traveling at *less than* the speed of light, and indeed a frame can be chosen such that $|\mathbf{x}_2 - \mathbf{x}_1| = 0$
2. $\Delta s^2 = 0$. This is **lightlike** separation. We have $c|t_2 - t_1| = |\mathbf{x}_2 - \mathbf{x}_1|$: the two points can only be connected by a signal traveling *at* the speed of light
3. $\Delta s^2 < 0$. This is **spacelike** separation, with $c|t_2 - t_1| < |\mathbf{x}_1 - \mathbf{x}_2|$. The two points cannot communicate, and indeed a frame exists in which $t_1 = t_2$

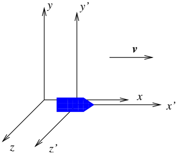




- Points that can be connected with the space-time origin by a light signal are said to lie on the **light cone**
- Points within the light cone can be causally connected with the origin, whilst those outside cannot

- The forward $(ct > 0)$ and backward $(t < 0)$ cones define absolute future and absolute past, and the ordering is preserved under Lorentz transformations

Simultaneity



- Consider a rocket moving with constant velocity v along the x direction relative to the lab frame K
- Let us denote the rest frame of the rocket by K'
- We assume that the axes of the frames are parallel, and the origins coincide at $t = 0$
- On the side of a rocket is a meter rule
- In the lab. frame, we have observers, each with a very accurate clock synchronized in the frame K
- **Simultaneity.** At time t, an observer in the lab frame, co-incident with one end of the meter rod, records his position $(ct, \mathbf{x}_1)$
- An observer coincident with the other end does likewise $(ct, \mathbf{x}_2)$
- $(ct, \mathbf{x}_1)$ and $(ct, \mathbf{x}_2)$ are two events, *simultaneous* in the lab. frame
- In the rocket rest frame K' we have

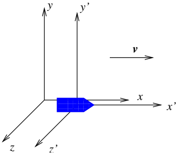
$\ell \equiv x_2 - x_1$

$$ct'_1 = \gamma(ct - \beta x_1) \quad , \quad x'_1 = \gamma(x_1 - \beta ct)$$

$$ct'_2 = \gamma(ct - \beta x_2) \quad , \quad x'_2 = \gamma(x_2 - \beta ct)$$

- We immediately see that $t'_1 = t'_2$ only if $x_1 = x_2$
- In general the points are *not simultaneous* in the rocket rest frame

Length Contraction



$$ct'_1 = \gamma(ct - \beta x_1) \quad , \quad x'_1 = \gamma(x_1 - \beta ct)$$
$$ct'_2 = \gamma(ct - \beta x_2) \quad , \quad x'_2 = \gamma(x_2 - \beta ct)$$

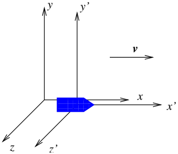
- In the rocket frame, our meter rule has length $x'_2 - x'_1$
- However, in the laboratory frame the length is obtained from

$$x'_2 - x'_1 = \gamma(x_2 - x_1) \quad , \quad \text{i.e.} \quad x_2 - x_1 = \frac{x'_2 - x'_1}{\gamma}$$

- Since $\gamma \geq 1$, we have that length is **contracted**
- In a frame, in which the meter rule is moving (lab frame), its length is smaller than in the frame where the meter rule is at rest (rocket frame)

$$\gamma \hat{=} \frac{1}{\sqrt{1 - v^2/c^2}}$$

Time Dilation



$$ct'_1 = \gamma(ct_1 - \beta x_1) \quad , \quad x'_1 = \gamma(x_1 - \beta ct_1)$$

$$ct'_2 = \gamma(ct_2 - \beta x_2) \quad , \quad x'_2 = \gamma(x_2 - \beta ct_2)$$

- We now imagine that the clocks in K, K' are synchronized at $t_1 = t'_1 = 0$ as the rocket observer (located at $x'_2 = 0$) passes origin in frame K
- An observer at x_2 in K records the time t_2 at which rocket observer passes x_2
- An observer in K' records time t'_2 at which he passes the observer in K
- The rocket observer is always at $x'_2 = 0$, so we have

$$0 = \gamma(x_2 - \beta ct_2) \implies x_2 = \beta ct_2 = vt_2$$

- Since $ct'_2 = \gamma(ct_2 - \beta x_2)$, we have

$$ct'_2 = \gamma(ct_2 - \beta x_2) = \gamma(ct_2 - \beta^2 ct_2) = \gamma\, ct_2\, (1 - \beta^2) = ct_2/\gamma \;,$$

$1/\gamma^2$

or

$$t'_2 = \frac{t_2}{\gamma}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

- Time is **dilated**: a clock that is at rest (lab frame) shows a larger time between two events than a moving clock (rocket frame)

Proper Time

- We now generalize the discussion to the case where the rocket is moving with a velocity $\mathbf{v}(t)$ along some path relative to the lab frame K
- We will now introduce K' as the **instantaneous rest frame** of the rocket
- Consider two closely separated points on the trajectory, with coordinates in the two frames $\{(ct, \mathbf{x}), (c[t+dt], \mathbf{x}+d\mathbf{x})\}$ and $\{(ct', \mathbf{x}'), (c[t'+dt'], \mathbf{x}'+d\mathbf{x}')\}$ respectively
- The interval between the points is the invariant, and we have

$$ds^2 = c^2 dt'^2 - \mathbf{dx}'^2 = c^2 dt^2 - \mathbf{dx}^2$$

- But $\mathbf{dx}' = 0$ in K', and furthermore $\mathbf{dx}^2 = \mathbf{v}^2 dt^2$, and thus

$$cdt' = cdt\sqrt{1-\beta^2(t)} \quad \text{, where} \quad \beta(t) = \frac{v(t)}{c}$$

- Then the elapsed time in the rocket between two events is

$$t_2' - t_1' = \int_{t_1}^{t_2} dt\sqrt{1-\beta^2(t)} < t_2 - t_1$$

- **Proper time** τ is the *elapsed time* in the frame in which the object is at rest
- Thus $cd\tau = ds$ where ds is the *interval* introduced earlier
- In this case we have

$$d\tau = dt\sqrt{1-\beta^2(t)}$$

- Note that proper time can only be defined for *time-like* quantities

Addition of Velocities

- Suppose now that a projectile is fired with velocity $\mathbf{u}'$ from the rocket, relative to the rocket
- Then the coordinates of the projectile in K' satisfy

$$\mathbf{u}' = \frac{d\mathbf{x}'}{dt'}$$

while in K we have

$$\mathbf{u} = \frac{d\mathbf{x}}{dt}$$

- Using the Lorentz transformation with $v \to -v$, we have

$$x_{\|} = \gamma_v[x'_{\|} + \beta ct'] \quad \Longrightarrow \quad u_{\|} \equiv \frac{dx_{\|}}{dt} = \gamma_v \left[\frac{dx'_{\|}}{dt'}\frac{dt'}{dt} + \beta c \frac{dt'}{dt} \right]$$

$$\Longrightarrow \quad u_{\|} = \gamma_v \left[\frac{dx'_{\|}}{dt'} + \beta c \right] \frac{dt'}{dt} = \gamma_v[u'_{\|} + v] \frac{dt'}{dt}$$

- We use $\|$ to denote the component along $\mathbf{v}$. We also have

$$ct = \gamma_v[ct' + \beta x'_{\|}] \quad \Longrightarrow \quad c = \gamma_v \left[c\frac{dt'}{dt} + \beta u'_{\|} \frac{dt'}{dt} \right] = \gamma_v[c + \beta u'_{\|}] \frac{dt'}{dt}$$

$$\Longrightarrow \frac{dt'}{dt} = \frac{1}{\gamma_v[1 + \beta u'_{\|}/c]}$$

Addition of Velocities, cont.

$$u_\parallel = \gamma_v [u'_\parallel + v] \frac{dt'}{dt} \quad , \quad \frac{dt'}{dt} = \frac{1}{\gamma_v [1 + \beta u'_\parallel / c]}$$

- Combining these two results, we find

$$u_\parallel = \frac{u'_\parallel + v}{1 + \beta u'_\parallel / c} = \frac{u'_\parallel + v}{1 + v u'_\parallel / c^2}$$

- Velocity of light is the same in both systems. Indeed, take $u'_\parallel = c$, then

$$u_\parallel = \frac{c + v}{1 + v/c} = c$$

- Similarly

$$u_\perp = \frac{dx_\perp}{dt} = \frac{dx'_\perp}{dt'} \frac{dt'}{dt} \quad \text{yielding} \quad u_\perp = \frac{u'_\perp}{\gamma (1 + \beta u'_\parallel / c)}$$

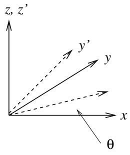
- In vector notation, this becomes

$$u_\parallel = \frac{u'_\parallel + v}{1 + \mathbf{v} \cdot \mathbf{u}' / c^2} , \quad \mathbf{u}_\perp = \frac{\mathbf{u}'_\perp}{\gamma (1 + \mathbf{v} \cdot \mathbf{u}' / c^2)}$$

- This reduces to the Galilean result $\mathbf{u} = \mathbf{u}' + \mathbf{v}$ for the case $u', v \ll c$

Euclidean Rotations: Vectors

- A much more convenient way is to use *four vectors*
- To see how these work, let us consider rotations in Euclidean space



- Consider two co-ordinate systems P, P'
- Their origins coincide, but they are related by rotation through an angle θ
- The coordinates of a point in the two systems are related through rotation matrix R

$$x'^i = R^i_j x^j$$

- Note that we have put the indices **upstairs** on the vectors
- For the specific case of a rotation through θ about the z axis

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Quantities that transform as

$$A'^i = R^i_j A^j = \frac{\partial x'^i}{\partial x^j} A^j$$

are called **vectors**

Euclidean Rotations: Co-Vectors

- A simple example of a vector is $d\mathbf{x}$, which transforms as

$$dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j = R^i_j dx^j$$

- A scalar is a quantity which transforms as $f' = f$.
- Let us now consider how the **gradient** of a function transforms:

$$\nabla'_i f = \frac{\partial f}{\partial x'^i} = \frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial f}{\partial x^j}$$

- This is an example of the transformation property

$$B'_i = \frac{\partial x^j}{\partial x'^i} B_j ,$$

  which is *different* from that for vectors
- Quantities that transform in this way are known as **covectors** or **forms**
- We put their indices downstairs
- Summarizing, we have

$$\left.\begin{array}{lll} \text{Vector:} & A'^i & = \frac{\partial x'^i}{\partial x^j} A^j \\ \text{Scalar:} & f' & = f \\ \text{Covector:} & B'_i & = \frac{\partial x^j}{\partial x'^i} B_j \end{array}\right\}$$

Euclidean Rotations: Tensors

- Finally, we have that a **tensor** is an object that transforms as a *vector* on each *upstairs* index, and a *covector* on each *downstairs* index

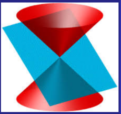
$$C'^{i'j'\dots}_{k'l'\dots} = \frac{\partial x'^{i'}}{\partial x^i} \frac{\partial x'^{j'}}{\partial x^j} \cdots \frac{\partial x^k}{\partial x'^{k'}} \frac{\partial x^l}{\partial x'^{l'}} \cdots C^{ij\dots}_{kl\dots}$$

- The **length** of a vector is a bilinear, and independent of the choice of frame
- Define the **inner product** of two vectors by

$$X \cdot Y = g_{ij} X^i Y^j .$$

- We call g_{ij} the **metric tensor**
- In Cartesian coordinates (x, y, z), we have $g_{ij} = \delta_{ij}$, since

$$(dl)^2 = (dx)^2 + (dy)^2 + (dz)^2$$



Metric Tensor

- In spherical coordinates (r, θ, φ), we have

$$(dl)^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2\theta (d\varphi)^2 \ ,$$

hence

$$g_{ij} = \text{diag}(1, r^2, r^2 \sin^2\theta)$$

- We can use the metric tensor to *raise* or *lower* indices:

$$X_i = g_{ij} X^j$$
$$X \cdot Y = X^i Y_i = X_i Y^i$$

- We only have the luxury of indentifying *vectors* with *covectors* in Cartesian coordinates in Euclidean space
- In that case, the components of the two are numerically equal
- For instance, in spherical coordinates, taking

$$dx^i = \{dr, d\theta, d\varphi\}$$

as a vector, we have

$$dx_i = \{dr, r^2 d\theta, r^2 \sin^2\theta d\varphi\}$$

as the corresponding co-vector

Minkowski Space-Time

- Apply these ideas to Lorentz transformations of four-dimensional space-time
- Denote "ct" as the coordinate x_0, and write a **contravariant** four vector as

$$x^\mu \equiv (ct, x, y, z) = (x^0, x^1, x^2, x^3)$$

- Its "length" is the **interval** left invariant under Lorentz transformations
- More generally, we define the inner product of two vectors by

$$x \cdot y = g_{\mu\nu} x^\mu y^\nu$$

- We immediately see that the metric tensor is

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

- It is conventional to use *Greek Letters* for the components of a four-vector
- Four vectors are not underlined or printed in bold
- In some areas of physics, time is introduced as the *fourth* component
- Furthermore, the metric can be defined such that the spatial components are positive, and the temporal component negative
- The convention we will be using is probably the most widely used, and essentially universal amongst particle physicists
- The summation convention is as follows:
  An index can appear no more than twice. Any index appearing twice must have one upper index and one lower index, and that index is summed over