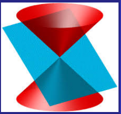


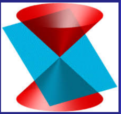
PHYSICS 453

Electromagnetism I
Lecture 17

Physics Department
Old Dominion University

April 3, 2025

Outline



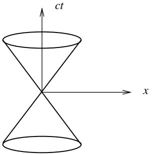
Light Cone

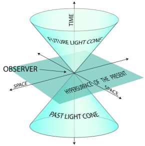
- Given two events $(ct_1, \mathbf{x}_1)$ and $(ct_2, \mathbf{x}_2)$, Lorentz transformations leave the **interval**

$$\Delta s^2 = -c^2(t_2 - t_1)^2 + (\mathbf{x}_2 - \mathbf{x}_1)^2$$

invariant. Thus we can classify the interval by the **sign** of Δs^2, as follows

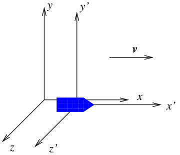
1. $\Delta s^2 < 0$. This is **timelike** separation. We have $c|t_2 - t_1| > |\mathbf{x}_2 - \mathbf{x}_1|$: the two points can communicate by a signal traveling at *less than* the speed of light, and indeed a frame can be chosen such that $|\mathbf{x}_2 - \mathbf{x}_1| = 0$
2. $\Delta s^2 = 0$. This is **lightlike** separation. We have $c|t_2 - t_1| = |\mathbf{x}_2 - \mathbf{x}_1|$: the two points can only be connected by a signal traveling *at* the speed of light
3. $\Delta s^2 0$. This is **spacelike** separation, with $c|t_2 - t_1| < |\mathbf{x}_1 - \mathbf{x}_2|$. The two points cannot communicate, and indeed a frame exists in which $t_1 = t_2$





- Points that can be connected with the space-time origin by a light signal are said to lie on the **light cone**
- Points within the light cone can be causally connected with the origin, whilst those outside cannot
- The forward $(ct > 0)$ and backward $(t < 0)$ cones define absolute future and absolute past, and the ordering is preserved under Lorentz transformations

Simultaneity



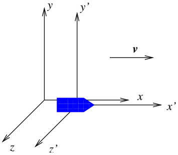
- Consider a rocket moving with constant velocity v along the x direction relative to the lab frame K
- Let us denote the rest frame of the rocket by K'
- We assume that the axes of the frames are parallel, and the origins coincide at $t = 0$
- On the side of a rocket is a meter rule
- In the lab. frame, we have observers, each with a very accurate clock synchronized in the frame K
- **Simultaneity.** At time t, an observer in the lab frame, co-incident with one end of the meter rod, records his position $(ct, \mathbf{x}_1)$
- An observer coincident with the other end does likewise $(ct, \mathbf{x}_2)$
- $(ct, \mathbf{x}_1)$ and $(ct, \mathbf{x}_2)$ are two events, *simultaneous* in the lab. frame
- In the rocket rest frame K' we have

$$ct'_1 = \gamma(ct - \beta x_1) \quad , \quad x'_1 = \gamma(x_1 - \beta ct)$$

$$ct'_2 = \gamma(ct - \beta x_2) \quad , \quad x'_2 = \gamma(x_2 - \beta ct)$$

- We immediately see that $t'_1 = t'_2$ only if $x_1 = x_2$
- In general the points are *not simultaneous* in the rocket rest frame

Length Contraction



$$ct'_1 = \gamma(ct - \beta x_1) \quad , \quad x'_1 = \gamma(x_1 - \beta ct)$$

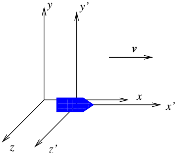
$$ct'_2 = \gamma(ct - \beta x_2) \quad , \quad x'_2 = \gamma(x_2 - \beta ct)$$

- In the rocket frame, our meter rule has length $x'_2 - x'_1$
- However, in the laboratory frame the length is obtained from

$$x'_2 - x'_1 = \gamma(x_2 - x_1) \quad , \quad \text{i.e.} \quad x_2 - x_1 = \frac{x'_2 - x'_1}{\gamma}$$

- Since $\gamma \geq 1$, we have that length is **contracted**
- In a frame, in which the meter rule is moving (lab frame), its length is smaller than in the frame where the meter rule is at rest (rocket frame)

Time Dilation



$$ct'_1 = \gamma(ct_1 - \beta x_1) \quad , \quad x'_1 = \gamma(x_1 - \beta ct_1)$$

$$ct'_2 = \gamma(ct_2 - \beta x_2) \quad , \quad x'_2 = \gamma(x_2 - \beta ct_2)$$

- We now imagine that the clocks in K, K' are synchronized at $t_1 = t'_1 = 0$ as the rocket observer (located at $x'_2 = 0$) passes origin in frame K
- An observer at x_2 in K records the time t_2 at which rocket observer passes x_2
- An observer in K' records time t'_2 at which he passes the observer in K
- The rocket observer is always at $x'_2 = 0$, so we have

$$0 = \gamma(x_2 - \beta ct_2) \implies x_2 = \beta ct_2$$

- Since $ct'_2 = \gamma(ct_2 - \beta x_2)$, we have

$$ct'_2 = \gamma(ct_2 - \beta x_2) = \gamma(ct_2 - \beta^2 ct_2) = \gamma\, ct_2\, (1 - \beta^2) = ct_2/\gamma \ ,$$

or

$$t'_2 = \frac{t_2}{\gamma}$$

- Time is **dilated**: a clock that is at rest (lab frame) shows a larger time between two events than a moving clock (rocket frame)

Proper Time

- We now generalize the discussion to the case where the rocket is moving with a velocity $\mathbf{v}(t)$ along some path relative to the lab frame K
- We will now introduce K' as the **instantaneous rest frame** of the rocket
- Consider two closely separated points on the trajectory, with coordinates in the two frames $\{(ct, \mathbf{x}), (c[t+dt], \mathbf{x}+d\mathbf{x})\}$ and $\{(ct', \mathbf{x}'), (c[t'+dt'], \mathbf{x}'+d\mathbf{x}')\}$ respectively
- The interval between the points is the invariant, and we have

$$-ds^2 = c^2 dt'^2 - \mathbf{dx}'^2 = c^2 dt^2 - \mathbf{dx}^2$$

- But $\mathbf{dx}' = 0$ in K', and furthermore $\mathbf{dx}^2 = \mathbf{v}^2 dt^2$, and thus

$$cdt' = cdt\sqrt{1-\beta^2(t)} \quad \text{, where} \quad \beta(t) = \frac{v(t)}{c}$$

- Then the elapsed time in the rocket between two events is

$$t_2' - t_1' = \int_{t_1}^{t_2} dt\sqrt{1-\beta^2(t)} < t_2 - t_1$$

- **Proper time** τ is the *elapsed time* in the frame in which the object is at rest
- Thus $cd\tau = ds$ where ds is the *interval* introduced earlier
- In this case we have

$$d\tau = dt\sqrt{1-\beta^2(t)}$$

- Note that proper time can only be defined for *time-like* quantities

Addition of Velocities

- Suppose now that a projectile is fired with velocity $\mathbf{u}'$ from the rocket, relative to the rocket
- Then the coordinates of the projectile in K' satisfy

$$\mathbf{u}' = \frac{d\mathbf{x}'}{dt'}$$

while in K we have

$$\mathbf{u} = \frac{d\mathbf{x}}{dt}$$

- Using the Lorentz transformation with $v \to -v$, we have

$$x_{\|} = \gamma_v [x'_{\|} + \beta c t'] \quad \Longrightarrow \quad u_{\|} \equiv \frac{dx_{\|}}{dt} = \gamma_v \left[\frac{dx'_{\|}}{dt'} \frac{dt'}{dt} + \beta c \frac{dt'}{dt} \right]$$

$$\Longrightarrow \quad u_{\|} = \gamma_v \left[\frac{dx'_{\|}}{dt'} + \beta c \right] \frac{dt'}{dt} = \gamma_v [u'_{\|} + v] \frac{dt'}{dt}$$

- We use $\|$ to denote the component along $\mathbf{v}$. We also have

$$ct = \gamma_v [ct' + \beta x'_{\|}] \quad \Longrightarrow \quad c = \gamma_v \left[c \frac{dt'}{dt} + \beta u'_{\|} \frac{dt'}{dt} \right] = \gamma_v [c + \beta u'_{\|}] \frac{dt'}{dt}$$

$$\Longrightarrow \frac{dt'}{dt} = \frac{1}{\gamma_v [1 + \beta u'_{\|} / c]}$$

Addition of Velocities, cont.

$$u_{\|} = \gamma_v [u'_{\|} + v] \frac{dt'}{dt} \quad , \quad \frac{dt'}{dt} = \frac{1}{\gamma_v [1 + \beta u'_{\|}/c]}$$

- Combining these two results, we find

$$u_{\|} = \frac{u'_{\|} + v}{1 + \beta u'_{\|}/c} = \frac{u'_{\|} + v}{1 + v u'_{\|}/c^2}$$

- Velocity of light is the same in both systems. Indeed, take $u'_{\|} = c$, then

$$u_{\|} = \frac{c + v}{1 + v/c} = c$$

- Similarly

$$u_{\perp} = \frac{dx_{\perp}}{dt} = \frac{dx'_{\perp}}{dt'} \frac{dt'}{dt} \quad \text{yielding} \quad u_{\perp} = \frac{u'_{\perp}}{\gamma(1 + \beta u'_{\|}/c)}$$

- In vector notation, this becomes

$$u_{\|} = \frac{u'_{\|} + v}{1 + \mathbf{v} \cdot \mathbf{u}'/c^2} \quad , \quad \mathbf{u}_{\perp} = \frac{\mathbf{u}'_{\perp}}{\gamma(1 + \mathbf{v} \cdot \mathbf{u}'/c^2)}$$

- This reduces to the Galilean result $\mathbf{u} = \mathbf{u}' + \mathbf{v}$ for the case $u', v \ll c$