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PHYSICS453

Electromagnetism II
Lecture 18

Physics Department
Old Dominion University

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Outline

1. Special Relativity and Four Vectors
   - Euclidean Rotations
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Euclidean Rotations: Vectors

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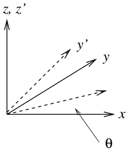
- A much more convenient way is to use *four vectors*
- To see how these work, let us consider rotations in Euclidean space



- Consider two co-ordinate systems P, P'
- Their origins coincide, but they are related by rotation through an angle θ
- The coordinates of a point in the two systems are related through rotation matrix R

$$x'^i = R^i_j x^j$$

- Note that we have put the indices **upstairs** on the vectors
- For the specific case of a rotation through θ about the z axis

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Quantities that transform as

$$A'^i = R^i_j A^j = \frac{\partial x'^i}{\partial x^j} A^j$$

are called **vectors**

Euclidean Rotations: Co-Vectors

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- A simple example of a vector is $d\mathbf{x}$, which transforms as

$$dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j = R^i_j dx^j$$

- A scalar is a quantity which transforms as $f' = f$.
- Let us now consider how the **gradient** of a function transforms:

$$\nabla'_i f = \frac{\partial f}{\partial x'^i} = \frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial f}{\partial x^j}$$

- This is an example of the transformation property

$$B'_i = \frac{\partial x^j}{\partial x'^i} B_j,$$

which is *different* from that for vectors
- Quantities that transform in this way are known as **covectors** or **forms**
- We put their indices downstairs
- Summarizing, we have

$$\left.\begin{array}{llll} \text{Vector:} & A'^i & = \frac{\partial x'^i}{\partial x^j} A^j \\ \text{Scalar:} & f' & = f \\ \text{Covector:} & B'_i & = \frac{\partial x^j}{\partial x'^i} B_j \end{array}\right\}$$

Euclidean Rotations: Tensors

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- Finally, we have that a **tensor** is an object that transforms as a *vector* on each *upstairs* index, and a *covector* on each *downstairs* index

$$C'^{i'j'\dots}_{k'l'\dots} = \frac{\partial x'^{i'}}{\partial x^i}\frac{\partial x'^{j'}}{\partial x^j}\cdots\frac{\partial x^k}{\partial x'^{k'}}\frac{\partial x^l}{\partial x'^{l'}}\cdots C^{ij\dots}_{kl\dots}$$

- The **length** of a vector is a bilinear, and independent of the choice of frame
- Define the **inner product** of two vectors by

$$X \cdot Y = g_{ij}X^iY^j.$$

- We call g_{ij} the **metric tensor**
- In Cartesian coordinates (x, y, z), we have $g_{ij} = \delta_{ij}$, since

$$(dl)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

Metric Tensor

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- In spherical coordinates (r, θ, φ), we have

$$(dl)^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2 \theta (d\varphi)^2 \ ,$$

  hence

$$g_{ij} = \text{diag}(1, r^2, r^2 \sin^2 \theta)$$

- We can use the metric tensor to *raise* or *lower* indices:

$$X_i = g_{ij} X^j$$
$$X \cdot Y = X^i Y_i = X_i Y^i$$

- We only have the luxury of indentifying *vectors* with *covectors* in Cartesian coordinates in Euclidean space
- In that case, the components of the two are numerically equal
- For instance, in spherical coordinates, taking

$$dx^i = \{dr, d\theta, d\varphi\}$$

  as a vector, we have

$$dx_i = \{dr, r^2 d\theta, r^2 \sin^2 \theta d\varphi\}$$

  as the corresponding co-vector

4-vectors in Minkowski Space-Time

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- Apply these ideas to Lorentz transformations of four-dimensional space-time
- Denote "ct" as the coordinate x_0, and write a **contravariant** four vector as

$$x^\mu \equiv (ct, x, y, z) = (x^0, x^1, x^2, x^3)$$

- Its "length" is the **interval** left invariant under Lorentz transformations
- We define the inner product of two vectors by

$$x \cdot y = -x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3 = g_{\mu\nu} x^\mu y^\nu$$

- We immediately see that the metric tensor is

$$g_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \qquad (*)$$

- It is conventional to use *Greek Letters* for the components of a four-vector
- Four vectors are not underlined or printed in bold
- In some areas of physics, time is introduced as the *fourth* component
- Furthermore, the metric can be defined such that the temporal components are positive, and the spatial component negative. Such convention $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is probably the most widely used. We will follow *Griffiths'* convention $(*)$.
- The summation convention is as follows:
  An index can appear no more than twice. Any index appearing twice must have one upper index and one lower index, and that index is summed over

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Lorentz Transformations and Four Vectors

- The **covariant four vector** or **form** can be obtained as before by using the raising and lowering properties of the metric tensor

$$x_\mu = g_{\mu\nu} x^\nu = (-x^0, x^1, x^2, x^3) = (-ct, x, y, z)$$

- $\Rightarrow$ For 4-vectors in Minkowski space *the components of a co-vector are numerically different to those of the vector*
- The relation between vectors (in 4-dimensional case of special relativity, they are called *contravariant vectors*) in the two frames is given by

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu = L^\mu{}_\nu x^\nu$$

- Let us assume a similar transformation law for the 4-dimensional analogs of covectors (called *covariant vectors*)

$$x'_\mu = L_\mu{}^\nu x_\nu$$

- We require that $x^\mu x_\mu$ is invariant under the Lorentz transformation

$$x^\mu x_\mu = x'^\mu x'_\mu = L^\mu{}_\nu L_\mu{}^\sigma x^\nu x_\sigma$$

and since this is true for all vectors, we have

$$L^\mu{}_\nu L_\mu{}^\sigma = \delta_\nu{}^\sigma \quad \text{where} \quad \delta_\nu{}^\sigma = \begin{cases} 1 & \text{if} \quad \nu = \sigma \\ 0 & \text{if} \quad \nu \neq \sigma \end{cases}$$

Lorentz Transformations and Four Vectors, cont.

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- To find $L_\mu^{\ \sigma}$, we note that, according to $x'_\mu = L_\mu^{\ \nu} x_\nu$, we have

$$L^\mu_{\ \nu} = \frac{\partial x'^\mu}{\partial x^\nu}$$

- Now, use the identity

$$\frac{\partial x^\sigma}{\partial x^\nu} = \delta_\nu^{\ \sigma}$$

- Write it through the chain rule as

$$\delta_\nu^{\ \sigma} = \frac{\partial x^\sigma}{\partial x^\nu} = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\nu} = \underbrace{\frac{\partial x'^\mu}{\partial x^\nu}}_{L^\mu_{\ \nu}}\frac{\partial x^\sigma}{\partial x'^\mu}$$

- Comparing with $L^\mu_{\ \nu} L_\mu^{\ \sigma} = \delta_\nu^{\ \sigma}$, we conclude that

$$L_\mu^{\ \sigma} = \frac{\partial x^\sigma}{\partial x'^\mu}$$

- This corresponds to the characteristic transformation property of a co-vector $\partial/\partial x^\mu$:

$$\frac{\partial f}{\partial x'^\mu} = \frac{\partial f}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu} = \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}}_{L_\mu^{\ \nu}}\frac{\partial f}{\partial x^\nu}$$

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- The various quantities we will encounter in the remainder of this course are
- **Contravariant Vectors**:

$$A'^{\mu} = L^{\mu}{}_{\nu} A^{\nu}$$

- **Covariant Vectors**:

$$B'_{\mu} = L_{\mu}{}^{\nu} B_{\nu}$$

- **Tensors**:

$$C'^{\mu'\nu'\ldots}{}_{\rho'\sigma'\ldots} = L^{\mu'}{}_{\mu} L^{\nu'}{}_{\nu} \ldots L_{\rho'}{}^{\rho} L_{\sigma'}{}^{\sigma} \ldots C^{\mu\nu\ldots}{}_{\rho\sigma\ldots}$$

- **Scalars**:

$$A \cdot B = A_{\mu} B^{\mu} = g_{\mu\nu} A^{\mu} B^{\nu}$$

Derivatives

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- As we have noted earlier, the derivatives transform as *covectors*

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \nabla\right)$$
$$\partial^\alpha = \frac{\partial}{\partial x_\alpha} = \left(-\frac{\partial}{\partial x^0}, \nabla\right)$$

- Suppose now that we have a four vector A^μ. Then

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \nabla \cdot \mathbf{A}$$

- The Laplacian is defined by

$$\Box = \partial_\alpha \partial^\alpha = -\frac{\partial^2}{\partial {x^0}^2} + \nabla^2$$

Four Velocity

- Define velocity in a usual way as $v^i = dx^i/dt$, and use that $t = x^0/c$
- This $v^i = c\,dx^i/dx^0$ cannot transform as a vector under Lorentz transformations
- A formal reason is that such a derivative is a $0i$ component of a 4-tensor
- Thus, it does not transform as an ith component of a 4-vector
- Indeed, let us assume that the object dx^μ/dt transforms as a 4-vector,

$$\mathcal{V}^\mu \equiv \frac{dx^\mu}{dt} = \frac{d}{dt}\{x_0, \mathbf{x}\} = \frac{d}{dt}\{ct, \mathbf{x}\} = \{c, \mathbf{v}\}$$

- Consider frames K and K', moving with velocity $\mathbf{V}$ with respect to K
- Take $\mathbf{V}$ along x^3 axis and split components parallel and transverse to $\mathbf{V}$

$$\mathcal{V}^\mu = \{c, \mathbf{0}_\perp, \mathcal{V}^3\} \quad , \quad \mathcal{V}'^\mu \equiv \frac{dx'^\mu}{dt'} = \{c, \mathbf{0}_\perp, \mathcal{V}'^3\}$$

- If $\mathcal{V}^\mu$ is a 4-vector, then, according to the Lorentz transformation,

$$\mathcal{V}'^3 = \gamma_V \left(\mathcal{V}^3 - \frac{V}{c}\mathcal{V}^0\right) = \gamma_V \left(\mathcal{V}^3 - V\right) \ ,$$

where $\gamma_V = 1/\sqrt{1 - V^2/c^2}$. This gives

$$\mathcal{V}'^3 = \left(\mathcal{V}^3 - V\right)/\sqrt{1 - V^2/c^2}$$

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$$\mathcal{V}'^3 = \gamma_V \left(\mathcal{V}^3 - \frac{V}{c} \mathcal{V}^0 \right) = \gamma_V \left(\mathcal{V}^3 - V \right) = \left(\mathcal{V}^3 - V \right) / \sqrt{1 - V^2/c^2} \, ,$$

- The correct result is that the velocity in the K' frame should be given by

$$v_3' = \frac{v_3 V}{1 - v_3 V/c^2}$$

- This is the relativistic velocity addition formula
- Note that we should take into account that K frame moves with respect to K' frame with the velocity $-\mathbf{V}$
- So, the question is whether it is possible to find a definition of a velocity that does indeed transform covariantly under Lorentz transformations, yet reduces to a Galilean transformation for $v \ll c$?

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- To construct a **four velocity**, we need to take the derivative of the 4-vector x^μ with respect to some time that, unlike dt or dt', is the same in all frames, i.e. is a *Lorentz Scalar*
- Such a scalar is provided by the **Proper Time** $d\tau$, or time measured in the frame that moves together with the particle
- This frame has velocity $\mathbf{v}$ in the K frame. Proper time is defined by

$$c^2 d\tau^2 = ds^2,$$

where ds is the Lorentz-invariant *interval*
- The proper time is a scalar, and a natural definition of the **four velocity** is

$$\eta^\alpha = \frac{dx^\alpha}{d\tau}$$

- Recalling that the proper time is related to the K frame time by

$$d\tau = dt\sqrt{1-\beta^2(t)} \quad \text{we have}$$

$$\eta^\alpha = \frac{1}{\sqrt{1-\beta^2}}\frac{d}{dt}(ct, \mathbf{x}) = \gamma(c, \mathbf{v}), \quad \text{or} \quad \eta^\alpha = (\gamma c, \gamma \mathbf{v})$$

- The spatial components of η^μ clearly reduce to our familiar definition of velocity in the non-relativistic (NR) limit
- Note that $\eta^\alpha \eta_\alpha = c^2$

Four Velocity, cont.

- Let us take now the component of $\mathbf{v}$ parallel to the relative velocity $\mathbf{V}$ and check that applying the Lorentz transformation to η^μ, namely

$$\eta'^3 = \gamma_V \left(\eta^3 - \frac{V}{c}\eta^0\right) \quad , \quad \eta'^0 = \gamma_V \left(\eta^0 - \frac{V}{c}\eta^3\right)$$

leads to the correct relativistic velocity addition formula

- Indeed, substituting

$$\eta^0 = \gamma_v c \ , \ \eta^3 = \gamma_v v_3$$

(where $\gamma_v = 1/\sqrt{1 - v^2/c^2}$) and

$$\eta'^0 = \gamma_{v'} c = \gamma_V \left(\eta^0 - \frac{V}{c}\eta^3\right) \ , \ \eta'^3 = \gamma_{v'} v'_3 = \gamma_V \left(\eta^3 - \frac{V}{c}\eta^0\right)$$

(where $\gamma_{v'} = 1/\sqrt{1 - v'^2/c^2}$), we get

$$\Rightarrow \quad v'_3 \gamma_{v'} = \gamma_V \gamma_v \left(v_3 - V\right) \quad , \quad c\gamma_{v'} = \gamma_V \gamma_v \left(c - \frac{V}{c} v_\parallel\right)$$

- Dividing the first of these equations by the second one gives

$$\frac{v'_3}{c} = \frac{v_3 - V}{c - \frac{V}{c} v_3} \quad \text{or} \quad v'_\parallel = \frac{v_3 - V}{1 - \frac{V}{c^2} v_3}$$

Four Momentum

- The definition of a Lorentz-covariant 4-momentum is now straightforward:

$$p^\mu = mv^\mu = (m\gamma c, m\gamma \mathbf{v}),$$

  where m is a **Lorentz scalar** that we will call the **rest mass**
- Spatial components of p^μ reduce to our usual definition of momentum
- To interpret the temporal component, we will look at its NR limit:

$$p^0 = m\gamma c = mc\left\{1 - v^2/c^2\right\}^{-1/2} = \frac{1}{c}\left\{mc^2 + \frac{1}{2}mv^2 + \mathcal{O}(v^4/c^2)\right\}$$

- The second term in braces is clearly the kinetic energy
- The first term we identify as the **rest energy**, and write

$$p^0 = E/c$$

  where E is the **energy**
- Thus the four momentum contains both the energy and the three momentum
- The “length” of p^μ is a Lorentz scalar

$$\begin{aligned} -p^\mu p_\mu &= m^2\gamma^2c^2 - m^2\gamma^2v^2 = m^2\gamma^2c^2\left[1 - v^2/c^2\right] \\ &= m^2\gamma^2c^2\gamma^{-2} = m^2c^2 \end{aligned}$$

- Thus we have $p^\mu p_\mu = p^2 = -m^2c^2$ confirming that the rest mass is a (frame-independent) scalar

$E = mc^2$

- Finally, if we now go back and write $-p^\mu p_\mu = m^2c^2$ in terms of our old-fashioned three vectors we have

$$\frac{1}{c^2}E^2 - \mathbf{p}^2 = m^2c^2$$
$$\implies E^2 = m^2c^4 + c^2\mathbf{p}^2.$$

- For a particle at rest, we have perhaps the most famous equation in physics

$$E = mc^2$$

- The use of four-vectors is **essential** to solve problems in special (and general...) relativity
- Whilst simple kinematical problems can be solved using three vectors, it is very clumsy indeed

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- Let us demonstrate that $g_{\mu\sigma}A^{\sigma}$ is indeed a covector, i.e. transforms according to $B'_{\mu} = L_{\mu}^{\ \nu} B_{\nu}$
- We need to show that

$$\left(g_{\mu\sigma}A^{\sigma}\right)' = L_{\mu}^{\ \nu}\left(g_{\nu\lambda}A^{\lambda}\right)$$

- The l.h.s. is $g_{\mu\sigma}(A^{\sigma})' = g_{\mu\sigma}L^{\sigma}_{\ \lambda}A^{\lambda}$ and therefore we must prove that

$$g_{\mu\sigma}L^{\sigma}_{\ \lambda} = L_{\mu}^{\ \nu}g_{\nu\lambda}$$

- Multiplying this relation by $L_{\rho}^{\ \lambda}$ and using $L^{\sigma}_{\ \lambda}L_{\rho}^{\ \lambda} = \delta^{\sigma}_{\rho}$ converts it into

$$g_{\mu\rho} = g_{\nu\lambda}L_{\mu}^{\ \nu}L_{\rho}^{\ \lambda}$$

- The last equation follows from the definition $L_{\mu}^{\ \nu} \equiv \partial x'^{\nu}/\partial x^{\mu}$ and

$$\frac{\partial^2}{\partial x^{\mu}\partial x^{\rho}} g_{\nu\lambda}x^{\nu}x^{\lambda} = \frac{\partial^2}{\partial x^{\mu}\partial x^{\rho}} g_{\nu\lambda}x'^{\nu}x'^{\lambda}$$

- Indeed, we have, first, $g_{\nu\lambda}x^{\nu}x^{\lambda} = x^2$, $g_{\nu\lambda}x'^{\nu}x'^{\lambda} = x'^2$, and $x'^2 = x^2$

Exercise, cont.

$$g_{\nu\lambda}x^\nu x^\lambda = x^2 \quad , \quad g_{\nu\lambda}x'^\nu x'^\lambda = x'^2 \quad , \quad \text{and} \quad x'^2 = x^2$$

- Then

$$\frac{\partial^2}{\partial x^\mu \partial x^\rho} g_{\nu\lambda}x^\nu x^\lambda = g_{\nu\lambda}[\delta^\nu_\mu \delta^\lambda_\rho + \delta^\nu_\rho \delta^\lambda_\mu] = 2g_{\mu\rho} \ ,$$

$$\frac{\partial^2}{\partial x^\mu \partial x^\rho} g_{\nu\lambda}x'^\nu x'^\lambda = g_{\nu\lambda}\left[\underbrace{\frac{\partial x'^\nu}{\partial x^\mu}}_{L_\mu^{\ \nu}} \underbrace{\frac{\partial x'^\lambda}{\partial x^\rho}}_{L_\rho^{\ \lambda}} + \{\mu \leftrightarrow \rho\}\right]$$

$$= g_{\nu\lambda}[L_\mu^{\ \nu} L_\rho^{\ \lambda} + L_\rho^{\ \nu} L_\mu^{\ \lambda}] = 2g_{\nu\lambda}L_\mu^{\ \nu} L_\rho^{\ \lambda}$$

- On the last step we used the fact that $g_{\nu\lambda}$ is a symmetric tensor
- The metric tensor with upper indices $g^{\mu\nu}$ defines the inner product

$$x \cdot y = g^{\mu\nu}x_\mu y_\nu$$

  in terms of covariant vectors. This product is invariant under Lorentz transformations if $g^{\mu\nu}y_\nu$ transforms as a contravariant vector y^μ
- Using $y_\nu = g_{\nu\sigma}y^\sigma$, we conclude that

$$g^{\mu\nu}g_{\nu\sigma} = \delta^\mu_\sigma \ ,$$

  i.e., the matrices $g^{\mu\nu}$ and $g_{\mu\nu}$ are inverse to each other
- In our case $g^{\mu\nu} = g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$