PHYSICS 453

Electromagnetism II
Lecture 20

Physics Department
Old Dominion University

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Outline

Newton's 2nd Law

- Newton's 2nd Law is correct provided we use relativistic momentum

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \qquad \mathbf{p} \equiv \textit{relativistic} \text{ momentum}$$

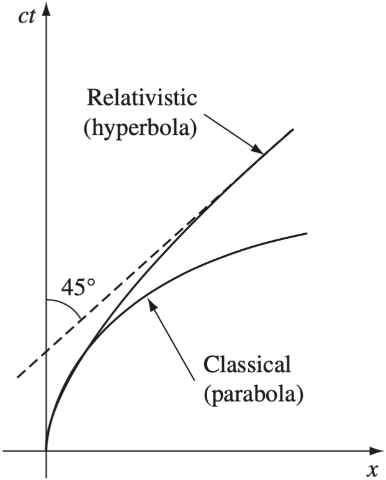
- Example 12.10: motion under constant force $\mathbf{F} = F\hat{e}_1$ starting from the origin

$$\frac{dp}{dt} = F \quad \Rightarrow \quad p = Ft + \text{const}$$

$$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} = Ft \quad \Rightarrow \quad v = \frac{tF/m}{\sqrt{1 + \left(\frac{Ft}{mc}\right)^2}}$$

$$\Rightarrow \quad x(t) = \frac{F}{m}\int_0^t dt' \frac{t'}{\sqrt{1 + \left(\frac{Ft}{mc}\right)^2}} = \frac{mc^2}{F}\left[\sqrt{1 + \left(\frac{Ft}{mc}\right)^2} - 1\right]$$

Newton's 2nd Law, cont.



Work-Energy Theorem

- Check of work-energy theorem

$$W = \int \mathbf{F} \cdot d\mathbf{l} = \int \frac{d\mathbf{p}}{dt} \cdot d\mathbf{l} = \int \frac{dp}{dt} \cdot \mathbf{v} dt$$

$$\frac{d\mathbf{p}}{dt} \cdot \mathbf{v} = \frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) \cdot \mathbf{v} = \frac{m\mathbf{v}}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} \cdot \frac{d\mathbf{v}}{dt}$$
$$= \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{dE}{dt}$$

$$\Rightarrow W = \int dt \frac{dE}{dt} = E_{\text{final}} - E_{\text{initial}}$$

- Newton's third law is correct only in contact interactions. In general, use momentum conservation (correct in any frame)

Lorentz transformation of a force

$$F'_y = \frac{dp'_y}{dt} = \frac{dp_y}{\gamma dt - \frac{\gamma\beta}{c}dx} = \frac{\frac{dp_y}{dt}}{\gamma\left(1 - \frac{\beta}{c}\frac{dx}{dt}\right)} = \frac{F_y}{\gamma\left(1 - \frac{\beta v_x}{c}\right)}$$

Similarly

$$F'_z = \frac{dp'_y}{dt} = \frac{F_z}{\gamma\left(1 - \frac{\beta v_x}{c}\right)}$$

and

$$F'_x = \frac{dp_x}{dt} = \frac{\gamma dp_x - \gamma\beta dp_0}{\gamma dt - \frac{\gamma\beta}{c}dx} = \frac{\frac{dp_x}{dt} - \beta\frac{dp_0}{dt}}{1 - \frac{\beta}{c}\frac{dx}{dt}} = \frac{F_x - \frac{\beta}{c}\frac{dE}{dt}}{1 - \frac{\beta v_x}{c}}$$

Since $\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v}$

$$F'_x = \frac{F_x - \frac{\beta}{c}\mathbf{F}\cdot\mathbf{v}}{1 - \frac{\beta v_x}{c}}$$

If the particle is instantaneously at rest

$$\mathbf{F}'_\perp = \frac{1}{\gamma}\mathbf{F}_\perp, \qquad \mathbf{F}'_\parallel = \mathbf{F}_\parallel$$

Minkowski force

- Minkowski force

$$K^\mu \equiv \frac{dp^\mu}{d\tau}$$

$$\mathbf{K} = \frac{d\mathbf{p}}{d\tau} = \frac{d\mathbf{p}}{dt}\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\mathbf{F}, \quad K^0 = \frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau}$$

- Q: Lorentz force

$$\mathbf{F} = q(E + \mathbf{v}\times\mathbf{B}) \quad \text{or} \quad \mathbf{K} = q(E + \mathbf{v}\times\mathbf{B})\ ?$$

- A: $\mathbf{F} = q(E + \mathbf{v}\times\mathbf{B})$ (to be proved later)

Total momentum and Center of Energy

For a set of particles

- In non-relativistic classical mechanics

$$\mathbf{P} = M\frac{d\mathbf{R}_m}{dt}, \qquad M = \sum m_i, \quad R_m = \frac{\sum m_i \mathbf{r}_i}{M}$$

- In relativistic mechanics

$$\mathbf{P} = \frac{E}{c^2}\frac{d\mathbf{R}_e}{dt}, \qquad E = \sum E_i, \quad \mathbf{R}_e = \frac{1}{E}\sum \mathbf{r}_i E_i$$

- Center of mass $\mathbf{R}_m = \frac{\sum m_i \mathbf{r}_i}{\sum m_i}$ is replaced by *center of energy* $\mathbf{R}_e = \frac{\sum \mathbf{r}_i E_i}{\sum E_i}$