PHYSICS 453

Electromagnetism II
Lecture 21

Physics Department
Old Dominion University

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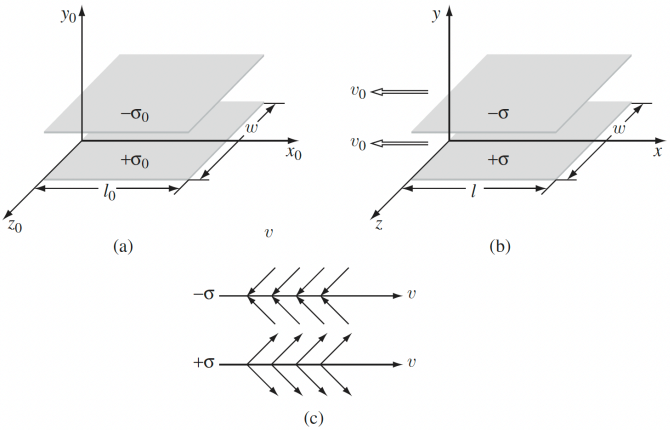
Outline

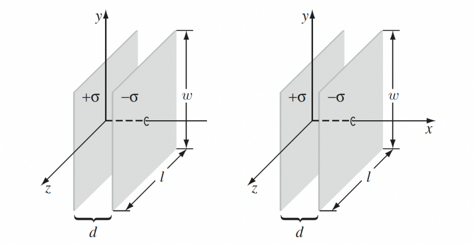
1. Lorentz Transformation of Fields
   - Lorentz Transformation of the electric field
   - Lorentz Transformation of $\mathbf{E}$ and $\mathbf{B}$
   - Lorentz transformation of fields

Lorentz transformation of the electric field



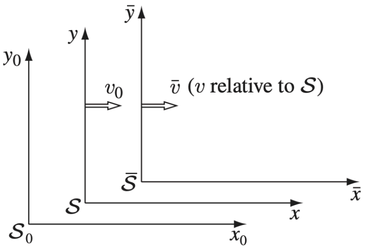
$$E^{(S_0)} = \frac{\sigma_0}{\epsilon_0}, \quad l = \frac{l_0}{\gamma_0} \;\Rightarrow\; \sigma^{S)} = \gamma_0 \sigma_0^{(S_0)} \;\Rightarrow\; \mathbf{E}_\perp^{(S)} = \gamma_0 \mathbf{E}_\perp^{(S_0)}, \quad \gamma_0 = \left(1 - \frac{v_0^2}{c^2}\right)^{-\frac{1}{2}}$$

Lorentz transformation of the electric field, cont.



$$E^{(S_0)} = \frac{\sigma_0}{\epsilon_0}, \quad E \text{ does not depend on } d \quad \Rightarrow \quad E_{\parallel}^{(S)} = E_{\parallel}^{(S_0)}$$

Lorentz transformation of $\mathbf{E}$ and $\mathbf{B}$



- Start from frame S:

$$E_y^{(S)} = \frac{\sigma^{(S)}}{\epsilon_0}, \quad \mathbf{K}_\pm^{(S)} = \mp\sigma^{(S)} v_0 \hat{e}_x \quad \Rightarrow \quad B_z^{(S)} = -\mu_0 \sigma^{(S)} v_0$$

- In the frame $\bar{S}$ moving with speed v relative to S

$$E_y^{(\bar{S})} = \frac{\sigma^{(\bar{S})}}{\epsilon_0}, \quad B_z^{(\bar{S})} = -\mu_0 \sigma^{(\bar{S})} \bar{v}$$

where $\bar{v} = \frac{v+v_0}{1+\frac{vv_0}{c^2}}$ = velocity of $\bar{S}$ with respect to S_0 and

$$\bar{\sigma}^{\bar{S}} = \bar{\gamma}\sigma_0, \qquad \bar{\gamma} = 1/\sqrt{1-\bar{v}^2/c^2}$$

Lorentz transformation of $\mathbf{E}$ and $\mathbf{B}$, cont.

- Transformation of $\mathbf{E}_y$ and $\mathbf{B}_z$

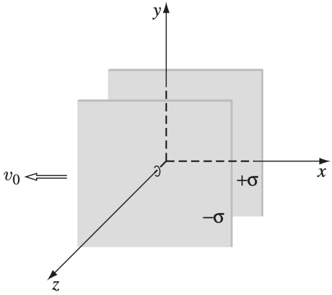
$$\bar{E}_y^{(\bar{S})} = \frac{\bar{\sigma}^{(\bar{S})}}{\epsilon_0} = \frac{\bar{\gamma}\sigma_0^{(S_0)}}{\epsilon_0} = \frac{\bar{\gamma}}{\gamma_0}\frac{\sigma^{(S)}}{\epsilon_0}, \quad B_z^{(\bar{S})} = -\mu_0\sigma^{(\bar{S})}\bar{v} = -\frac{\bar{\gamma}}{\gamma_0}\mu_0\sigma^{(S)}\bar{v}$$

$$\frac{\bar{\gamma}}{\gamma_0} = \frac{\sqrt{1-v_0^2/c^2}}{\sqrt{1-\bar{v}^2/c^2}} = \gamma\left(1+\frac{vv_0}{c^2}\right), \qquad \gamma = 1/\sqrt{1-v^2/c^2}$$

$$\bar{E}_y^{(\bar{S})} = \gamma\left(1+\frac{vv_0}{c^2}\right)\frac{\sigma^{(S)}}{\epsilon_0} = \gamma\left(E_y^{(S)} - vB_z^{(S)}\right),$$

$$\bar{B}_z^{(\bar{S})} = -\gamma\left(1+\frac{vv_0}{c^2}\right)\mu_0\sigma^{(S)}\left(\frac{v+v_0}{1+vv_0/c^2}\right) = \gamma\left(B_z^{(S)} - vE_y^{(S)}\right)$$

- Transformation of $\mathbf{E}_z$ and $\mathbf{B}_y$



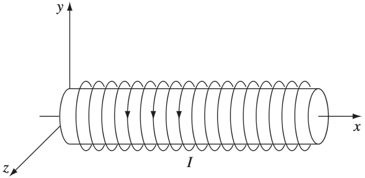
$$E_z^{(S)} = \frac{\sigma^{(S)}}{\epsilon_0}, \quad B_y^{(S)} = \mu_0 \sigma^{(S)} v_0$$

Same derivation with $E_y \to E_z$ and $B_z \to -B_y$

$$\Rightarrow \quad \bar{E}_z^{(\bar{S})} = \gamma(E_z^{(S)} + vB_y^{(S)}), \quad \bar{B}_y^{(\bar{S})} = \gamma\left(B_y^{(S)} + \frac{v}{c^2} B_y^{(S)}\right)$$

Lorentz Transformation of $\mathbf{E}$ and $\mathbf{B}$, cont.

- Transformation of $\mathbf{E}_x$ and $\mathbf{B}_x$
  First, we know that $E_x^{(\bar{S})} = E_x^{(S)}$
  Too get transformation of B_x, consider a moving solenoid



- In S frame: $B_x^{(S)} = \mu_0 n^{(S)} I^{(S)}$
- In $\bar{S}$ frame:

$$\bar{n}^{(\bar{S})} = \gamma n^{(S)}, I = \frac{dQ}{dt} \Rightarrow \bar{I}^{(\bar{S})} = \frac{1}{\gamma} I^{(S)} \Rightarrow \bar{B}_x^{(\bar{S})} = B_x^{(S)}$$

Lorentz transformation of fields

- General formulas

$$E'_x = E_x \qquad E'_y = \gamma(E_y - vB_z) \qquad E'_z = \gamma(E_z + vB_y)$$

$$B'_x = B_x \qquad B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \qquad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

- If $\mathbf{B} = 0$ in S

$$\mathbf{B}' = \frac{\gamma v}{c^2}(E_z\hat{e}_y - E_y\hat{e}_z) = \frac{v}{c^2}(E'_z\hat{e}_y - E'_y\hat{e}_z) \;\Leftrightarrow\; \mathbf{B}' = -\frac{1}{c^2}\mathbf{v}\times\mathbf{E}$$

- If $\mathbf{E} = 0$ in S

$$\mathbf{E}' = -\gamma v(B_z\hat{e}_y - B_y\hat{e}_z) = -v(B'_z\hat{e}_y - B'_y\hat{e}_z) \;\Leftrightarrow\; \mathbf{E}' = \mathbf{v}\times\mathbf{E}$$