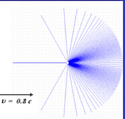
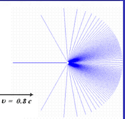


PHYSICS 704/804

Electromagnetism II
Lecture 23

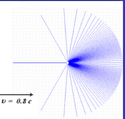
Physics Department
Old Dominion University

April 24, 2025



Outline

1. Covariant Formulation of Maxwell's Equations
   - Jacobi identity
   - Transformation Properties of EM Field
   - Electric and magnetic fields of relativistically moving point charge



Jacobi identity

Jacobi identity: $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$

Proof:

- Math formula

$$\epsilon^{\mu\nu\lambda\rho}\epsilon_{\alpha\beta\gamma\rho} = -\det\begin{vmatrix} \delta^\mu_\alpha & \delta^\mu_\beta & \delta^\mu_\gamma \\ \delta^\nu_\alpha & \delta^\nu_\beta & \delta^\nu_\gamma \\ \delta^\lambda_\alpha & \delta^\lambda_\beta & \delta^\lambda_\gamma \end{vmatrix}$$

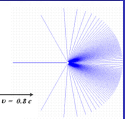
- Start from $\partial_\nu \tilde{F}^{\rho\nu} = 0$

$$\partial_\nu \tilde{F}^{\rho\nu} = 0 \;\Leftrightarrow\; \epsilon^{\rho\nu\lambda\mu}\partial_\nu F_{\lambda\mu} = 0 \Leftrightarrow\; \epsilon^{\mu\nu\lambda\rho}\partial_\nu F_{\lambda\mu} = 0$$

- Multiply by $-\epsilon_{\alpha\beta\gamma\rho}$

$$0 = -\epsilon_{\alpha\beta\gamma\rho}\epsilon^{\mu\nu\lambda\rho}\partial_\nu F_{\lambda\mu} = \det\begin{vmatrix} \delta^\mu_\alpha & \delta^\mu_\beta & \delta^\mu_\gamma \\ \delta^\nu_\alpha & \delta^\nu_\beta & \delta^\nu_\gamma \\ \delta^\lambda_\alpha & \delta^\lambda_\beta & \delta^\lambda_\gamma \end{vmatrix}\partial_\nu F_{\lambda\mu}$$

$$= 2(\partial_\beta F_{\gamma\alpha} + \partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta})$$



Lorentz Invariants

- There are two invariants we can construct from the field-strength tensor

$$F_{\mu\nu}F^{\mu\nu} = F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij} = 2(\mathbf{B}^2 - \mathbf{E}^2)$$

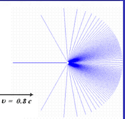
- On the last step, we used explicit forms of $F^{\mu\nu}$ and $F_{\mu\nu}$:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix},$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Thus $\mathbf{B}^2 - \mathbf{E}^2$ is a **Lorentz Scalar**

$$\mathbf{B}^2 - \mathbf{E}^2 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$$



Lorentz Invariants

- The result $F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\mathbf{E}\cdot\mathbf{B}$ can be checked by using explicit form of $\tilde{F}^{\mu\nu}$:

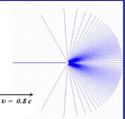
$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix},$$

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

- Thus $\mathbf{E}\cdot\mathbf{B}$ is also a **Lorentz Scalar** (more precisely, a *pseudoscalar*)

$$\mathbf{E}\cdot\mathbf{B} = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}$$

- These are the *only* Lorentz invariants built from electromagnetic fields



Transformation Properties of EM Field

- Since $F^{\mu\nu}$ is a second-rank tensor, it transforms according to

$$F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} F^{\alpha\beta} \frac{\partial x'^{\nu}}{\partial x^{\beta}},$$

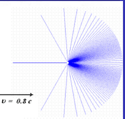
- This we can write as

$$F' = \Lambda F \Lambda^T, \quad \text{where } \Lambda^{\mu}_{\ \nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}$$

- Specifically, let us consider a boost from K to K' where K' has velocity v in x-direction w.r.t. K, and origins coincide at $t = t' = 0$. Then

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$



Transformation Properties of EM Field, cont.

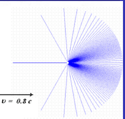
$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Using this expression in $F' = \Lambda F \Lambda^T$, we find

$$F' = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & B_3 & -B_2 \\ -E_2/c & -B_3 & 0 & B_1 \\ -E_3/c & B_2 & -B_1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Multiplying last two factors

$$F' = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\gamma\beta E_1/c & \gamma E_1/c & E_2/c & E_3/c \\ -\gamma E_1/c & \gamma\beta E_1/c & B_3 & -B_2 \\ -\gamma\left(\frac{E_2}{c} - \beta B_3\right) & \gamma\left(\beta\frac{E_2}{c} - B_3\right) & 0 & B_1 \\ -\gamma\left(\frac{E_3}{c} + \beta B_2\right) & \gamma\left(B_2 + \beta\frac{E_3}{c}\right) & -B_1 & 0 \end{pmatrix}$$



Transformation Properties of EM Field, cont.

$$F' = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\gamma\beta E_1/c & \gamma E_1/c & E_2/c & E_3/c \\ -\gamma E_1/c & \gamma\beta E_1/c & B_3 & -B_2 \\ -\gamma\left(\frac{E_2}{c} - \beta B_3\right) & \gamma\left(\beta\frac{E_2}{c} - B_3\right) & 0 & B_1 \\ -\gamma\left(\frac{E_3}{c} + \beta B_2\right) & \gamma\left(B_2 + \beta\frac{E_3}{c}\right) & -B_1 & 0 \end{pmatrix}$$

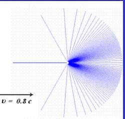
- Finally

$$F' = \begin{pmatrix} 0 & E_1 & \gamma\left(\frac{E_2}{c} - \beta B_3\right) & \gamma\left(\frac{E_3}{c} + \beta B_2\right) \\ -E_1 & 0 & \gamma\left(B_3 - \beta\frac{E_2}{c}\right) & -\gamma\left(B_2 + \beta\frac{E_3}{c}\right) \\ -\gamma\left(\frac{E_2}{c} - \beta B_3\right) & \gamma\left(B_3 - \beta\frac{E_2}{c}\right) & 0 & B_1 \\ -\gamma\left(\frac{E_3}{c} + \beta B_2\right) & \gamma\left(B_2 + \beta\frac{E_3}{c}\right) & -B_1 & 0 \end{pmatrix}$$

- Writing out the individual vector components, we find

$$\left.\begin{aligned} E_1' &= E_1; & B_1' &= B_1 \\ E_2' &= \gamma(E_2 - \beta c B_3); & B_2' &= \gamma(B_2 + \tfrac{\beta}{c}E_3) \\ E_3' &= \gamma(E_3 + \beta c B_2); & bB_3' &= \gamma(B_3 - \tfrac{\beta}{c}E_2) \end{aligned}\right\}$$

- Thus the $\mathbf{E}$ and $\mathbf{B}$ fields **mix** under a Lorentz transformation



Transformation Properties of EM Field, cont.

$$\left.\begin{array}{rr} E_1' = E_1; & B_1' = B_1 \\ E_2' = \gamma(E_2 - \beta c B_3); & B_2' = \gamma(B_2 + \frac{\beta}{c} E_3) \\ E_3' = \gamma(E_3 + \beta c B_2); & bB_3' = \gamma(B_3 - \frac{\beta}{c} E_2) \end{array}\right\}$$

- We can express this in (three) vector form as

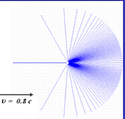
$$\mathbf{E}' = \gamma[\mathbf{E} + c\boldsymbol{\beta} \times \mathbf{B}] - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E})$$

$$\mathbf{B}' = \gamma\left(\mathbf{B} - \frac{1}{c}\boldsymbol{\beta} \times \mathbf{E}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B})$$

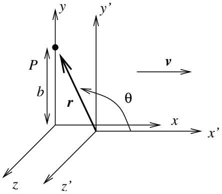
- $\boldsymbol{\beta} = \mathbf{v}/c$
- Check: take the component of $\mathbf{E}'$ parallel to $\mathbf{v}$. This gives

$$\mathbf{E}' \cdot \mathbf{v} \equiv E_v' = \gamma E_v - \frac{\gamma^2\beta^2}{\gamma+1} E_v = \frac{\gamma^2 + \gamma - \gamma^2\beta^2}{\gamma+1} E_v = \frac{1+\gamma}{\gamma+1} E_v = E_v$$

since $\gamma^2 - \gamma^2\beta^2 = 1$



Electric and magnetic fields of relativistically moving point charge



- Charge q moves along a line at velocity $\mathbf{v} = v\mathbf{e_1}$ in K
- The charge is at rest in the frame K'
- At $t = t' = 0$, the origins of the two frames coincide
- We have an observer P at impact parameter b (i.e. distance of closest approach) as shown
- Write electric and magnetic fields at point P *in frame* K' at time t'. P has coordinates

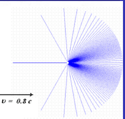
$$x' = -vt' \quad , \quad y' = b \quad , \quad z' = 0$$

- Thus, from Coulomb's law

$$\begin{array}{rlcrlcrl} 4\pi\epsilon_0 E_1' & = -qvt'/r'^3 & ; & 4\pi\epsilon_0 E_2' & = qb/r'^3 & ; & E_3' & = 0 \\ B_1' & = 0 & ; & B_2' & = 0 & ; & B_3' & = 0. \end{array}$$

- To express this in terms of coordinates in K, we note that $r'^2 = b^2 + v^2 t'^2$
- But we also have $ct' = \gamma(ct - \beta x) = \gamma ct$
- Thus $r'^2 = b^2 + v^2\gamma^2 t^2$ and we have

$$4\pi\epsilon_0 E_1' = -\frac{q\gamma vt}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \quad , \quad 4\pi\epsilon_0 E_2' = \frac{qb}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \quad , \quad E_3' = 0$$



Electric and magnetic fields of relativistically moving point charge, cont.

$$E_1' = -\frac{q\gamma v t}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \ , \ E_2' = \frac{qb}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \ , \ E_3' = 0$$

- We now use transformation laws

$$\left.\begin{array}{rl} E_1' = E_1; & B_1' = B_1 \\ E_2' = \gamma(E_2 - \beta c B_3); & B_2' = \gamma(B_2 + \frac{\beta}{c}E_3) \\ E_3' = \gamma(E_3 + \beta c B_2); & bB_3' = \gamma(B_3 - \frac{\beta}{c}E_2) \end{array}\right\}$$

changing there $\beta \to -\beta$:

$$\left.\begin{array}{rl} E_1 = E_1'; & B_1 = B_1' \\ E_2 = \gamma(E_2' + \beta c B_3'); & B_2 = \gamma(B_2' - \frac{\beta}{c}E_3') \\ E_3 = \gamma(E_3' - \beta c B_2'); & B_3 = \gamma(B_3' + \frac{\beta}{c}E_2') \end{array}\right\}$$

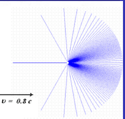
to write

$$E_1 = E_1' = -\frac{q\gamma v t}{(b^2 + v^2\gamma^2 t^2)^{3/2}}$$

$$E_2 = \gamma E_2' = \frac{\gamma q b}{(b^2 + v^2\gamma^2 t^2)^{3/2}}$$

$$E_3 = \gamma E_3' = 0 \ , \ B_1 = 0 \ , \ B_2 = \gamma B_2' = 0 \ , \ B_3 = \gamma \frac{\beta}{c} E_2' = \frac{\beta}{c} E_2$$

- Thus in the laboratory frame we see a magnetic induction



Electric and magnetic fields of relativistically moving point charge, cont.

$$E_1 = E_1' = -\frac{q\gamma vt}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \ , \ E_2 = \gamma E_2' = \frac{\gamma qb}{(b^2 + v^2\gamma^2 t^2)^{3/2}}$$

$$E_3 = \gamma E_3' = 0 \ , \ B_1 = 0 \ , \ B_2 = \gamma B_2' = 0 \ , \ B_3 = \gamma \frac{\beta}{c} E_2' = \frac{\beta}{c} E_2$$

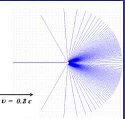
- Note that in the limit $v \to c$, we have $\beta \to 1$ and the magnetic induction equals the transverse electric field
- In the nonrelativistic limit $v \to 0$,

$$B_3 = \frac{v}{c} \frac{\gamma qb}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \longrightarrow \frac{vqb}{c(b^2 + v^2 t^2)^{3/2}} \implies \mathbf{B} \sim \frac{q}{c} \frac{\mathbf{v} \times \mathbf{r}}{r^3}$$

- We have used $vb = vr \sin\theta$.
- The result is just the *Biot-Savart Law*
- Finally, let us look at the field lines. We have

$$\frac{E_2}{E_1} = -\frac{b}{vt}$$

- The electric field is still a central field in the frame K



Electric and magnetic fields of relativistically moving point charge, cont.

$$E_1 = E_1' = -\frac{q\gamma v t}{(b^2 + v^2\gamma^2 t^2)^{3/2}} \;,\; E_2 = \gamma E_2' = \frac{\gamma q b}{(b^2 + v^2\gamma^2 t^2)^{3/2}}$$

$$E_3 = \gamma E_3' = 0 \;,\; B_1 = 0 \;,\; B_2 = \gamma B_2' = 0 \;,\; B_3 = \gamma \frac{\beta}{c} E_2' = \frac{\beta}{c} E_2$$

- If we now look at the *magnitude* of the field, however, we find

$$|\mathbf{E}| = \frac{\gamma q}{(b^2 + v^2\gamma^2 t^2)^{3/2}} (b^2 + v^2 t^2)^{1/2}$$

- Setting $b = r\sin\theta, vt = r\cos\theta$, we have

$$|\mathbf{E}| = \frac{\gamma q r}{r^3(\sin^2\theta + \gamma^2\cos^2\theta)^{3/2}} = \frac{q}{r^2\gamma^2(\sin^2\theta/\gamma^2 + \cos^2\theta)^{3/2}}$$

- This gives

$$|\mathbf{E}| = \frac{q}{\gamma^2 r^2}(1 - \beta^2\sin^2\theta)^{-3/2}$$

- The lines of force, whilst central, are no longer *isotropic*