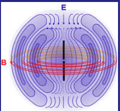

PHYSICS 453

Electromagnetism II
Lecture 10_1

Physics Department
Old Dominion University

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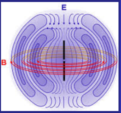
Outline

Set of Maxwell's Equations

- Modification of Ampere's law plus Faraday's law, result in set of Maxwell's equations

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho \quad \text{(ME1) } \textit{Coulomb's Law} \\
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \quad \text{(ME2) } \textit{Faraday's Law} \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{(ME3) } \textit{Ampere's Law + Maxwell} \\
\nabla \cdot \mathbf{B} &= 0 \quad \text{(ME4)}
\end{aligned}$$

- Unification of electrical and magnetic phenomena through these equations represents the crowning achievement of classical, 19^{th} century physics
- Addition of electric displacement to r.h.s. of Ampere's law was essential to showing that the solutions admit wave propagation at the speed of light



Vector and Scalar Potentials

- Maxwell's equations comprise a set of coupled, first-order PDE's
- In particularly simple cases, they can be solved directly
- In electrostatics and magnetostatics we have seen the efficacy of using vector and scalar potentials
- Introduce potentials so that the two homogenous equations (Faraday's law and the solenoidal condition) are satisfied automatically
- Since $\nabla \cdot \mathbf{B} = 0$, we can introduce a **vector potential** $\mathbf{A}$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

- Substituting into Faraday's law $\nabla \times \mathbf{E} + \partial \mathbf{B}/\partial t = 0$ gives

$$\nabla \times \mathbf{E} + \frac{\partial}{\partial t}\left[\nabla \times \mathbf{A}\right] = 0 \Rightarrow \nabla \times \left[\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right] = 0$$

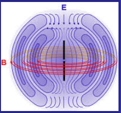
- Introduce a **scalar potential** Φ such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

- Electric and magnetic fields can be then written as

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$



Two remaining equations

- ME1 and ME3 determine dynamical dependence of $\mathbf{A}$ and Φ on t and $\mathbf{x}$
- We need constitutive relation between $(\mathbf{D}, \mathbf{H})$ and $(\mathbf{E}, \mathbf{B})$
- Consider first the case of the vacuum, where we have

$$\begin{aligned}\mathbf{D} &= \epsilon_0 \mathbf{E} \\ \mathbf{H} &= \frac{1}{\mu_0} \mathbf{B}\end{aligned}$$

- Coulomb's law, ME1, is thus

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

- Ampère's law, ME3, is

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

- In terms of potentials $(\Phi, \mathbf{A})$, ME1 becomes

$$\nabla^2 \Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0}$$

- Substituting for potentials in ME3, we have

$$\frac{1}{\mu_0} \nabla \times (\nabla \times \mathbf{A}) = \mathbf{J} + \epsilon_0 \left\{ -\nabla \frac{\partial \Phi}{\partial t} - \frac{\partial^2 \mathbf{A}}{\partial t^2} \right\}$$

Two remaining equations, cont.

$$\begin{aligned}\frac{1}{\mu_0}\nabla\times(\nabla\times\mathbf{A}) &= \mathbf{J}+\epsilon_0\left\{-\nabla\frac{\partial\Phi}{\partial t}-\frac{\partial^2\mathbf{A}}{\partial t^2}\right\}\\ \Longrightarrow \nabla[\nabla\cdot\mathbf{A}]-\nabla^2\mathbf{A} &= \mu_0\mathbf{J}+\mu_0\epsilon_0\left\{-\nabla\frac{\partial\Phi}{\partial t}-\frac{\partial^2\mathbf{A}}{\partial t^2}\right\}.\end{aligned}$$

- We now write $\epsilon_0\mu_0 = 1/c^2$ (we of course all know what c will be!) to get

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\nabla\left[\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right] = -\mu_0\mathbf{J}$$

- Combining with the first equation

$$\nabla^2\Phi+\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -\frac{\rho}{\epsilon_0}$$

- We have two coupled second-order PDE's
- With definitions of potentials

$$\begin{aligned}\mathbf{B} &= \nabla\times\mathbf{A}\\ \mathbf{E} &= -\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}\end{aligned}$$

they are equivalent to the original four Maxwell equations

Gauge Transformations Revisited

- To decouple these two equations through a clever choice of *gauge transformation*
- Recall that the physical fields are not $(\mathbf{A}, \Phi)$, but rather $(\mathbf{B}, \mathbf{E})$
- Gauge transformation of $(\mathbf{A}, \Phi)$ leaves the physics unaltered
- Gauge transformations in magnetostatics; the substitution

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \nabla\Lambda$$

  leaves $\mathbf{B} = \nabla \times \mathbf{A}$ invariant
- Now $\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t$ and $\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \nabla\Lambda$ will change $\mathbf{E}$
- Unless we make a suitable change $\Phi \longrightarrow \Phi'$
- In terms of the transformed potentials $(\mathbf{A}', \Phi')$

$$\mathbf{E} = -\nabla\Phi' - \frac{\partial\mathbf{A}'}{\partial t} = -\nabla\Phi' - \frac{\partial}{\partial t}\left[\mathbf{A} + \nabla\Lambda\right]$$

- But we also have

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t},$$

- Equating two expressions gives

$$\nabla\left[\Phi' + \frac{\partial\Lambda}{\partial t} - \Phi\right] = 0 \Longrightarrow \Phi' = \Phi - \frac{\partial\Lambda}{\partial t}$$

- Note: the potential is only defined up to an additive constant

Coulomb Gauge

- Gauge transformation of Maxwell's equations takes the form

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \nabla\Lambda$$

$$\Phi \longrightarrow \Phi' = \Phi - \frac{\partial\Lambda}{\partial t}$$

- We will now discuss some particular choice of gauges
- Original equations for scalar and vector potentials

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left[\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right] = -\mu_0\mathbf{J}$$

- In magnetostatics, we have introduced the *Coulomb gauge*

$$\nabla\cdot\mathbf{A} = 0$$

- It is not manifestly Lorentz covariant, but has the property that the scalar potential satisfies Poisson's equation (Coulomb's law!)

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$$

Coulomb Gauge, cont.

- The solution of

$$\nabla^2 \Phi(\mathbf{x}, t) = -\frac{\rho(\mathbf{x}, t)}{\epsilon_0}$$

is of course

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|}$$

- The vector potential satisfies the **inhomogeneous wave equation**

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \frac{1}{c^2}\nabla\frac{\partial \Phi}{\partial t}$$

- Note that the scalar potential $\Phi(\mathbf{x}, t)$ is the **instantaneous** Coulomb potential due to a charge density $\rho(\mathbf{x}, t)$ at the same time
- Equation for the vector potential contains a gradient operator, $\nabla \partial\Phi/\partial t$, with Φ taken from the solution of Poisson's equation for the scalar potential

Lorentz Gauge

- Suppose we can find a gauge transformation such that

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$$

- This is known as the **Lorentz condition**
- The dynamical equations then assume the form

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}$$

- $\mathbf{A}$ and Φ fields have become decoupled
- Simplified equations are just the **wave equations**, with an inhomogeneous source

Lorentz Gauge, cont.

- We need to show that it is possible to *find* a gauge transformation that transforms original potentials Φ, $\mathbf{A}$ into potentials Φ', $\mathbf{A}'$ satisfying

$$\nabla \cdot \mathbf{A}' + \frac{1}{c^2}\frac{\partial \Phi'}{\partial t} = 0$$

- Using $\mathbf{A}' = \mathbf{A} + \nabla\Lambda$ and $\Phi' = \Phi - \partial\Lambda/\partial t$, we have

$$\nabla \cdot \mathbf{A} + \nabla^2\Lambda + \frac{1}{c^2}\left[\frac{\partial \Phi}{\partial t} - \frac{\partial^2\Lambda}{\partial t^2}\right] = 0$$

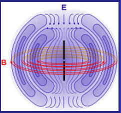
- We need to find Λ satisfying

$$\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = -\left[\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right]$$

- If such a function is found, then $\mathbf{A}'$ and Φ' satisfy the Lorentz condition
- Note that the solution of this differential equation is not unique
- One can always add to Λ a function $\delta\Lambda$ that satisfies the homogeneous equation

$$\nabla^2(\delta\Lambda) - \frac{1}{c^2}\frac{\partial^2(\delta\Lambda)}{\partial t^2} = 0\,,$$

- New function $\Lambda' = \Lambda + \delta\Lambda$ also satisfies the inhomogeneous equation



Lorentz Gauge, cont.

- Lorentz condition does not specify a gauge uniquely
- Indeed, let $(\mathbf{A}, \Phi)$ satisfy the Lorentz condition
- Now consider the transformation

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \nabla(\delta\Lambda) \quad , \quad \Phi \longrightarrow \Phi' = \Phi - \frac{\partial(\delta\Lambda)}{\partial t}$$

- Then the combination entering the Lorentz condition transforms as

$$\nabla \cdot \mathbf{A}' + \frac{1}{c^2}\frac{\partial \Phi'}{\partial t} = \nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} + \nabla^2(\delta\Lambda) - \frac{1}{c^2}\frac{\partial^2(\delta\Lambda)}{\partial t^2}$$

- By assumption, the fields $(\mathbf{A}, \Phi)$ satisfy the Lorentz condition $\nabla \cdot \mathbf{A} + (1/c^2)\,\partial\Phi/\partial t = 0$
- Thus, the fields $(\mathbf{A}', \Phi')$ will also satisfy the Lorentz condition if

$$\nabla^2(\delta\Lambda) - \frac{1}{c^2}\frac{\partial^2(\delta\Lambda)}{\partial t^2} = 0$$

- The Lorentz gauge is important because:
- The wave equation is manifest explicitly,
- $(\mathbf{A}, \Phi)$ are treated on equal footing and, when we discuss *Special Relativity*, we will see that the Lorentz condition is **Lorentz covariant**, i.e. independent of the choice of the 4-dimensional $(ct, \mathbf{x})$ coordinate system