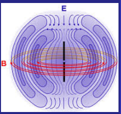

PHYSICS 453

Electromagnetism II
Lecture 10_2

Physics Department
Old Dominion University

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Outline

Retarded Solutions for the Fields

- In the Lorentz gauge scalar Φ and vector potential $\mathbf{A}$ satisfy wave equations

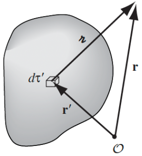
$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \qquad \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} \qquad (*)$$

- In electrostatics and magnetostatics

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\mathbf{r}')}{\varsigma}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{r}')}{\varsigma}$$

In electrodynamics the effect of change of ϕ and J feels after time $\frac{|r-r'|}{c} = \frac{\varsigma}{c}$ elapsed



Direct check of solution in terms of the retarded potentials

- We can make a guess

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\rho(\mathbf{r}',t_{\rm ret})}{\varsigma}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\mathbf{J}(\mathbf{r}',t_{\rm ret})}{\varsigma}$$

where $\varsigma \equiv |\mathbf{r}-\mathbf{r}'|$ and

$$t_{\rm ret} = t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'| = t - \frac{\varsigma}{c}$$

- The subscript $\texttt{ret}$ denotes that the functions $\rho, \mathbf{J}$ are evaluated at an earlier time
- The difference is equal to the time interval necessary for propagation over the distance $\varsigma = |\mathbf{r}-\mathbf{r}'|$ with the velocity of light c
- Mathematically, $t_{\rm ret}$ is a function of t, $\mathbf{r}$ and $\mathbf{r}'$

Direct check of solution in terms of the retarded potentials

- Check directly that expressions for Φ and $\mathbf{A}$ satisfy the wave equations $(*)$

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\mathbf{r}',t_{\rm ret})}{|\mathbf{r}-\mathbf{r}'|}, \quad \mathbf{A}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\mathbf{J}(\mathbf{r}',t_{\rm ret})}{|\mathbf{r}-\mathbf{r}'|}$$

- Integrand of the $\mathbf{r}'$-integral depends on $\mathbf{r}$ in two places:
  (a) *Explicitly*, in the denominator $|\mathbf{r}-\mathbf{r}'| \equiv \varsigma$
  (b) *Implicitly*, through $t_{\rm ret} = t - \varsigma/c$, in the numerator

$$\nabla\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \left[\rho\nabla\left(\frac{1}{\varsigma}\right) + (\nabla\rho)\,\frac{1}{\varsigma}\right]$$

- Using chain rule, we have (recall $t_{\rm ret} = t - \frac{\varsigma}{c}$ and $\nabla\varsigma = \hat{\mathbf{e}}_\varsigma$)

$$\nabla\rho(\mathbf{r}',t_{\rm ret}) = \dot{\rho}(\mathbf{r}',t_{\rm ret})\nabla t_{\rm ret} = -\frac{\dot{\rho}}{c}\,\nabla\varsigma$$

  where the dot denotes differentiation with respect to time
- Now, $\nabla\varsigma = \hat{\mathbf{e}}_\varsigma \equiv \boldsymbol{\varsigma}/\varsigma$ and $\nabla(1/\varsigma) = -\hat{\mathbf{e}}_\varsigma/\varsigma^2$. Hence,

$$\nabla\rho = -\frac{\dot{\rho}}{c}\,\nabla\varsigma = -\frac{\dot{\rho}}{c}\,\hat{\mathbf{e}}_\varsigma$$

$$\nabla\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \left[-\rho\frac{\hat{\mathbf{e}}_\varsigma}{\varsigma^2} - \frac{\dot{\rho}}{c}\,\frac{\hat{\mathbf{e}}_\varsigma}{\varsigma}\right]$$

Direct check of solution through retarded potentials, cont.

- Taking the divergence,

$$\nabla^2\Phi(\mathbf{r},t) = -\frac{1}{4\pi\epsilon_0}\int d^3x'\left[\nabla\cdot\rho\frac{\mathbf{e}_\varsigma}{\varsigma^2} + \nabla\cdot\frac{\dot\rho}{c}\frac{\mathbf{e}_\varsigma}{\varsigma}\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\int d^3x'\left[\rho\nabla\cdot\left(\frac{\mathbf{e}_\varsigma}{\varsigma^2}\right) + \frac{\mathbf{e}_\varsigma}{\varsigma^2}\cdot(\nabla\rho) + \frac{\dot\rho}{c}\cdot\nabla\left(\frac{\mathbf{e}_\varsigma}{\varsigma}\right) + \frac{\mathbf{e}_\varsigma}{\varsigma c}\cdot(\nabla\dot\rho)\right]$$

- Now, using

$$\nabla\cdot\left(\frac{\hat{\mathbf{e}}_\varsigma}{\varsigma^2}\right) = -\nabla^2\left(\frac{1}{\varsigma}\right) = 4\pi\delta^3(\varsigma)\,, \qquad \nabla\cdot\boldsymbol{\varsigma} = 3,$$

$$\nabla\cdot\left(\frac{\hat{\mathbf{e}}_\varsigma}{\varsigma}\right) = \nabla\cdot\left(\frac{\varsigma}{\varsigma^2}\right) = \frac{\nabla\cdot\varsigma}{\varsigma^2} - 2\varsigma\cdot\frac{\nabla\varsigma}{\varsigma^3} = \frac{3}{\varsigma^2} - 2\varsigma\cdot\frac{\hat{\mathbf{e}}_\varsigma}{\varsigma^3} = \frac{1}{\varsigma^2},$$

$$\nabla\dot\rho(\mathbf{r},t_{\rm ret}) = \ddot\rho\nabla t_{\rm ret} = \ddot\rho\nabla\left(t-\frac{\varsigma}{c}\right) = -\frac{1}{c}\ddot\rho\nabla\varsigma = -\frac{1}{c}\,\ddot\rho\,\hat{\mathbf{e}}_\varsigma$$

we get

Direct check of solution through retarded potentials, cont.

- We get the result

$$\begin{aligned}\nabla^2\Phi(\mathbf{r},t) = &-\frac{1}{4\pi\epsilon_0}\int d^3x' \left\{\left[\rho(4\pi\delta^3(\varsigma)) + \frac{\hat{\mathbf{e}}_\varsigma}{\varsigma^2}\cdot\left(-\frac{1}{c}\dot{\rho}\,\hat{\mathbf{e}}_\varsigma\right)\right]\right. \\ &\left.+\frac{1}{c}\left[\dot{\rho}\frac{1}{\varsigma^2} + \frac{\hat{\mathbf{e}}_\varsigma}{\varsigma}\cdot\left(-\frac{1}{c}\,\ddot{\rho}\,\hat{\mathbf{e}}_\varsigma\right)\right]\right\} \\ &= \frac{1}{4\pi\epsilon_0}\int d^3x'\left[-4\pi\rho\,\delta^3(\varsigma) + \frac{1}{c^2}\frac{\ddot{\rho}}{\varsigma}\right] \\ &= -\frac{1}{\epsilon_0}\rho(\mathbf{r},t) + \frac{\partial^2}{c^2\partial t^2}\left(\frac{1}{4\pi\epsilon_0}\int d^3x'\frac{\rho}{\varsigma}\right) \\ &= -\frac{1}{\epsilon_0}\rho(\mathbf{r},t) + \frac{\partial^2}{c^2\partial t^2}\,\Phi(\mathbf{r},t)\end{aligned}$$

- Confirms that the retarded potential $\Phi(\mathbf{r},t)$ satisfies the inhomogenuous wave equation $(*)$
- Similar derivation for each component A_i leads to

$$\nabla^2 A_i(\mathbf{r},t) \;=\; -\frac{\mu_0}{4\pi}J_i(\mathbf{r},t) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}A_i(\mathbf{r},t) \;\Rightarrow\; \text{Eq. } (*)$$

Generalization of Coulomb and Biot-Savart Laws

- Given the retarded potentials

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\mathbf{r}',t_{\rm ret})}{\varsigma}$$

and

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{r}',t_{\rm ret})}{\varsigma}$$

- Determine the fields:

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \qquad , \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

- Remember that the integrands depend on $\mathbf{r}$ *explicitly*, through $\varsigma = |\mathbf{r}-\mathbf{r}'|$ in the denominator, and *implicitly*, through the retarded time $t_{\rm ret} = t - \varsigma/c$ in the argument of numerator . We already have expression

$$\nabla\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \left[-\rho\frac{\hat{\mathbf{e}}_\varsigma}{\varsigma^2} - \frac{\dot{\rho}}{c}\frac{\hat{\mathbf{e}}_\varsigma}{\varsigma}\right]$$

for the gradient of Φ. Use

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0}{4\pi}\int d^3x' \frac{\dot{\mathbf{J}}(\mathbf{r}',t_{\rm ret})}{\varsigma} = \frac{1}{4\pi\epsilon_0 c^2}\int d^3x' \frac{\dot{\mathbf{J}}(\mathbf{r}',t_{\rm ret})}{\varsigma}$$

Generalization of Coulomb and Biot-Savart Laws (Jefimenko Equations)

- The result is the time-dependent generalization of Coulomb's law

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int d^3x'\left[\frac{\rho(\mathbf{r}',t_{\rm ret})}{\varsigma^2}\hat{\mathbf{e}}_\varsigma+\frac{\dot\rho(\mathbf{r}',t_{\rm ret})}{c\varsigma}\,\hat{\mathbf{e}}_\varsigma-\frac{\dot{\mathbf{J}}(\mathbf{r}',t_{\rm ret})}{c^2\varsigma}\right]$$

- In the static case, the second and third terms drop out and the first term loses its dependence on time $t_{\rm ret}$
- For $\mathbf{B}$, the curl of $\mathbf{A}$ contains two terms

$$\nabla\times\mathbf{A}=\frac{\mu_0}{4\pi}\int d^3x'\left[-\mathbf{J}\times\nabla\left(\frac{1}{\varsigma}\right)+\frac{1}{\varsigma}(\nabla\times\mathbf{J})\right]$$

- The chain rule that gave $\nabla\rho=-\frac{1}{c}\hat{\mathbf{e}}_\varsigma\,\dot\rho$, in case of curl gives

$$\nabla\times\mathbf{J}\left(t-\frac{\varsigma}{c}\right)=-\frac{1}{c}\,\hat{\mathbf{e}}_\varsigma\times\dot{\mathbf{J}}=\frac{1}{c}\,\dot{\mathbf{J}}\times\hat{\mathbf{e}}_\varsigma\ .$$

- Recalling $\nabla(1/\varsigma)=-\hat{\mathbf{e}}_\varsigma/\varsigma^2$ gives

$$\mathbf{B}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int d^3x'\left[\frac{\mathbf{J}(\mathbf{r}',t_{\rm ret})}{\varsigma^2}+\frac{\dot{\mathbf{J}}(\mathbf{r}',t_{\rm ret})}{c\varsigma}\right]\times\hat{\mathbf{e}}_\varsigma\ .$$

- Note that in the expressions for the fields $\mathbf{E},\mathbf{B}$ contain completely new terms involving derivatives of ρ and $\mathbf{J}$

Generalization of Coulomb and Biot-Savart Laws

- In the case of a slowly changing current density, namely, when one can neglect all the higher derivatives in the Taylor expansion

$$\mathbf{J}(t_{\mathrm{ret}}) = \mathbf{J}(t) + (t_{\mathrm{ret}} - t)\dot{\mathbf{J}}(t) + \dots$$

(we suppress here the $\mathbf{r}'$-dependence, which is not an issue) one can write

$$\mathbf{J}(t_{\mathrm{ret}}) = \mathbf{J}(t) - \frac{\varsigma}{c}\dot{\mathbf{J}}(t)$$

The result is

$$B(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int d^3x' \left[\frac{\mathbf{J}(\mathbf{r}', t) \times \hat{\mathbf{e}}_\varsigma}{\varsigma^2}\right] + \dots$$

the Biot-Savart law with $\mathbf{J}$ evaluated at the *non-retarded* time.

The Accuracy of Quasistatic Approximation

- Let us take into account next term in the Taylor expansion

$$\mathbf{J}(\mathbf{r}', t_{\rm ret}) = \mathbf{J}(\mathbf{r}', t) + (t_{\rm ret} - t)\dot{\mathbf{J}}(\mathbf{r}', t) + \ldots \; = \; \mathbf{J}(\mathbf{r}', t) - \frac{\varsigma}{c}\dot{\mathbf{J}}(\mathbf{r}', t) + O\Big(\frac{1}{c^2}\Big)$$

- We get

$$\begin{aligned} B(\mathbf{r}, t) \; &= \; \frac{\mu_0}{4\pi}\int d^3x' \Big(\frac{\mathbf{J}(\mathbf{r}', t_{\rm ret})}{\varsigma^2} + \frac{\dot{\mathbf{J}}(\mathbf{r}', t_{\rm ret})}{c\varsigma}\Big) \times \hat{\mathbf{e}}_\varsigma \\ &= \frac{\mu_0}{4\pi}\int d^3x' \Big(\frac{\mathbf{J}(\mathbf{r}', t)}{\varsigma^2} - \frac{\varsigma}{c}\dot{\mathbf{J}}(\mathbf{r}', t) + \frac{\dot{\mathbf{J}}(\mathbf{r}', t)}{c\varsigma} + O\Big(\frac{1}{c^2}\Big)\Big) \times \hat{\mathbf{e}}_\varsigma \\ &= \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{r}', t) \times \hat{\mathbf{e}}_\varsigma}{\varsigma^2} \; + O\Big(\frac{1}{c^2}\Big) \end{aligned}$$

- The accuuracy of the quasistatic approximation is $O(\frac{v^2}{c^2})$ rather than $O(\frac{v}{c})$.