PHYSICS 453

Electromagnetism II
Lecture 10_3

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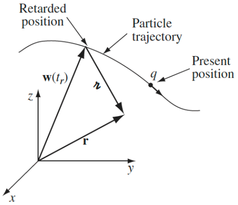
Liénard-Wiechert Potentials

- Consider a point charge moving along the trajectory $\mathbf{r} = \mathbf{w}(t)$



- As usual, it is convenient to start with the potentials. In the Lorentz gauge

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \int dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-t+\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3x' \int dt' \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-t+\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)$$

Liénard-Wiechert Potentials, cont.

- For a point charge

$$\rho(\mathbf{r}',t) \;=\; q\delta(\mathbf{r}'-\mathbf{w}(t)), \quad \mathbf{J}(\mathbf{r}',t) \;=\; q\mathbf{v}(t)\delta(\mathbf{r}'-\mathbf{w}(t))$$

- First, let us find the scalar potential. We have

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\int d^3x'\int dt'\,\frac{\delta(\mathbf{r}'-\mathbf{w}(t'))}{|\mathbf{r}-\mathbf{r}'|}\delta\Big(1+\frac{|\mathbf{r}-\mathbf{r}'|}{c}\Big)$$

$$= \frac{q}{4\pi\epsilon_0}\int dt'\,\frac{\delta\Big(t'-t+\frac{|\mathbf{r}-\mathbf{w}(t')|}{c}\Big)}{|\mathbf{r}-\mathbf{w}(t')|}$$

$$= \frac{q}{4\pi\epsilon_0}\int dt'\,\frac{1}{\frac{\partial}{\partial t'}\Big(t'-t+\frac{|\mathbf{r}-\mathbf{w}(t')|}{c}\Big)\Big|_{t'=t_r}}\,\frac{\delta(t'-t_r)}{|\mathbf{r}-\mathbf{w}(t')|}\;,$$

where t_r is the solution of the equation $c(t-t_r) = |\mathbf{r}-\mathbf{w}(t_r)|$

- We used formula

$$\delta(F(x)) = \frac{1}{|F'(x)|}\delta(x-x_*), \qquad F(x_*) = 0$$

Scalar Potential

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\int dt' \frac{1}{\frac{\partial}{\partial t'}\left(t'-t+\frac{|\mathbf{r}-\mathbf{w}(t')|}{c}\right)\Big|_{t'=t_r}} \frac{\delta(t'-t_r)}{|\mathbf{r}-\mathbf{w}(t')|}$$

- Calculate the derivative

$$\frac{\partial}{\partial t'}\left(t'-t+\frac{|\mathbf{r}-\mathbf{w}(t')|}{c}\right) = 1 - \frac{\mathbf{v}(t')\cdot(\mathbf{r}-\mathbf{w}(t'))}{c|\mathbf{r}-\mathbf{w}(t')|}$$

- $\mathbf{v}(t) \equiv \frac{\partial}{\partial t}\mathbf{w}(t)$ is the velocity of the particle, and we have

$$\begin{aligned}\Phi(\mathbf{r},t) &= \frac{q}{4\pi\epsilon_0}\int dt' \frac{\delta(t'-t_r)}{|\mathbf{r}-\mathbf{w}(t')| - \frac{1}{c}\mathbf{v}(t')\cdot(\mathbf{r}-\mathbf{w}(t'))} \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{w}(t_r)| - \mathbf{v}(t_r)\cdot(\mathbf{r}-\mathbf{w}(t_r))/c}\end{aligned}$$

Vector Potential

- Similarly,

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0 q}{4\pi}\mathbf{v}(t_r)\frac{1}{|\mathbf{r}-\mathbf{w}(t_r)| - \mathbf{v}(t_r)\cdot(\mathbf{r}-\mathbf{w}(t_r))/c}$$

- The potentials

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\;\frac{1}{|\mathbf{r}-\mathbf{w}(t_r)| - \mathbf{v}(t_r)\cdot(\mathbf{r}-\mathbf{w}(t_r))/c}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mathbf{v}(t_r)}{c^2}\,\Phi(\mathbf{r},t)$$

  are called the Lenard-Wiechert potentials for a point charge
- Introducing the notation $\boldsymbol{\varsigma}(t) \equiv \mathbf{r}-\mathbf{w}(t)$, we obtain

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\;\frac{1}{\varsigma(t_r) - \mathbf{v}(t_r)\cdot\boldsymbol{\varsigma}(t_r)/c}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mathbf{v}(t_r)}{c^2}\,\Phi(\mathbf{r},t) = \frac{q\mu_0}{4\pi}\;\frac{\mathbf{v}(t_r)}{\varsigma(t_r) - \mathbf{v}(t_r)\cdot\boldsymbol{\varsigma}(t_r)/c}$$

- According to these formulas, the field at the point of observation at time t is determined by the state of motion of the charge at the earlier time t_r
- Also, $\varsigma(t) = \mathbf{r}-\mathbf{w}(t)$ is the radius vector from the charge q to the observation point P; like $\mathbf{w}(t)$ it is a given function of the time
- The time t_r is determined by the equation $t_r + \varsigma(t_r)/c = t$

Electric and Magnetic Fields

- To calculate the intensities of the electric and magnetic fields from

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \quad , \quad \mathbf{B} = \nabla \times \mathbf{A} \,,$$

  we must differentiate Φ and $\mathbf{A}$ with respect to the coordinates x, y, z of the point P, and the time t of observation
- But the potentials are implicit functions of x, y, z, t through $t_r + \varsigma(t_r)/c = t$
- Therefore we must first calculate the derivatives of t_r
- Differentiating the relation $\varsigma(t_r) = c(t - t_r)$ with respect to t, we get

$$\frac{\partial \varsigma}{\partial t} = c\left(1 - \frac{\partial t_r}{\partial t}\right) = \frac{\partial \varsigma}{\partial t_r}\frac{\partial t_r}{\partial t} = -\frac{\boldsymbol{\varsigma}\cdot\mathbf{v}}{\varsigma}\frac{\partial t_r}{\partial t}$$

- $\partial\varsigma/\partial t_r$ is obtained from using $\varsigma^2 = \boldsymbol{\varsigma}^2$ and substituting $\partial\boldsymbol{\varsigma}(t_r)/\partial t_r = -\mathbf{v}(t_r)$
- The minus sign is present because $\boldsymbol{\varsigma} = \mathbf{r} - \mathbf{w}$. Thus,

$$c\left(1 - \frac{\partial t_r}{\partial t}\right) = -\frac{\boldsymbol{\varsigma}\cdot\mathbf{v}}{\varsigma}\frac{\partial t_r}{\partial t} \Rightarrow \frac{\partial t_r}{\partial t} = \frac{1}{1 - (\boldsymbol{\varsigma}\cdot\mathbf{v})/c\varsigma}$$

- Differentiating the same relation with respect to the coordinates, we find

$$\nabla t_r = -\frac{1}{c}\nabla\varsigma(t_r) = -\frac{1}{c}\left(\frac{\partial\varsigma}{\partial t_r}\nabla t_r + \frac{\boldsymbol{\varsigma}}{\varsigma}\right) \Rightarrow -\frac{\boldsymbol{\varsigma}}{c(\varsigma - (\boldsymbol{\varsigma}\cdot\mathbf{v})/c)}$$

Electric and Magnetic Fields, cont.

- With the help of these formulas, one can calculate the fields $\mathbf{E}$ and $\mathbf{B}$
- Omitting the intermediate calculations, we give the final results:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 \left(\varsigma - \boldsymbol{\varsigma}\cdot\mathbf{v}/c\right)^3}\left\{\left(1 - v^2/c^2\right)\left(\boldsymbol{\varsigma} - \frac{\mathbf{v}}{c}\varsigma\right) + \frac{1}{c^2}\boldsymbol{\varsigma}\times\left[\left(\boldsymbol{\varsigma} - \frac{\mathbf{v}}{c}\varsigma\right)\times\mathbf{a}\right]\right\}$$

$$\mathbf{B} = \frac{1}{\varsigma}\boldsymbol{\varsigma}\times\mathbf{E}$$

Here, $\mathbf{a} = \partial\mathbf{v}/\partial t_r$

- All quantities on the right sides of the equations refer to the time t_r
- Note that the magnetic field is everywhere perpendicular to the electric
- In the non-relativistic limit, the electric field $\mathbf{E}$ reduces to the Coulomb field

$$\mathbf{E}(\mathbf{r},t)|_{v/c\to 0} \quad\to\quad \frac{q\boldsymbol{\varsigma}}{4\pi\epsilon_0\varsigma^3}\ ,$$

while the magnetic field tends to zero as v/c

- Introduce the unit vector $\hat{\boldsymbol{\varsigma}} \equiv \boldsymbol{\varsigma}/\varsigma$, and also the notation $\mathbf{u} \equiv \hat{\boldsymbol{\varsigma}} - \mathbf{v}(t_r)/c$
- This vector tends to $\hat{\boldsymbol{\varsigma}}$ in the non-relativistic limit $v/c \to 0$
- Using $\varsigma - \boldsymbol{\varsigma}\cdot\mathbf{v}/c = (\boldsymbol{\varsigma}\cdot\mathbf{u})$, we can write the electric and magnetic fields as

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \frac{q}{4\pi\epsilon_0}\,\frac{\varsigma}{(\boldsymbol{\varsigma}\cdot\mathbf{u})^3}\left[\mathbf{u}(1 - v^2/c^2) + \boldsymbol{\varsigma}\times(\mathbf{u}\times\mathbf{a})/c^2\right] \\ \mathbf{B}(\mathbf{r},t) &= \hat{\boldsymbol{\varsigma}}\times\mathbf{E}(\mathbf{r},t)/c\end{aligned}$$

Electric and Magnetic Fields, cont.

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\,\frac{\varsigma}{(\boldsymbol{\varsigma}\cdot\mathbf{u})^3}\left[\mathbf{u}(1-v^2/c^2) + \boldsymbol{\varsigma}\times(\mathbf{u}\times\mathbf{a})/c^2\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\hat{\boldsymbol{\varsigma}}}{c}\times\mathbf{E}(\mathbf{r},t)$$

- As usual, velocity and acceleration are taken at $t = t_r$
- Recall that t_r is defined as a solution to the equation $c(t - t_r) = \varsigma$
- The electric field consists of two parts of different type
- The first term ($\sim \mathbf{u}$) is called the velocity field
- The second ($\sim \mathbf{a}$) is called the acceleration or the radiation field
- The first term varies at large distances like $1/\varsigma^2$
- Since it is independent of the acceleration, it corresponds to the field produced by a uniformly moving charge
- For large distances ς, the velocity field decreases as $1/\varsigma^2$
- The total field $\mathbf{E}(\mathbf{r},t)$ contains also a radiation component $\mathbf{E}^{\rm rad}(\mathbf{r},t)$
- It decreases as $1/\varsigma$ for large ς, and is given by

$$\mathbf{E}_{\rm rad}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2\varsigma}\,\frac{\hat{\boldsymbol{\varsigma}}\times(\mathbf{u}\times\mathbf{a})}{(\hat{\boldsymbol{\varsigma}}\cdot\mathbf{u})^3} = \frac{q}{4\pi\epsilon_0 c^2\varsigma}\,\frac{\hat{\boldsymbol{\varsigma}}\times(\mathbf{u}\times\mathbf{a})}{(1-\hat{\boldsymbol{\varsigma}}\cdot\mathbf{v}/c)^3}$$

- The radiation magnetic field also decreases as $1/\varsigma$ for large ς

$$\mathbf{B}_{\rm rad}(\mathbf{r},t) = \frac{\hat{\boldsymbol{\varsigma}}}{c}\times\mathbf{E}_{\rm rad}(\mathbf{r},t)$$

Electromagnetic fields due to a point charge moving with constant velocity

- In this case, $w(t) = t\mathbf{v}$ and $\boldsymbol{\varsigma}(t) = \mathbf{r} - t\mathbf{v}$. Recalling that $\varsigma(t_r) = c(t - t_r)$,

$$\boldsymbol{\varsigma}(t_r) - \frac{\mathbf{v}}{c}\,\varsigma(t_r) = [\mathbf{r} - t_r\mathbf{v}] - (t - t_r)\,\mathbf{v} = \mathbf{r} - t\mathbf{v} = \boldsymbol{\varsigma}(t)$$

  is the distance $\boldsymbol{\varsigma}(t)$ from the charge to the point of observation at precisely the moment t of observation.

- The retarded time may be also calculated explicitly:

$$c(t - t_r) = |\mathbf{r} - t_r\mathbf{v}| \quad \Rightarrow \quad c^2(t^2 - 2tt_r + t_r^2) = r^2 - 2t_r\mathbf{v}\cdot\mathbf{r} + v^2t_r^2$$

$$\Rightarrow\; t_r = \frac{c^2t - \mathbf{v}\cdot\mathbf{r} - \sqrt{(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)}}{c^2 - v^2}$$

- This gives

$$c^2t - \mathbf{v}\cdot\mathbf{r} - (c^2 - v^2)t_r = \sqrt{(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)}$$

- The Liénard-Wiechert potentials take the form

$$\Phi(\mathbf{r}, t) = \frac{qc}{4\pi\epsilon_0}\,\frac{1}{c|\mathbf{r} - t_r\mathbf{v}| - \mathbf{v}\cdot(\mathbf{r} - t_r\mathbf{v})} = \frac{qc}{4\pi\epsilon_0[c^2t - (c^2 - v^2)t_r - \mathbf{v}\cdot\mathbf{r}]}$$

$$= \frac{qc}{4\pi\epsilon_0}\left[(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)\right]^{-1/2}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c^2}\,\Phi(\mathbf{r}, t)$$

Potentials for a point charge moving with constant velocity

$$\Phi(\mathbf{r},t) = \frac{qc}{4\pi\epsilon_0}\left[(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)\right]^{-1/2}$$

- Let us demonstrate that $\Phi(\mathbf{r},t)$ can be rewritten as

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 R}\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{-1/2}$$

$\mathbf{R} = \mathbf{r} - t\mathbf{v}$ and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$

- For a constant velocity R is the distance to the position of the moving charge at the time of measurement of the fields. We have

$$\begin{aligned}
&(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2) \\
&= [c^2t - \mathbf{v}\cdot(\mathbf{R} + t\mathbf{v})]^2 - (c^2 - v^2)[c^2t^2 - (\mathbf{R} + t\mathbf{v})^2] \\
&= [(c^2 - v^2)t - \mathbf{v}\cdot\mathbf{R}]^2 - (c^2 - v^2)[(c^2 - v^2)t^2 - 2t\mathbf{v}\cdot\mathbf{R} - R^2)] \\
&= (c^2 - v^2)R^2 + (\mathbf{v}\cdot\mathbf{R})^2 = c^2R^2 - v^2R^2\sin^2\theta
\end{aligned}$$

- Therefore

$$\sqrt{(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)} = Rc\sqrt{1 - \frac{v^2}{c^2}\sin^2\theta}$$

Fields for a point charge moving with constant velocity

- Potentials in this case are given by explicit functions of t and $\mathbf{r}$, and the calculation of

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \quad , \quad \mathbf{B} = \nabla \times \mathbf{A}$$

is straightforward. We need

$$\begin{aligned} -\frac{1}{c^2}\frac{\partial}{\partial t}\left[(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)\right]^{-1/2} &= \frac{c^2t - \mathbf{v}\cdot\mathbf{r} - (c^2 - v^2)t}{[(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)]^{3/2}} \\ &= \frac{-\mathbf{v}\cdot\mathbf{r} + v^2t}{[(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)]^{3/2}} \end{aligned}$$

and

$$\begin{aligned} -\nabla\left[(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)\right]^{-1/2} \\ = \frac{-(c^2t - \mathbf{v}\cdot\mathbf{r})\mathbf{v} + (c^2 - v^2)\mathbf{r}}{[(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)]^{3/2}} \end{aligned}$$

Fields for a point charge moving with constant velocity, cont.

- The fields are then given by

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} = \frac{qc}{4\pi\epsilon_0}\frac{-(c^2t - \mathbf{v}\cdot\mathbf{r})\mathbf{v} + (c^2 - v^2)\mathbf{r} + \mathbf{v}(-\mathbf{v}\cdot\mathbf{r} + v^2 t)}{[(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)]^{3/2}}$$

$$= \frac{qc}{4\pi\epsilon_0}\frac{(c^2 - v^2)(\mathbf{r} - t\mathbf{v})}{[(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)]^{3/2}}$$

$$= \frac{qc(c^2 - v^2)}{4\pi\epsilon_0}\frac{\mathbf{R}}{(R^2c^2 - R^2v^2\sin^2\theta)^{3/2}}$$

or

$$\mathbf{E} = \frac{q\mathbf{R}}{4\pi\epsilon_0 R^3}\frac{1 - v^2/c^2}{\left(1 - v^2\sin^2\theta/c^2\right)^{3/2}}$$

- For the magnetic field, we get

$$\mathbf{B} = -\frac{q}{4\pi\epsilon_0 c}\frac{(c^2 - v^2)(\mathbf{r}\times\mathbf{v})}{[(c^2t - \mathbf{v}\cdot\mathbf{r})^2 - (c^2 - v^2)(c^2t^2 - r^2)]^{3/2}}$$

$$= \frac{q(1 - v^2/c^2)}{4\pi\epsilon_0 R^3 c^2}\frac{\mathbf{v}\times\mathbf{R}}{(1 - v^2\sin^2\theta/c^2)^{3/2}} = \frac{1}{c^2}\mathbf{v}\times\mathbf{E}(\mathbf{r},t)$$

- It can be demonstrated that these fields are Lorentz transforms of the usual Coulomb field of a point charge ($\mathbf{E}(\mathbf{r},t) = q\mathbf{R}/4\pi\epsilon_0 R^3$, $\mathbf{B} = 0$)