PHYSICS 453

Electromagnetism II
Lecture 11-1

Physics Department
Old Dominion University

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Outline

1. Radiating Systems
   - Radiation
   - Dipole Radiation

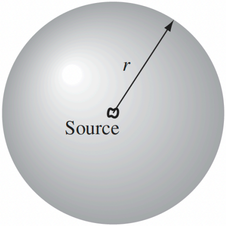
What is radiation

- Energy flow thru the surface of radius $R \to \infty$

$$P = \oint d\mathbf{a} \cdot \mathbf{S} = \oint d\mathbf{a} \cdot \mathbf{E} \times \mathbf{H}$$

- Since elecromagnetic fields travel with a speed of light

$$P_{\text{rad}}(t_0) = \lim_{r \to \infty} P\left(r, t_0 + \frac{r}{c}\right)$$



- $P_{\text{rad}} = 0$: no radiation
- $P_{\text{rad}} \neq 0$: radiation

Radiation, cont.

- In electro/magnetostatics

$$\left.\begin{aligned}\mathbf{E}(\mathbf{x},t) &= \tfrac{1}{4\pi\epsilon_0}\int d^3x'\,\tfrac{\rho(\mathbf{x}')}{\varsigma^2}\hat{\mathbf{e}}_\varsigma \;\sim\; O\big(\tfrac{1}{r^2}\big)\\ \mathbf{B}(\mathbf{x},t) &= \tfrac{\mu_0}{4\pi}\int d^3x'\,\tfrac{\mathbf{J}(\mathbf{x}')}{\varsigma^2}\times\hat{\mathbf{e}}_\varsigma \;\sim\; O\big(\tfrac{1}{r^2}\big)\end{aligned}\right\}\Rightarrow \begin{aligned}P_{\mathrm{rad}} &= \lim_{r\to\infty}\oint d\mathbf{a}\cdot\tfrac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\\ \sim r^2\times\tfrac{1}{r^4} &= 0\end{aligned}$$

- In electrodynamics

$$\mathbf{E}(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\left[\frac{\rho(\mathbf{x}',t_{\mathrm{ret}})}{\varsigma^2}\hat{\mathbf{e}}_\varsigma + \frac{\dot\rho(\mathbf{x}',t_{\mathrm{ret}})}{c\varsigma}\hat{\mathbf{e}}_\varsigma - \frac{\dot{\mathbf{J}}(\mathbf{x}',t_{\mathrm{ret}})}{c^2\varsigma}\right] \sim O\big(\frac{1}{r}\big)$$

$$\mathbf{B}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\left[\frac{\mathbf{J}(\mathbf{x}',t_{\mathrm{ret}})}{\varsigma^2} + \frac{\dot{\mathbf{J}}(\mathbf{x}',t_{\mathrm{ret}})}{c\varsigma}\right]\times\hat{\mathbf{e}}_\varsigma \;\sim\; O\big(\frac{1}{r}\big)$$

$$\Rightarrow P_{\mathrm{rad}} = \lim_{r\to\infty}\oint d\mathbf{a}\cdot\tfrac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \sim r^2\times\tfrac{1}{r^2} \neq 0$$

$\Rightarrow$ radiation is possible

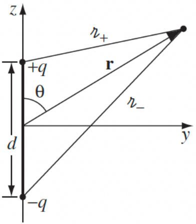
Electric Dipole Radiation

- Vector and scalar potentials due to an arbitrary source are:

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\mathbf{x}',t_r)}{|\mathbf{x}-\mathbf{x}'|}, \qquad \mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\mathbf{J}(\mathbf{x}',t_r)}{|\mathbf{x}-\mathbf{x}'|},$$

  where $t_r = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$ is the retarded time

- Consider the electric dipole with $q(t) = q_0 \cos\omega t$ so that $\mathbf{p} = \mathbf{p}_0 \cos\omega t$



$$\begin{aligned} \varsigma_+ &= \mathbf{r} - \frac{d}{2}\hat{\mathbf{e}}_3, & t_r^+ &= t - \frac{\varsigma_+}{c} \\ \varsigma_- &= \mathbf{r} + \frac{d}{2}\hat{\mathbf{e}}_3, & t_r^- &= t - \frac{\varsigma_-}{c} \end{aligned}$$

- For a point charge $\rho(\mathbf{x},t) = q\delta(\mathbf{x}-\mathbf{w}(t))$ where $\mathbf{w}(t)$ is the trajectory.

$$\Rightarrow \quad \rho(\mathbf{r}',t_r) \;=\; q(t_r^+)\delta(\varsigma_+) - q(t_r^-)\delta(\varsigma_-))$$

Electric Dipole Radiation , cont.

- We will study case when wavelength $\lambda = \frac{2\pi c}{\omega} \gg$ size of the dipole d. Also, we consider fields in so-called *radiation zone* $r \gg \lambda$
- Expand at large r and $\frac{\omega d}{c} \sim \frac{d}{\lambda} \ll 1$

$$\varsigma_\pm = \sqrt{r^2 \mp 2rd\cos\theta + \frac{d^2}{4}} \simeq r\left(1 \mp \frac{d}{2r}\right) \Rightarrow \frac{1}{\varsigma_\pm} \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

$$\Rightarrow \cos\omega\left(t - \frac{\varsigma_\pm}{c}\right) \simeq \cos\omega\left(t - \frac{r}{c} \pm \frac{\omega d}{2c}\cos\theta\right)$$

$$\simeq \cos\omega\left(t - \frac{r}{c}\right) \mp \frac{\omega d}{2c}\cos\theta \sin\left(t - \frac{\varsigma_\pm}{c}\right) + O\left(\frac{d^2}{\lambda^2}\right)$$

- We get scalar potential in the form

$$\Rightarrow \Phi(\mathbf{r}, t) = \frac{q_0 \cos\left(\omega - \frac{\varsigma_+}{c}\right)}{4\pi\epsilon_0 \varsigma_+} - \frac{q_0 \cos\left(\omega - \frac{\varsigma_-}{c}\right)}{4\pi\epsilon_0 \varsigma_-}$$

$$= -\frac{p_0 \omega \cos\theta}{4\pi\epsilon_0 c r} \sin\omega\left(t - \frac{r}{c}\right) + \frac{p_0 \cos\theta}{4\pi\epsilon_0 r^2} \cos\omega\left(t - \frac{r}{c}\right)$$

$$= -\frac{p_0 \omega \cos\theta}{4\pi\epsilon_0 c r} \sin\omega\left(t - \frac{r}{c}\right)\left[1 + O\left(\frac{c}{\omega r} \sim \frac{\lambda}{r}\right)\right]$$

Electric Dipole Radiation , cont.

- Vector potential is determined by the current flowing in the wire

$$\mathbf{I}(t) \;=\; \hat{\mathbf{e}}_3 \frac{dq(t)}{dt} \;=\; -q_0 \hat{\mathbf{e}}_3 \omega \sin \omega t$$

- For the vector potential, the first term in the expansions is sufficient:

$$\mathbf{A}(\mathbf{r}, t) \;=\; \frac{\mu_0}{4\pi} \int d^3x' \frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \xrightarrow{\text{line currents}} \mathbf{A}(\mathbf{r}, t) \;=\; \frac{\mu_0}{4\pi} \int dl' \frac{\mathbf{I}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|}$$

$$= \;-\; \frac{\mu_0}{4\pi} \hat{\mathbf{e}}_3 q_0 \omega \int_{-\frac{d}{2}}^{\frac{d}{2}} dz \frac{\sin\left(t - \frac{\varsigma}{c}\right)}{\varsigma} \;\simeq\; -\frac{\mu_0}{4\pi r} \hat{\mathbf{e}}_3 q_0 \omega \sin\left(t - \frac{r}{c}\right)$$

- Fields

$$\mathbf{E}(\mathbf{r}, t) \;= -\nabla\Phi(\mathbf{r}, t) - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}, t), \qquad \mathbf{B}(\mathbf{r}, t) \;=\; \nabla \times \mathbf{A}(\mathbf{r}, t)$$

We need to differentiate potentials keeping only $\sim \frac{1}{r}$ terms and neglecting $\mathcal{O}\left(\frac{1}{r^2}\right)$ contributions

Electric Dipole Radiation , cont.

- In spherical coordinates

$$
\begin{aligned}
-\nabla\Phi(r,\theta) &= \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}\right)\frac{p_0\omega\cos\theta}{4\pi\epsilon_0 cr}\sin\omega\left(t-\frac{r}{c}\right) \\
&= \hat{\mathbf{r}}p_0\frac{\omega\cos\theta}{4\pi\epsilon_0 cr}\frac{\partial}{\partial r}\sin\omega\left(t-\frac{r}{c}\right) + \mathcal{O}\left(\frac{1}{r^2}\right) = -\hat{\mathbf{r}}\frac{p_0\omega^2\cos\theta}{4\pi\epsilon_0 c^2 r}\cos\omega\left(t-\frac{r}{c}\right), \\
-\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) &= \frac{\partial}{\partial t}\frac{\mu_0}{4\pi r}\hat{\mathbf{e}}_3 q_0\omega\sin\omega\left(t-\frac{r}{c}\right) = \hat{\mathbf{e}}_3\frac{p_0\omega^2\cos\theta}{4\pi\epsilon_0 c^2 r}\cos\omega\left(t-\frac{r}{c}\right) \\
\Rightarrow \mathbf{E}(\mathbf{r},t) &= -\frac{p_0\omega^2\sin\theta}{4\pi\epsilon_0 c^2 r}\cos\omega\left(t-\frac{r}{c}\right)\hat{\theta} \qquad \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{e}}_3 = \hat{\theta}\sin\theta
\end{aligned}
$$

- Magnetic vector potential in spherical coordinates is

$$
A_r = -\frac{\mu_0 p_0\omega}{4\pi r}\cos\theta\sin\omega\left(t-\frac{r}{c}\right), \quad A_\theta = \frac{\mu_0 p_0\omega}{4\pi r}\sin\theta\sin\omega\left(t-\frac{r}{c}\right), \quad A_\phi = 0
$$

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t) = \frac{\hat{\phi}}{r}\Big(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\Big)$$

$$= \frac{\hat{\phi}}{r}\Big[\frac{\partial}{\partial r}\,\frac{\mu_0 p_0\omega}{4\pi}\sin\theta\sin\omega\big(t-\frac{r}{c}\big) + \frac{\partial}{\partial\theta}\,\frac{\mu_0 p_0\omega}{4\pi r}\cos\theta\sin\omega\big(t-\frac{r}{c}\big)\Big]$$

$$= -\frac{\hat{\phi}}{r}\,\frac{\mu_0 p_0\omega^2}{4\pi c}\sin\theta\cos\omega\big(t-\frac{r}{c}\big)$$

- We get

$$\left.\begin{aligned}\mathbf{E}(\mathbf{r},t) &= -\frac{p_0\omega^2}{4\pi\epsilon_0 c^2 r}\sin\theta\cos\omega\big(t-\tfrac{r}{c}\big)\hat{\theta}\\ \mathbf{B}(\mathbf{r},t) &= -\frac{\mu_0 p_0\omega^2}{4\pi c r}\sin\theta\cos\omega\big(t-\tfrac{r}{c}\big)\hat{\phi}\end{aligned}\right\}\ \text{spherical wave with } \mathbf{B} = \frac{\hat{\mathbf{r}}}{c}\times\mathbf{E}$$

- The Poynting vector is then

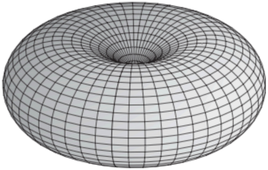
$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{\mu_0 p_0^2\omega^4}{16\pi^2 c r^2}\sin^2\theta\cos^2\omega\big(t-\frac{r}{c}\big)\hat{\mathbf{r}}$$

- Time average of the Poynting vector

$$\langle\mathbf{S}\rangle = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c r^2}\sin^2\theta\,\hat{\mathbf{r}}$$

Electric Dipole Radiation, cont.

- Intensity profile of the dipole radiation:



- $\Rightarrow$ the total radiated power takes the form

$$\langle P \rangle = \int \langle \mathbf{S} \rangle \cdot \hat{\mathbf{n}} \, dA \overset{\mathbf{n}=\mathbf{r}}{=} \frac{\mu_0}{32\pi^2 c} p_0^2 \omega^4 \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \underbrace{\int_0^{\pi} d\theta \, \sin^3\theta}_{4/3} = \frac{\mu_0}{12\pi c} p_0^2 \omega^4$$

- The total radiated power

$$P = \frac{\mu_0}{12\pi c} p_0^2 \omega^4$$