PHYSICS 453

Electromagnetism II
Lecture 11-2

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Outline

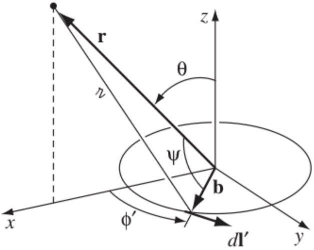
Magnetic dipole radiation

- Consider a circular loop with the current $I(t) = I_0 \cos\omega t$
- It is a typical example of an oscillating magnetic dipole

$$\mathbf{m}(t) = \pi b^2 I(t)\hat{\mathbf{e}}_3 = m_0 \hat{\mathbf{e}}_3 \cos\omega t, \qquad m_0 = \pi b^2 I_0$$

- First, the scalar potential is obviously zero.
- Vector potential

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0 I_0}{4\pi}\int d\mathbf{l}' \frac{\cos\omega\left(t-\frac{\varsigma}{c}\right)}{\varsigma}$$



For simplicity talke $\mathbf{r}$ in the XZ plane: $\mathbf{r} = r\sin\theta\hat{\mathbf{e}}_1 + r\cos\theta\hat{\mathbf{e}}_3$.

$$\mathbf{r}' = b\cos\phi\hat{\mathbf{e}}_1 + b\sin\phi\hat{\mathbf{e}}_2 \Rightarrow$$

$$(\mathbf{r}-\mathbf{r}')^2 = r^2 + b^2 - 2br\sin\theta\cos\phi$$

$$d\mathbf{l}' = \hat{\phi}dl' = (\hat{\mathbf{e}}_2\cos\phi - \hat{\mathbf{e}}_1\sin\phi)bd\phi$$

Magnetic dipole radiation, cont.

$$\mathbf{A}(\mathbf{r},t) \;=\; \frac{\mu_0 I_0 b}{4\pi}\int_0^{2\pi} d\phi\,\frac{\cos\omega\Big(t-\frac{1}{c}\sqrt{r^2+b^2-2br\sin\theta\cos\phi}\Big)}{\sqrt{r^2+b^2-2br\sin\theta\cos\phi}}\,(\hat{\mathbf{e}}_2\cos\phi-\hat{\mathbf{e}}_1\sin\phi)$$

$$= \frac{\mu_0 I_0 b}{4\pi}\hat{\mathbf{e}}_2\int_0^{2\pi} d\phi\;\cos\phi\,\frac{\cos\omega\Big(t-\frac{1}{c}\sqrt{r^2+b^2-2br\sin\theta\cos\phi}\Big)}{\sqrt{r^2+b^2-2br\sin\theta\cos\phi}}$$

- In the radiation zone we can expand in $\frac{d}{r}$

$$\sqrt{r^2+b^2-2bd\sin\theta\cos\phi} \simeq r\big(1-\frac{b}{r}\sin\theta\cos\phi\big) = r - b\sin\theta\cos\phi$$

$$\mathbf{A}(\mathbf{r},t) \overset{b\lll r}{=} \frac{\mu_0 I b}{4\pi r}\hat{\mathbf{e}}_2\int_0^{2\pi} d\phi\;\cos\phi\cos\Big[\omega\big(t-\frac{r}{c}\big)+\frac{\omega b}{c}\sin\theta\cos\phi\Big]$$

$$\overset{b\lll\lambda}{=} \frac{\mu_0 I b}{4\pi r}\hat{\mathbf{e}}_2\int_0^{2\pi} d\phi\;\cos\phi\Big[\cos\omega\big(t-\frac{r}{c}\big)-\frac{\omega b}{c}\sin\theta\cos\phi\sin\omega\big(t-\frac{r}{c}\big)\Big]$$

$$= -\frac{\mu_0 m_0\omega}{4\pi r c}\hat{\mathbf{e}}_2\sin\theta\sin\omega\big(t-\frac{r}{c}\big)$$

- For an arbitrary point $\mathbf{r}$ one should replace $\hat{\mathbf{e}}_2 \rightarrow \hat{\phi}$

$$\mathbf{A}(\mathbf{r},t) \;=\; -\frac{\mu_0 m_0\omega}{4\pi r c}\hat{\mathbf{e}}_\phi\sin\theta\sin\omega\big(t-\frac{r}{c}\big) \;=\; \frac{\mathbf{m}_0\times\hat{\mathbf{r}}}{4\pi r c}\mu_0\omega\sin\omega\big(t-\frac{r}{c}\big)$$

Magnetic dipole radiation, cont.

- In spherical polars $A_r = A_\theta = 0$ and $A_\phi = -\frac{m_0 \mu_0 \omega}{4\pi r c} \sin\theta \sin\omega(t - \frac{r}{c})$
- The fields are

$$\mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi r c} \hat{\mathbf{e}}_\phi \sin\theta \cos\omega(t - \frac{r}{c})$$

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A} = \frac{\hat{\mathbf{r}}}{r\sin\theta} \frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\hat{\boldsymbol{\theta}}}{r} \frac{\partial}{\partial r}(r A_\phi)$$

$$= -\frac{\mu_0 m_0 \omega^2 \hat{\boldsymbol{\theta}}}{4\pi r c^2} \sin\theta \cos\omega(t - \frac{r}{c}) - \frac{\mu_0 m_0 \omega^2 \hat{\mathbf{r}}}{2\pi r^2 c} \cos\theta \sin\omega(t - \frac{r}{c})$$

- We got a *spherical wave*

$$\left.\begin{aligned} \mathbf{E}(\mathbf{r},t) &= \frac{\mu_0 m_0 \omega^2}{4\pi r c} \hat{\mathbf{e}}_\phi \sin\theta \cos\omega(t - \frac{r}{c}) \\ \mathbf{B}(\mathbf{r},t) &= \frac{\hat{\mathbf{r}}}{c} \times \mathbf{E} \end{aligned}\right\} \Rightarrow \text{Spherical wave}$$

Magnetic dipole radiation, cont.

- Poynting vector

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \;=\; \frac{\mu_0}{c}\hat{\mathbf{r}}\left\{\frac{m_0\omega^2}{4\pi c}\,\frac{\sin\theta}{r}\cos\omega\left(t-\frac{r}{c}\right)\right\}^2$$

- Intensity

$$I \;=\; \langle\mathbf{S}\rangle \;=\; \frac{\mu_0 m_0^2\omega^4}{32\pi^2c^3}\,\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}}$$

- Radiated power

$$P \;=\; R^2\int_0^1 \sin\theta\int_0^{2\pi} d\phi\,\frac{\mu_0 m_0^2\omega^4}{32\pi^2c^3}\,\frac{\sin^2\theta}{r^2} \;=\; \frac{\mu_0 m_0^2\omega^4}{12\pi c^3}$$

- It is much smaller than power radiated by comparable electric dipole.
- Example: $+q_0$ and $-q_0$ electric dipole rotating with angular velocity ω. Electric dipole power $P_{\mathrm{el}} = \frac{\mu_0 p_0^2}{12\pi c}$ where $p_0 = q_0 d$. Also, in this model $m_0 = \pi\frac{d^2}{4}I_0$ and $I_0 = q_0\omega$ so

$$\frac{P_{\mathrm{mag}}}{P_{\mathrm{el}}} \;=\; \left(\frac{m_0}{p_0 c}\right)^2 \;\sim\; \left(\frac{\pi d\omega}{4c}\right)^2 \;\sim\; \frac{v^2}{c^2} \;=\; \text{Relativistic correction}$$

Dipole Radiation From an Arbitrary Source

- In this section we will derive the formulas for the dipole radiation from an arbitrary source.
- Vector and scalar potentials due to an arbitrary source are:

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\rho(\mathbf{x}',t_r)}{|\mathbf{x}-\mathbf{x}'|}$$

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\mathbf{J}(\mathbf{x}',t_r)}{|\mathbf{x}-\mathbf{x}'|}\ ,$$

  where $t_r = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$ is the retarded time
- To study the behavior of these expressions in the radiation zone $|\mathbf{x}| \gg |\mathbf{x}'|$, we choose the origin somewhere inside the radiating body and expand the denominators in a usual way:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r}\left(1 + \frac{\hat{\mathbf{n}}\cdot\mathbf{x}'}{r} + \ldots\right)$$

  where $r \equiv |\mathbf{x}|$ and $\hat{\mathbf{n}} \equiv \hat{\mathbf{r}}$ is the propagation vector for our would-be spherical wave
- We need also to expand the retarded time in powers of r'/r:

$$t_r = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c} \simeq t - \frac{r}{c} + \frac{\hat{\mathbf{n}}\cdot\mathbf{x}'}{c}$$

Dipole Radiation From an Arbitrary Source cont.

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{r}\left(1+\frac{\hat{\mathbf{n}}\cdot\mathbf{x}'}{r}+\ldots\right),\quad \hat{\mathbf{n}}\equiv\hat{\mathbf{r}}\ ,\quad t_r=t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\simeq t-\frac{r}{c}+\frac{\hat{\mathbf{n}}\cdot\mathbf{x}'}{c}$$

- Introduce $t_0\equiv t-r/c$ as the retarded time for our origin. Then

$$\rho(\mathbf{x}',t_r)=\rho(\mathbf{x}',t_0)+\frac{\hat{\mathbf{n}}\cdot\mathbf{x}'}{c}\dot{\rho}(\mathbf{x}',t_0)+\ldots$$

- The parameter of the expansion here is $d/\lambda\ll 1$ (see previous Section)
- Indeed, $\dot{\rho}\sim\omega_{\rm char}\rho$ where $\omega_{\rm char}$ are the characteristic frequencies of the emitted radiation, hence $\frac{d\dot{\rho}}{c\rho}\sim\frac{d\omega}{c}=\frac{d}{\lambda}\ll 1$
- Substituting the expansions in the expression for $\Phi(\mathbf{x},t)$, one obtains:

$$\Phi(\mathbf{x},t)=\frac{1}{4\pi\epsilon_0 r}\int d^3x'\left[\rho(\mathbf{x}',t_0)+\frac{\hat{\mathbf{n}}\cdot\mathbf{x}'}{c}\dot{\rho}(\mathbf{x}',t_0)\right]\left(1+\frac{\hat{\mathbf{n}}\cdot\mathbf{x}'}{r}+\ldots\right)$$
$$=\frac{Q}{4\pi\epsilon_0 r}+\frac{\hat{\mathbf{n}}\cdot\mathbf{p}(t_0)}{4\pi\epsilon_0 r^2}+\frac{\hat{\mathbf{n}}\cdot\dot{\mathbf{p}}(t_0)}{4\pi\epsilon_0 rc}+\ldots$$

- For the vector potential, the first term in the expansions is sufficient:

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int d^3x'\frac{\mathbf{J}(\mathbf{x}',t_r)}{|\mathbf{x}-\mathbf{x}'|}\ \simeq\ \frac{\mu_0}{4\pi r}\int d^3x'\mathbf{J}(\mathbf{x}',t_0)$$

Dipole Radiation From an Arbitrary Source, cont.

- It can be demonstrated that

$$\int d^3x' \mathbf{J}(\mathbf{x}', t) \;=\; \dot{\mathbf{p}}(t)$$

- $\Rightarrow$ the dipole potentials in the radiation zone take the form ($\hat{\mathbf{n}} = \hat{\mathbf{r}}$)

$$\Phi(\mathbf{x}, t) = \frac{Q}{4\pi\epsilon_0 r} + \frac{\hat{\mathbf{n}} \cdot \mathbf{p}(t_0)}{4\pi\epsilon_0 r^2} + \frac{\hat{\mathbf{n}} \cdot \dot{\mathbf{p}}(t_0)}{4\pi\epsilon_0 rc} + \ldots\,, \qquad \mathbf{A}(\mathbf{x}, t) = \frac{\mu_0 \dot{\mathbf{p}}(t_0)}{4\pi r} + \ldots$$

- Next we calculate the electric and magnetic field in the radiation zone
- Discard terms $\sim \frac{1}{r^2}$ and use $\nabla f(t_0) = \dot{f}(t_0)\nabla t_0$ and $\nabla t_0 = -\frac{\hat{\mathbf{n}}}{c}$, i.e., $\nabla f(t_0) = -\dot{f}(t_0)\hat{\mathbf{n}}/c$. Then

$$\nabla\Phi(\mathbf{x}, t) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \frac{x_j \dot{p}_j(t_0)}{4\pi r^2 c} \simeq \hat{\mathbf{e}}_i \frac{x_j}{4\pi r^2 c} \frac{\partial}{\partial x_i} \dot{p}_j(t_0) = \hat{\mathbf{e}}_i \frac{x_j}{4\pi r^2 c} \ddot{p}_j(t_0) \frac{\partial}{\partial x_i} t_0$$

$$= \frac{\hat{\mathbf{n}} \cdot \ddot{\mathbf{p}}(t_0)}{4\pi\epsilon_0 r c^2} \nabla t_0 = -\frac{\mu_0 \hat{\mathbf{n}}}{4\pi r} (\hat{\mathbf{n}} \cdot \ddot{\mathbf{p}}(t_0))\,; \qquad \frac{\partial}{\partial t}\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0 \ddot{\mathbf{p}}(t_0)}{4\pi r}$$

- Electric field in the radiation zone is

$$\Rightarrow \; \mathbf{E}(\mathbf{x}, t) \;=\; \frac{\mu_0}{4\pi r} [\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \ddot{\mathbf{p}}(t_0)) - \ddot{\mathbf{p}}(t_0)] \;=\; \frac{\mu_0}{4\pi r} \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \ddot{\mathbf{p}}(t_0))$$

Fields in Radiation Zone

- Magnetic field in the radiation zone is $\mathbf{B}(\mathbf{x},t) = \nabla \times \mathbf{A}(\mathbf{x},t)$

$$
\begin{aligned}
B_i(\mathbf{x},t) &= \epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\mu_0}{4\pi r} \dot{p}_k(t_0) \simeq \frac{\mu_0}{4\pi r} \epsilon_{ijk} \frac{\partial}{\partial x_j} \dot{p}_k(t_0) \\
&= \frac{\mu_0}{4\pi r} \epsilon_{ijk} \ddot{p}_k \frac{\partial t_0}{\partial x_j} = -\frac{\mu_0}{4\pi r c} \epsilon_{ijk} \ddot{p}_k(t_0) \mathbf{n}_j = -\left(\frac{\mu_0}{4\pi c r} \hat{\mathbf{n}} \times \ddot{\mathbf{p}}(t_0) \right)_i \\
\Rightarrow \ \mathbf{B}(\mathbf{x},t) &= -\frac{\mu_0}{4\pi c r} \hat{\mathbf{n}} \times \ddot{\mathbf{p}}(t_0) = \frac{\hat{\mathbf{n}}}{c} \times \mathbf{E}(\mathbf{x},t)
\end{aligned}
$$

- Choose the frame with OZ axis collinear to $\ddot{\mathbf{p}}(t_0)$, the fields take the form

$$
\mathbf{E}(r,\theta,\varphi) = \frac{\mu_0 \ddot{p}(t_0)}{4\pi} \frac{\sin\theta}{r} \hat{\boldsymbol{\theta}}, \qquad \mathbf{B}(r,\theta,\varphi) = \frac{\mu_0 \ddot{p}(t_0)}{4\pi c} \frac{\sin\theta}{r} \hat{\boldsymbol{\varphi}}
$$

- The Poynting vector is then

$$
\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = \frac{\mu_0}{16\pi^2 c} (\ddot{p}(t_0))^2 \frac{\sin^2\theta}{r^2} \hat{\mathbf{n}}
$$

- $\Rightarrow$ the total radiated power takes the form

$$
P = \int \mathbf{S} \cdot \hat{\mathbf{n}} \, dA = \frac{\mu_0}{16\pi^2 c} (\ddot{p}(t_0))^2 \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \underbrace{\int_0^{\pi} d\theta \, \sin^3\theta}_{4/3} = \frac{\mu_0}{6\pi c} (\ddot{p}(t_0))^2
$$

Larmor Formula

- The total radiated power

$$P = \frac{\mu_0}{6\pi c}(\ddot{p}(t_0))^2$$

- For a single point charge q we have $\mathbf{p}(t) = q\mathbf{x}(t)$, so we get the Larmor formula

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

- The Larmor formula can be also obtained using the Liénard-Wiechert potentials of the moving point charge