PHYSICS 453

Electromagnetism II
Lecture 11-3

Physics Department
Old Dominion University

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Outline

Liénard-Wiechert potentials

- The Liénard-Wiechert potentials for a point charge are

$$\Phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\,\frac{1}{|\mathbf{r}-\mathbf{w}(t_r)| - \mathbf{v}(t_r)\cdot(\mathbf{r}-\mathbf{w}(t_r))/c}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mathbf{v}(t_r)}{c^2}\,\Phi(\mathbf{r},t)$$

- Introducing the notation $\boldsymbol{\varsigma}(t) \equiv \mathbf{r}-\mathbf{w}(t)$, we obtain

$$\begin{aligned}
\Phi(\mathbf{r},t) &= \frac{q}{4\pi\epsilon_0}\,\frac{1}{\varsigma(t_r) - \mathbf{v}(t_r)\cdot\boldsymbol{\varsigma}(t_r)/c} \\
\mathbf{A}(\mathbf{r},t) &= \frac{\mathbf{v}(t_r)}{c^2}\,\Phi(\mathbf{r},t) = \frac{q\mu_0}{4\pi}\,\frac{\mathbf{v}(t_r)}{\varsigma(t_r) - \mathbf{v}(t_r)\cdot\boldsymbol{\varsigma}(t_r)/c}
\end{aligned}$$

- According to these formulas, the field at the point of observation at time t is determined by the state of motion of the charge at the earlier time t_r
- Also, $\varsigma(t) = \mathbf{r}-\mathbf{w}(t)$ is the radius vector from the charge q to the observation point P; like $\mathbf{w}(t)$ it is a given function of the time
- The time t_r is determined by the equation $t_r + \varsigma(t_r)/c = t$

Electric and Magnetic Fields, cont.

- In Lecture 10-3 we obtained the corresponding expressions for the fields

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 \left(\varsigma - \boldsymbol{\varsigma} \cdot \mathbf{v}/c\right)^3} \left\{ (1 - v^2/c^2) \left(\boldsymbol{\varsigma} - \frac{\mathbf{v}}{c} \varsigma \right) + \frac{1}{c^2} \boldsymbol{\varsigma} \times \left[\left(\boldsymbol{\varsigma} - \frac{\mathbf{v}}{c} \varsigma \right) \times \mathbf{a} \right] \right\}$$

$$\mathbf{B} = \frac{1}{\varsigma} \boldsymbol{\varsigma} \times \mathbf{E}$$

  Here, $\mathbf{a} = \partial \mathbf{v} / \partial t_r$
- All quantities on the right sides of the equations refer to the time t_r
- Note that the magnetic field is everywhere perpendicular to the electric
- In the non-relativistic limit, the electric field $\mathbf{E}$ reduces to the Coulomb field

$$\mathbf{E}(\mathbf{r}, t)|_{v/c \to 0} \quad \to \quad \frac{q \boldsymbol{\varsigma}}{4\pi\epsilon_0 \varsigma^3} \ ,$$

  while the magnetic field tends to zero as v/c
- Introduce the unit vector $\hat{\boldsymbol{\varsigma}} \equiv \boldsymbol{\varsigma}/\varsigma$, and also the notation $\mathbf{u} \equiv \hat{\boldsymbol{\varsigma}} - \mathbf{v}(t_r)/c$
- This vector tends to $\hat{\boldsymbol{\varsigma}}$ in the non-relativistic limit $v/c \to 0$
- Using $\varsigma - \boldsymbol{\varsigma} \cdot \mathbf{v}/c = (\boldsymbol{\varsigma} \cdot \mathbf{u})$, we can write the electric and magnetic fields as

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \frac{q}{4\pi\epsilon_0} \frac{\varsigma}{(\boldsymbol{\varsigma} \cdot \mathbf{u})^3} \left[\mathbf{u}(1 - v^2/c^2) + \boldsymbol{\varsigma} \times (\mathbf{u} \times \mathbf{a})/c^2 \right] \\ \mathbf{B}(\mathbf{r}, t) &= \hat{\boldsymbol{\varsigma}} \times \mathbf{E}(\mathbf{r}, t)/c \end{aligned}$$

Electric and Magnetic Fields, cont.

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\,\frac{\varsigma}{(\boldsymbol{\varsigma}\cdot\mathbf{u})^3}\left[\mathbf{u}(1-v^2/c^2) + \boldsymbol{\varsigma}\times(\mathbf{u}\times\mathbf{a})/c^2\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\hat{\boldsymbol{\varsigma}}}{c}\times\mathbf{E}(\mathbf{r},t)$$

- As usual, velocity and acceleration are taken at $t = t_r$.
  (recall that t_r is defined as a solution to the equation $c(t - t_r) = \varsigma$)
- The electric field consists of two parts of different type:
  the first term ($\sim \mathbf{u}$) is called the velocity field, and
  the second ($\sim \mathbf{a}$) is called the acceleration or the radiation field.
- The first term varies at large distances like $1/\varsigma^2$.
  Since it is independent of the acceleration, it corresponds to the field produced by a uniformly moving charge. For large distances ς, the velocity field decreases as $1/\varsigma^2$
- The total field $\mathbf{E}(\mathbf{r},t)$ contains also a radiation component $\mathbf{E}^{\mathrm{rad}}(\mathbf{r},t)$. It decreases as $1/\varsigma$ for large ς, and is given by

$$\mathbf{E}_{\mathrm{rad}}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0 c^2\varsigma}\,\frac{\hat{\boldsymbol{\varsigma}}\times(\mathbf{u}\times\mathbf{a})}{(\hat{\boldsymbol{\varsigma}}\cdot\mathbf{u})^3} = \frac{q}{4\pi\epsilon_0 c^2\varsigma}\,\frac{\hat{\boldsymbol{\varsigma}}\times(\mathbf{u}\times\mathbf{a})}{(1-\hat{\boldsymbol{\varsigma}}\cdot\mathbf{v}/c)^3}$$

- The radiation magnetic field also decreases as $1/\varsigma$ for large ς

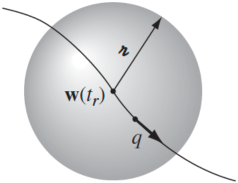
$$\mathbf{B}_{\mathrm{rad}}(\mathbf{r},t) = \frac{\hat{\boldsymbol{\varsigma}}}{c}\times\mathbf{E}_{\mathrm{rad}}(\mathbf{r},t)$$

Power radiated by a point charge

- The Poynting vector for the radiation component is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0 c}\mathbf{E}\times(\hat{\boldsymbol{\varsigma}}\times\mathbf{E}) = \frac{1}{\mu_0 c}[E^2\hat{\boldsymbol{\varsigma}} - (\hat{\boldsymbol{\varsigma}}\cdot\mathbf{E})\mathbf{E}]$$

- Some of the energy is radiation; another part is just a field energy carried along by the particle as it moves
- To calculate the power radiated by the particle at time t_*, we draw a large sphere with radius $\varsigma = R$, wait for $t - t_* = \frac{R}{c}$, and integrate Poynting vector over the surface



- The velocity field is $\sim 1/R^2$. The corresponding $P_{\rm rad}$ is $\sim R^2\frac{1}{R^4} = \frac{1}{R^2}$ so it does not contribute to the radiated power at large R

Larmor formula

$$\mathbf{S} = \frac{1}{\mu_0 c}[E^2 \hat{\boldsymbol{\varsigma}} - (\hat{\boldsymbol{\varsigma}} \cdot \mathbf{E})\mathbf{E}]$$

- The power due to the acceleration field ($\sim 1/R$) is finite: $P_{\text{rad}} \sim R^2 \frac{1}{R^2} = 1$

$$\mathbf{E}_{\text{rad}}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2 \varsigma} \frac{\hat{\boldsymbol{\varsigma}} \times (\mathbf{u} \times \mathbf{a})}{(\hat{\boldsymbol{\varsigma}} \cdot \mathbf{u})^3}$$

$$\rightarrow \boldsymbol{\varsigma} \cdot \mathbf{E}_{\text{rad}}(\mathbf{r}, t) = 0 \Rightarrow \mathbf{S}_{\text{rad}} = \frac{\hat{\boldsymbol{\varsigma}}}{\mu_0 c} E^2_{\text{rad}}$$

- For simplicity, consider the charge which is instantaneously at rest at $t = t_*$
- Since $\mathbf{v}(t_*) = 0$, $\mathbf{u}(t_*) = \hat{\boldsymbol{\varsigma}}$, so

$$\mathbf{S}_{\text{rad}} = \frac{\hat{\boldsymbol{\varsigma}}}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi R}\right)^2 [a^2 - (\hat{\boldsymbol{\varsigma}} \cdot \mathbf{a})^2] = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\sin^2\theta}{R^2} \hat{\boldsymbol{\varsigma}}$$

- The total power is given by the following Larmor formula

$$P_{\text{rad}} = \oint_S \mathbf{S}_{\text{rad}} \cdot \mathbf{dS} = \frac{\mu_0 q^2 a^2}{16\pi^2 c R^2} \int \frac{\sin^2\theta}{R^2} R^2 \sin\theta d\theta d\varphi = \frac{\mu_0 q^2 a^2}{6\pi c}$$

- We have already obtained this using the electric dipole radiation formulas

Larmor formula, cont.

- We have derived the Larmor formula under the assumption that $v = 0$
- One can demonstrate that it holds true as long as $v \ll c$
- In the general case of arbitrary velocity,
  the radiation is given by the Lienard formula

$$P_{\mathrm{rad}} = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \frac{(\mathbf{v}\cdot\mathbf{a})^2}{c^2}\right)$$

where $\gamma \equiv 1/\sqrt{1 - \frac{v^2}{c^2}}$