PHYSICS 453

Electromagnetism II
Lecture 11-4

Physics Department
Old Dominion University

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Outline

Abraham-Lorentz formula

- The radiation exerts a force $\mathbf{F}_{\mathrm{rad}}$ back on the charge - a *recoil* force.
- First, we derive the **radiation reaction** force from conservation of energy.
- Larmor formula:

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

- Conservation of energy $\Rightarrow$ this is also the rate at which the particle loses energy due to radiation reaction force $\mathbf{F}_{\mathrm{rad}}$:

$$\mathbf{F}_{\mathrm{rad}} \cdot \mathbf{v} = \frac{\mu_0 q^2 a^2}{6\pi c}$$

- However, the contribution of it velocity field was neglected

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\varsigma}{(\boldsymbol{\varsigma} \cdot \mathbf{u})^3} [(c^2 - v^2)\mathbf{u} + \boldsymbol{\varsigma} \times (\mathbf{u} \times \mathbf{a})]$$

Abraham-Lorentz formula, cont.

- If the system returns to its initial state, then the energy in the velocity fields is the same at both ends, and the only net loss is in the form of radiation.
- If the state of the system is identical at t_1 and t_2

$$\int_{t_1}^{t_2} dt\, \mathbf{F}_{\mathrm{rad}} \cdot \mathbf{v} \;=\; -\frac{\mu_0 q^2}{6\pi c} \int_{t_1}^{t_2} dt\, a^2(t)$$

- Integrate by parts

$$\int_{t_1}^{t_2} dt\, \frac{d\mathbf{v}}{dt} \cdot \frac{d\mathbf{v}}{dt} \;=\; \left(\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}\right)\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\, \frac{d^2\mathbf{v}}{dt^2} \cdot \mathbf{v}$$

- For identical states at t_1 and t_2

$$\int_{t_1}^{t_2} dt \left(\mathbf{F}_{\mathrm{rad}} - \frac{\mu_0 q^2}{6\pi c} \frac{d\mathbf{a}}{dt}\right) \cdot \mathbf{v} \;=\; 0$$

- $\Rightarrow$ Abraham-Lorentz formula

$$\mathbf{F}_{\mathrm{rad}} \;=\; \frac{\mu_0 q^2}{6\pi c} \frac{d\mathbf{a}}{dt}$$

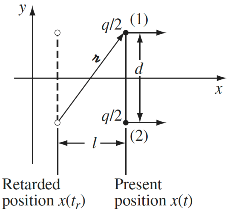
Mechanism for the Radiation Reaction

- The dumbbell moves in the x direction, and is (instantaneously) at rest at the retarded time. The electric field at (1) due to (2) is

$$\mathbf{E}_1(\mathbf{r},t) = \frac{q/2}{4\pi\epsilon_0}\frac{\varsigma}{(\boldsymbol{\varsigma}\cdot\mathbf{u})^3}[(c^2-v^2)\mathbf{u}+\boldsymbol{\varsigma}\times(\mathbf{u}\times\mathbf{a})]$$

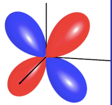
$$\mathbf{u} = c\hat{\boldsymbol{\varsigma}}, \quad \boldsymbol{\varsigma} = l\hat{\mathbf{e}}_1 + d\hat{\mathbf{e}}_2, \quad \varsigma = \sqrt{l^2+d^2},$$

$$\boldsymbol{\varsigma}\cdot\mathbf{u} = c\varsigma \quad \boldsymbol{\varsigma}\cdot\mathbf{a} = la, \quad \boldsymbol{\varsigma}\times(\mathbf{u}\times\mathbf{a}) = c(\hat{\boldsymbol{\varsigma}}la - \mathbf{a}\varsigma)$$



$$l = x(t) - x(t_r)$$

$$\Rightarrow \mathbf{E}_1(\mathbf{r},t) = \frac{q}{8\pi\epsilon_0 c^2}\frac{1}{(l^2+d^2)^{\frac{3}{2}}}[(c^2 l - d^2 a)\hat{\mathbf{e}}_1 + (c^2 - al)d\hat{\mathbf{e}}_2]$$



Mechanism for the Radiation Reaction, cont.

- By symmetry, the electric field at (2) due to (1) is

$$\Rightarrow \mathbf{E}_2(\mathbf{r},t) = \frac{q}{8\pi\epsilon_0 c^2}\frac{1}{(l^2+d^2)^{\frac{3}{2}}}[(c^2 l - d^2 a)\hat{\mathbf{e}}_1 + (c^2 - al)d\hat{\mathbf{e}}_2]$$

$$\Rightarrow \mathbf{F}_{\text{self}} = \frac{q}{2}(\mathbf{E}_1+\mathbf{E}_2) = \frac{q^2}{8\pi\epsilon_0 c^2}\frac{c^2 l - d^2 a}{(l^2+d^2)^{\frac{3}{2}}}\hat{\mathbf{e}}_1$$

- Expand at small $\tau \equiv t - t_r$

$$l = x(t) - x(t_r) = \frac{a}{2}\tau^2 + \frac{\dot{a}}{6}\tau^3 + ..., \qquad \tau \equiv t - t_r$$

- Retarded time condition $c^2\tau^2 = l^2 + d^2$

$$\Rightarrow d = \sqrt{c^2\tau^2 - l^2} = c\tau\sqrt{1 - (\frac{a\tau}{2c} + \frac{\dot{a}\tau^2}{6c} + ...)} = c\tau - \frac{a^2}{8c}\tau^3 + O(\tau^4)$$

- Convert into expansion at small d

$$d = c\tau - \frac{a^2}{8c}\tau^3 + O(\tau^4) \Leftrightarrow \tau = \frac{1}{c}d + \frac{a^2}{8c^5}d^3 + O(d^4)$$

Mechanism for the Radiation Reaction, cont.

- Expand l at small d

$$l = x(t) - x(t_r) = \frac{a}{2}\tau^2 + \frac{\dot{a}}{6}\tau^3 + O(\tau^4) = \frac{a}{2c^2}d^2 + \frac{\dot{a}}{6c^3}d^3 + O(d^4) \Rightarrow$$

$$\mathbf{F}_{\text{self}} = \frac{q\hat{\mathbf{e}}_1}{2}(\mathbf{E}_1 + \mathbf{E}_2) = \frac{q^2}{8\pi\epsilon_0 c^2}\frac{c^2 l - d^2 a(t_r)}{(l^2 + d^2)^{\frac{3}{2}}} = \frac{q^2\hat{\mathbf{e}}_1}{4\pi\epsilon_0}\left[\frac{-a(t_r)}{4c^2 d} + \frac{\dot{a}(t_r)}{12c^3} + O(d)\right]$$

- Finally, use $a(t_r) = a(t) - \tau\dot{a}(t) + O(\tau^2)$ to get

$$\mathbf{F}_{\text{self}} = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{a(t_r)}{4c^2 d} + \frac{\dot{a}(t_r)}{3c^3} + O(d)\right]\hat{\mathbf{e}}_1$$

- First term is a "mass renormalization" $m = 2m_0 + \frac{U_{\text{pot}}}{c^2}$

$$2m_0\mathbf{a} = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{a(t_r)}{4c^2 d} + \frac{\dot{a}(t_r)}{3c^3}\right]\hat{\mathbf{e}}_1 \Leftrightarrow \left(2m_0 + \frac{q^2}{16\pi\epsilon_0 d c^2}\right)\mathbf{a} = \frac{q^2}{4\pi\epsilon_0}\frac{\dot{a}(t_r)}{3c^3}\hat{\mathbf{e}}_1$$

- Second term is the radiation reaction

$$\mathbf{F}_{\text{rad}}^{\text{int}} = \frac{\mu_0 q^2\dot{\mathbf{a}}}{12\pi c},$$

- We've got *half* of $\mathbf{F}_{\text{rad}}$

$$\mathbf{F}_{\text{self}}^{\text{int}} = \frac{1}{2}\mathbf{F}_{\text{self}} = \frac{\mu_0 q^2\dot{\mathbf{a}}}{6\pi c}$$