



## Lecture 12-3

### Four Vectors

Euclidean Rotations

Vectors

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Rotations:

Co-Vectors

Euclidean

Rotations: Tensors

Metric Tensor

4-vectors in

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Lorentz

Transformations

Derivatives

Exercise

# PHYSICS453

## Electromagnetism II

### Lecture 12-3

Physics Department  
Old Dominion University

April 9, 2026



# Outline

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## Special Relativity and Four Vectors

- Euclidean Rotations
  - Vectors
  - Co-Vectors
  - Tensors
  - Metric Tensor
- 4-vectors in Minkowski Space-Time
  - Lorentz Transformations and Four Vectors
  - Derivatives
- Exercise



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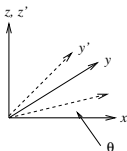
## Lorentz

## Transformations

## Derivatives

## Exercise

- Reminder: rotations in Euclidean space



- Consider two co-ordinate systems  $P, P'$
- Their origins coincide, but they are related by rotation through an angle  $\theta$
- The coordinates of a point in the two systems are related through rotation matrix  $R$

$$x'^i = R_j^i x^j$$

- Note that we have put the indices **upstairs** on the vectors
- For the specific case of a rotation through  $\theta$  about the  $z$  axis

$$R = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Quantities that transform as

$$A'^i = R_j^i A^j = \frac{\partial x'^i}{\partial x^j} A^j$$

are called **vectors**



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- A simple example of a vector is  $dx$ , which transforms as

$$dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j = R_j^i dx^j$$

- A scalar is a quantity which transforms as  $f' = f$ .
- Let us now consider how the **gradient** of a function transforms:

$$\nabla'_i f = \frac{\partial f}{\partial x'^i} = \frac{\partial f}{\partial x^j} \frac{\partial x^j}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i} \frac{\partial f}{\partial x^j}$$

- This is an example of the transformation property

$$B'_i = \frac{\partial x^j}{\partial x'^i} B_j,$$

which is *different* from that for vectors

- Quantities that transform in this way are known as **covectors** or **forms**
- We put their indices downstairs
- Summarizing, we have

$$\left. \begin{array}{l} \text{Vector: } A'^i = \frac{\partial x'^i}{\partial x^j} A^j \\ \text{Scalar: } f' = f \\ \text{Covector: } B'_i = \frac{\partial x^j}{\partial x'^i} B_j \end{array} \right\}$$



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- Finally, we have that a **tensor** is an object that transforms as a *vector* on each *upstairs* index, and a *covector* on each *downstairs* index

$$C_{k'l'...}^{i'j'...} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^{j'}}{\partial x^j} \cdots \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^l}{\partial x^{l'}} \cdots C_{kl...}^{ij...}$$

- The **length** of a vector is a bilinear, and independent of the choice of frame
- Define the **inner product** of two vectors by

$$X \cdot Y = g_{ij} X^i Y^j.$$

- We call  $g_{ij}$  the **metric tensor**
- In Cartesian coordinates  $(x, y, z)$ , we have  $g_{ij} = \delta_{ij}$ , since

$$(dl)^2 = (dx)^2 + (dy)^2 + (dz)^2$$



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- In spherical coordinates  $(r, \theta, \varphi)$ , we have

$$(dl)^2 = (dr)^2 + r^2(d\theta)^2 + r^2 \sin^2 \theta (d\varphi)^2,$$

hence

$$g_{ij} = \text{diag}(1, r^2, r^2 \sin^2 \theta)$$

- We can use the metric tensor to *raise* or *lower* indices:

$$X_i = g_{ij} X^j$$

$$X \cdot Y = X^i Y_i = X_i Y^i$$

- We only have the luxury of identifying *vectors* with *covectors* in Cartesian coordinates in Euclidean space
- In that case, the components of the two are numerically equal
- For instance, in spherical coordinates, taking

$$dx^i = \{dr, d\theta, d\varphi\}$$

as a vector, we have

$$dx_i = \{dr, r^2 d\theta, r^2 \sin^2 \theta d\varphi\}$$

as the corresponding co-vector



# 4-vectors in Minkowski Space-Time

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#### Exercise

- Apply these ideas to Lorentz transformations of four-dimensional space-time
- Denote “ $ct$ ” as the coordinate  $x_0$ , and write a **contravariant** four vector as

$$x^\mu \equiv (ct, x, y, z) = (x^0, x^1, x^2, x^3)$$

- Its “length” is the **interval** left invariant under Lorentz transformations
- We define the inner product of two vectors by

$$x \cdot y = -x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3 = g_{\mu\nu} x^\mu y^\nu$$

- We immediately see that the metric tensor is

$$g_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \quad (*)$$

- It is conventional to use *Greek Letters* for the components of a four-vector
- Four vectors are not underlined or printed in bold
- In some areas of physics, time is introduced as the *fourth* component
- Furthermore, the metric can be defined such that the temporal components are positive, and the spatial component negative. Such convention  $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$  is probably the most widely used. We will follow *Griffiths’* convention (\*).
- The summation convention is as follows:  
*An index can appear no more than twice. Any index appearing twice must have one upper index and one lower index, and that index is summed over*



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- The **covariant four vector** or **form** can be obtained as before by using the raising and lowering properties of the metric tensor

$$x_\mu = g_{\mu\nu} x^\nu = (-x^0, x^1, x^2, x^3) = (-ct, x, y, z)$$

- $\Rightarrow$  For 4-vectors in Minkowski space *the components of a co-vector are numerically different to those of the vector*
- The relation between vectors (in 4-dimensional case of special relativity, they are called *contravariant vectors*) in the two frames is given by

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu = L^\mu_\nu x^\nu$$

- Let us assume a similar transformation law for the 4-dimensional analogs of covectors (called *covariant vectors*)

$$x'_\mu = L_\mu^\nu x_\nu$$

- We require that  $x^\mu x_\mu$  is invariant under the Lorentz transformation

$$x^\mu x_\mu = x'^\mu x'_\mu = L^\mu_\nu L_\mu^\sigma x^\nu x_\sigma$$

and since this is true for all vectors, we have

$$L^\mu_\nu L_\mu^\sigma = \delta_\nu^\sigma \quad \text{where} \quad \delta_\nu^\sigma = \begin{cases} 1 & \text{if } \nu = \sigma \\ 0 & \text{if } \nu \neq \sigma \end{cases}$$



# Lorentz Transformations and Four Vectors, cont.

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- To find  $L_{\mu}^{\sigma}$ , we note that, according to  $x'_{\mu} = L_{\mu}^{\nu} x_{\nu}$ , we have

$$L_{\nu}^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}$$

- Now, use the identity

$$\frac{\partial x^{\sigma}}{\partial x^{\nu}} = \delta_{\nu}^{\sigma}$$

- Write it through the chain rule as

$$\delta_{\nu}^{\sigma} = \frac{\partial x^{\sigma}}{\partial x^{\nu}} = \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial x'^{\mu}}{\partial x^{\nu}} = \underbrace{\frac{\partial x'^{\mu}}{\partial x^{\nu}}}_{L_{\nu}^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}}$$

- Comparing with  $L_{\nu}^{\mu} L_{\mu}^{\sigma} = \delta_{\nu}^{\sigma}$ , we conclude that

$$L_{\mu}^{\sigma} = \frac{\partial x^{\sigma}}{\partial x'^{\mu}}$$

- This corresponds to the characteristic transformation property of a co-vector  $\partial/\partial x^{\mu}$ :

$$\frac{\partial f}{\partial x'^{\mu}} = \frac{\partial f}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x'^{\mu}} = \underbrace{\frac{\partial x^{\nu}}{\partial x'^{\mu}}}_{L_{\mu}^{\nu}} \frac{\partial f}{\partial x^{\nu}}$$



# Lorentz Transformations and Four Vectors, cont.

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Exercise

- The various quantities we will encounter in the remainder of this course are
- **Contravariant Vectors:**

$$A'^{\mu} = L^{\mu}_{\nu} A^{\nu}$$

- **Covariant Vectors:**

$$B'_{\mu} = L^{\nu}_{\mu} B_{\nu}$$

- **Tensors:**

$$C'^{\mu'\nu'\dots}_{\rho'\sigma'\dots} = L^{\mu'}_{\mu} L^{\nu'}_{\nu} \dots L^{\rho}_{\rho'} L^{\sigma}_{\sigma'} \dots C^{\mu\nu\dots}_{\rho\sigma\dots}$$

- **Scalars:**

$$A \cdot B = A_{\mu} B^{\mu} = g_{\mu\nu} A^{\mu} B^{\nu}$$



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- As we have noted earlier, the derivatives transform as *covectors*

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha} = \left( \frac{\partial}{\partial x^0}, \nabla \right)$$

$$\partial^\alpha = \frac{\partial}{\partial x_\alpha} = \left( -\frac{\partial}{\partial x^0}, \nabla \right)$$

- Suppose now that we have a four vector  $A^\mu$ . Then

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \nabla \cdot \mathbf{A}$$

- The Laplacian is defined by

$$\square = \partial_\alpha \partial^\alpha = -\frac{\partial^2}{\partial x^{02}} + \nabla^2$$



- Let us demonstrate that  $g_{\mu\sigma} A^\sigma$  is indeed a covector, i.e. transforms according to  $B'_\mu = L_\mu^\nu B_\nu$
- We need to show that

$$(g_{\mu\sigma} A^\sigma)' = L_\mu^\nu (g_{\nu\lambda} A^\lambda)$$

- The l.h.s. is  $g_{\mu\sigma} (A^\sigma)' = g_{\mu\sigma} L_\sigma^\lambda A^\lambda$  and therefore we must prove that

$$g_{\mu\sigma} L_\sigma^\lambda = L_\mu^\nu g_{\nu\lambda}$$

- Multiplying this relation by  $L_\rho^\lambda$  and using  $L_\sigma^\lambda L_\rho^\lambda = \delta_\rho^\sigma$  converts it into

$$g_{\mu\rho} = g_{\nu\lambda} L_\mu^\nu L_\rho^\lambda$$

- The last equation follows from the definition  $L_\mu^\nu \equiv \partial x'^\nu / \partial x^\mu$  and

$$\frac{\partial^2}{\partial x'^\mu \partial x'^\rho} g_{\nu\lambda} x^\nu x^\lambda = \frac{\partial^2}{\partial x'^\mu \partial x'^\rho} g_{\nu\lambda} x'^\nu x'^\lambda$$

- Indeed, we have, first,  $g_{\nu\lambda} x^\nu x^\lambda = x^2$ ,  $g_{\nu\lambda} x'^\nu x'^\lambda = x'^2$ , and  $x'^2 = x^2$



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$$g_{\nu\lambda} x^\nu x^\lambda = x^2 \quad , \quad g_{\nu\lambda} x'^\nu x'^\lambda = x'^2 \quad , \quad \text{and} \quad x'^2 = x^2$$

- Then

$$\frac{\partial^2}{\partial x'^\mu \partial x'^\rho} g_{\nu\lambda} x'^\nu x'^\lambda = g_{\nu\lambda} [\delta_\mu^\nu \delta_\rho^\lambda + \delta_\rho^\nu \delta_\mu^\lambda] = 2g_{\mu\rho} \quad ,$$

$$\begin{aligned} \frac{\partial^2}{\partial x'^\mu \partial x'^\rho} g_{\nu\lambda} x^\nu x^\lambda &= g_{\nu\lambda} \left[ \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}}_{L_\mu^\nu} \underbrace{\frac{\partial x^\lambda}{\partial x'^\rho}}_{L_\rho^\lambda} + \{\mu \leftrightarrow \rho\} \right] \\ &= g_{\nu\lambda} [L_\mu^\nu L_\rho^\lambda + L_\rho^\nu L_\mu^\lambda] = 2g_{\nu\lambda} L_\mu^\nu L_\rho^\lambda \end{aligned}$$

- On the last step we used the fact that  $g_{\nu\lambda}$  is a symmetric tensor
- The metric tensor with upper indices  $g^{\mu\nu}$  defines the inner product

$$x \cdot y = g^{\mu\nu} x_\mu y_\nu$$

in terms of covariant vectors. This product is invariant under Lorentz transformations if  $g^{\mu\nu} y_\nu$  transforms as a contravariant vector  $y^\mu$

- Using  $y_\nu = g_{\nu\sigma} y^\sigma$ , we conclude that

$$g^{\mu\nu} g_{\nu\sigma} = \delta_\sigma^\mu \quad ,$$

i.e., the matrices  $g^{\mu\nu}$  and  $g_{\mu\nu}$  are inverse to each other

- In our case  $g^{\mu\nu} = g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$