Lecture 12-4

PHYSICS 453

Electromagnetism II
Lecture 12-4

Physics Department
Old Dominion University

April 16, 2026

Outline

Four Velocity

- Define velocity in a usual way as $v^i = dx^i/dt$, and use that $t = x^0/c$
- This $v^i = c\, dx^i/dx^0$ cannot transform as a vector under Lorentz transformations
- A formal reason is that such a derivative is a $0i$ component of a 4-tensor
- Thus, it does not transform as an ith component of a 4-vector
- Indeed, let us assume that the object dx^μ/dt transforms as a 4-vector,

$$\mathcal{V}^\mu \equiv \frac{dx^\mu}{dt} = \frac{d}{dt}\{x_0, \mathbf{x}\} = \frac{d}{dt}\{ct, \mathbf{x}\} = \{c, \mathbf{v}\}$$

- Consider frames K and K', moving with velocity $\mathbf{V}$ with respect to K
- Take $\mathbf{V}$ and $\mathbf{v}$ along x^3 axis

$$\mathcal{V}^\mu = \{c, \mathbf{0}_\perp, \mathcal{V}^3\} \quad , \quad \mathcal{V}'^\mu \equiv \frac{dx'^\mu}{dt'} = \{c, \mathbf{0}_\perp, \mathcal{V}'^3\}$$

- If $\mathcal{V}^\mu$ is a 4-vector, then, according to the Lorentz transformation,

$$\mathcal{V}'^3 = \gamma_V \left(\mathcal{V}^3 - \frac{V}{c}\mathcal{V}^0\right) = \gamma_V \left(\mathcal{V}^3 - V\right) ,$$

where $\gamma_V = 1/\sqrt{1 - V^2/c^2}$. This gives

$$\mathcal{V}'^3 = \left(\mathcal{V}^3 - V\right)/\sqrt{1 - V^2/c^2}$$

Four Velocity, cont.

$$\mathcal{V}'^3 = \gamma_V \left(\mathcal{V}^3 - \frac{V}{c} \mathcal{V}^0 \right) = \gamma_V \left(\mathcal{V}^3 - V \right) = \left(\mathcal{V}^3 - V \right) / \sqrt{1 - V^2/c^2} \,,$$

- The correct result is that the velocity in the K' frame should be given by

$$v' = \frac{v - V}{1 - vV/c^2}$$

- This is the relativistic velocity addition formula. (Note that we should take into account that K frame moves with respect to K' frame with the velocity $-\mathbf{V}$)
- So, the question is whether it is possible to find a definition of a velocity that does indeed transform covariantly under Lorentz transformations, yet reduces to a Galilean transformation for $v \ll c$?

Four Velocity, cont.

- To construct a **four velocity**, we need to take the derivative of the 4-vector x^μ with respect to some time that, unlike dt or dt', is the same in all frames, i.e. is a *Lorentz Scalar*
- Such a scalar is provided by the **Proper Time** $d\tau$, or time measured in the frame that moves together with the particle
- This frame has velocity $\mathbf{v}$ in the K frame. Proper time is defined by

$$c^2 d\tau^2 = -ds^2,$$

  where ds is the Lorentz-invariant *interval*
- The proper time is a scalar, and a natural definition of the **four velocity** is

$$\eta^\alpha = \frac{dx^\alpha}{d\tau}$$

- Recalling that the proper time is related to the K frame time by

$$d\tau = dt\sqrt{1-\beta^2(t)} \quad \text{we have}$$

$$\eta^\alpha = \frac{1}{\sqrt{1-\beta^2}}\frac{d}{dt}(ct, \mathbf{x}) = \gamma(c, \mathbf{v}), \quad \text{or} \quad \eta^\alpha = (\gamma c, \gamma \mathbf{v})$$

- The spatial components of v^μ clearly reduce to our familiar definition of velocity in the non-relativistic (NR) limit
- Note that $\eta^\alpha \eta_\alpha = c^2$

- Let us take now the component of $\mathbf{v}$ parallel to the relative velocity $\mathbf{V}$ and check that applying the Lorentz transformation to η^μ, namely

$$\eta'^3 = \gamma_V \left(\eta^3 - \frac{V}{c}\eta^0\right) \quad , \quad \eta'^0 = \gamma_V \left(\eta^0 - \frac{V}{c}\eta^3\right)$$

leads to the correct relativistic velocity addition formula

- Indeed, substituting

$$\eta^0 = \gamma_v c \ , \ \ \eta^3 = \gamma_v v$$

(where $\gamma_v = 1/\sqrt{1 - v^2/c^2}$) and

$$\eta'^0 = \gamma_{v'} c \ , \ \ \eta'^3 = \gamma_{v'} v'$$

(where $\gamma_{v'} = 1/\sqrt{1 - v'^2/c^2}$), we get

$$\eta'^3 = v'\gamma_{v'} = \gamma_V \gamma_v (v - V) \quad , \quad c\gamma_{v'} = \gamma_V \gamma_v \left(c - \frac{V}{c} v\right)$$

- Dividing the first of these equations by the second one gives

$$\frac{v'}{c} = \frac{v - V}{c - \frac{V}{c} v} \quad \text{or} \quad v' = \frac{v - V}{1 - \frac{V}{c^2} v}$$

Four Momentum

- The definition of a Lorentz-covariant 4-momentum is now straightforward:

$$p^\mu = m\eta^\mu = (m\gamma c, m\gamma \mathbf{v}),$$

  where m is a **Lorentz scalar** that we will call the **rest mass**
- Spatial components of p^μ reduce to our usual definition of momentum at $v \ll c$
- To interpret the temporal component, we will look at its NR limit:

$$p^0 = m\gamma c = mc\left\{1 - v^2/c^2\right\}^{-1/2} = \frac{1}{c}\left\{mc^2 + \frac{1}{2}mv^2 + \mathcal{O}(v^4/c^2)\right\}$$

- The second term in braces is clearly the kinetic energy
- The first term we identify as the **rest energy**, and write

$$p^0 = E/c$$

  where E is the **energy**
- Thus the four momentum contains both the energy and the three momentum
- The "length" of p^μ is a Lorentz scalar

$$\begin{aligned} -p^\mu p_\mu &= m^2\gamma^2c^2 - m^2\gamma^2v^2 = m^2\gamma^2c^2\left[1 - v^2/c^2\right] \\ &= m^2\gamma^2c^2\gamma^{-2} = m^2c^2 \end{aligned}$$

- Thus we have $-p^\mu p_\mu = -p^2 = m^2c^2$ confirming that the rest mass is a (frame-independent) scalar

$E = mc^2$

- Finally, if we now go back and write $p^\mu p_\mu = m^2c^2$ in terms of our old-fashioned three vectors we have

$$\frac{1}{c^2}E^2 - \mathbf{p}^2 = m^2c^2$$
$$\Longrightarrow E^2 = m^2c^4 + c^2\mathbf{p}^2.$$

- For a particle at rest, we have perhaps the most famous equation in physics

$$E = mc^2$$

- The use of four-vectors is **essential** to solve problems in special (and general...) relativity
- While simple kinematical problems can be solved using three vectors, it is very clumsy indeed

- NB: Experimental fact - the momentum and energy defined as above are *conserved* (for closed systems)

Energy-momentum conservation in application to $1 \to 2$ decay process

- Consider a particle of mass M that decays at rest into two particles of masses m_1 and m_2
- Energy-momentum conservation requires that in any frame

$$P = p_1 + p_2$$

- P^μ is 4-momentum of the initial particle, $-P^2 = M^2c^2$
- $p_{1,2}$ are 4-momenta of final particles, $-p_{1,2}^2 = m_{1,2}^2c^2$
- In the rest frame of the decaying particle we have

$$P = (Mc, \mathbf{0}) \ , \ p_1 = (\frac{E_1}{c}, \mathbf{p}_1) \ , \ p_2 = (\frac{E_2}{c}, \mathbf{p}_2)$$

- Hence,

$$E_1 + E_2 = Mc^2 \qquad , \qquad \mathbf{p}_1 = -\mathbf{p}_2 \equiv \mathbf{p}$$

- Find first the energies of the final particles. Writing $p_2 = P - p_1$, we have

$$p_2^2 = P^2 - 2(Pp_1) + p_1^2 \ ,$$

$$m_2^2c^2 = M^2c^2 - 2(ME_1 - \mathbf{0}\cdot\mathbf{p}_1) + m_1^2 \ \Rightarrow \ m_2^2c^2 = M^2c^2 - 2ME_1 + m_1^2c^2$$

which gives

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}c^2 = \frac{Mc^2}{2} + \frac{m_1^2 - m_2^2}{2M}c^2$$

$1 \to 2$ decay process, cont.

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} c^2 = \frac{Mc^2}{2} + \frac{m_1^2 - m_2^2}{2M} c^2 \, .$$

- Interchanging $1 \leftrightarrow 2$, we get

$$E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M} c^2 = \frac{Mc^2}{2} - \frac{m_1^2 - m_2^2}{2M} c^2$$

- Here $E_{1,2}$ are relativistic energies that include the rest mass term
- The kinetic energy of the first final particle is given by

$$\begin{aligned} E_1^{\text{kin}} &= \frac{M^2 + m_1^2 - m_2^2}{2M} c^2 - m_1 c^2 = \frac{M^2 - 2Mm_1 + m_1^2 - m_2^2}{2M} c^2 \\ &= \frac{(M - m_1)^2 - m_2^2}{2M} c^2 = \frac{(M - m_1 - m_2)(M - m_1 + m_2)}{2M} c^2 \\ &= \boxed{\frac{\Delta M}{2}} \left[1 - \frac{m_1 - m_2}{M} \right] c^2 = \boxed{\Delta M} \left[1 - \frac{\Delta M}{2M} - \frac{m_1}{M} \right] c^2 \end{aligned}$$

- ΔM is the energy release. Similarly,

$$E_2^{\text{kin}} = \frac{\Delta M}{2} \left[1 + \frac{m_1 - m_2}{M} \right] c^2 = \Delta M \left[1 - \frac{\Delta M}{2M} - \frac{m_2}{M} \right] c^2$$

$1 \to 2$ decay process, cont.

- The magnitude of final particles' 3-momentum $|\mathbf{p}_1| = |\mathbf{p}_2| \equiv |\mathbf{p}|$ may be calculated from

$$|\mathbf{p}|^2 c^2 = E_1^2 - m_1^2 c^4 = \left(\frac{M^2 + m_1^2 - m_2^2}{2M}\right)^2 c^4 - m_1^2 c^4$$
$$= \frac{M^4 + m_1^4 + m_2^4 - 2m_1^2 m_2^2 - 2M^2 m_1^2 - 2M^2 m_2^2}{4M^2} c^4 = \frac{\lambda(M^2, m_1^2, m_2^2)}{4M^2} c^4$$

- Important symmetric function of all its three arguments

$$\lambda(a, b, c) \equiv a^2 + b^2 + c^2 - 2ab - 2bc - 2ac$$

- Thus, $|\mathbf{p}| = \frac{\sqrt{\lambda(M^2, m_1^2, m_2^2)}}{2M} c$
- If the decay occurs in flight, then we can use

$$P^2 = p_1^2 + p_2^2 + 2(p_1 p_2) = p_1^2 + p_2^2 - \frac{2}{c^2}(E_1 E_2) + 2(\mathbf{p}_1 \mathbf{p}_2)$$

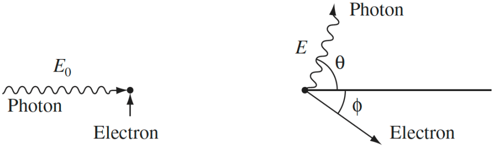
- Using $-P^2 = M^2$, $-p_i^2 = m_i^2$ and $(\mathbf{p}_1 \mathbf{p}_2) = |\mathbf{p}_1||\mathbf{p}_2| \cos\theta$ we get

$$M^2 c^2 = m_1^2 c^2 + m_2^2 c^2 + \frac{2}{c^2} E_1 E_2 - 2|\mathbf{p}_1||\mathbf{p}_2| \cos\theta$$

where θ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$

Compton Scattering

Example 12.9: A photon of energy E_0 bounces off the electron, initially at rest. Find the energy E of the outgoing photon, as a function of the scattering angle θ.



(before) (after)

- Conservation of momentum in the “vertical” direction gives
  $p_e \sin\phi = p_p \sin\theta \quad \Rightarrow \quad \sin\phi = \frac{E}{p_e c}\sin\theta$
- Conservation of momentum in the “horisontal” direction gives

$$\frac{E_0}{c} = p_p \cos\theta + p_e \cos\phi = \frac{E}{c}\cos\theta + p_e\sqrt{1-\left(\frac{E\sin\theta}{p_e c}\right)^2}$$

$$\Rightarrow \quad p_e^2 c^2 = (E_0 - E\cos\theta)^2 + E^2\sin^2\theta = E_0^2 - 2EE_0\cos\theta + E^2$$

Compton Scattering

- Conservation of energy

$$E._0 + mc^2 = E + E_e = E + \sqrt{m^2c^4 + p_e^2c^2}$$
$$= E + \sqrt{m^2c^4 + E_0^2 - 2EE_0\cos\theta + E^2}$$

- We get

$$E = \left[\frac{1-\cos\theta}{mc^2} + \frac{1}{E_0}\right]^{-1}$$

In terms of photon wavelength $\lambda = \frac{hc}{E}$

$$\lambda = \lambda_0 + \frac{h}{mc}(1-\cos\theta)$$

Addendum: 2$\rightarrow$ 2 particle scattering

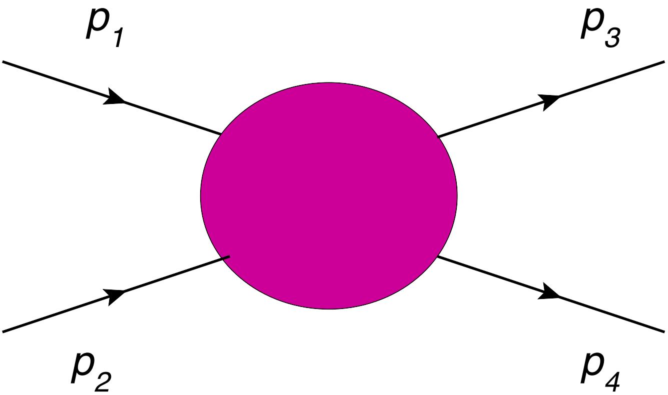


Figure: 2$\rightarrow$ 2 particle scattering

Energy-momentum conservation in application to $2 \to 2$ scattering process

- Two initial particles with 4-momenta p_1, p_2 and masses m_1,m_2 convert into two final particles with 4-momenta p_3, p_4 and masses m_3,m_4
- Using the momenta involved in this process, one can form several Lorentz invariants
- First, we have four invariants involving one of the momenta: $-p_1^2 = m_1^2,\ -p_2^2 = m_2^2,\ -p_3^2 = m_3^2,\ -p_4^2 = m_4^2,$ ($c = 1$ in this section)
- Combining momenta in pairs (and using the conservation law $p_1 + p_2 = p_3 + p_4$), we can form three *Mandelstam* invariants

$$-(p_1 + p_2)^2 \equiv s \equiv (-p_3 + p_4)^2$$
$$-(p_1 - p_3)^2 \equiv t \equiv -(p_2 - p_4)^2$$
$$-(p_1 - p_4)^2 \equiv u \equiv -(p_2 - p_3)^2$$

- These invariants are not independent
- There exists a linear relation between them

$$s + t + u = \sum_{i=1}^{4} m_i^2$$

Mandelstam invariants

$$s + t + u = \sum_{i=1}^{4} m_i^2$$

- Indeed,

$$\begin{aligned}
s + t + u &= -(p_1 + p_2)^2 - (p_1 - p_3)^2 - (p_1 - p_4)^2 \\
&= m_1^2 + m_2^2 - 2(p_1 p_2) \\
&+ m_1^2 + m_3^2 + 2(p_1 p_3) \\
&+ m_1^2 + m_4^2 + 2(p_1 p_4) \\
&= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 \\
&+ 2p_1 \cdot \underbrace{(-p_2 + p_3 + p_4)}_{p_1} \\
&= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 - 2m_1^2 \\
&= m_1^2 + m_2^2 + m_3^2 + m_4^2
\end{aligned}$$

Laboratory and center-of-mass frames

- There are two natural frames to study $2 \rightarrow 2$ process
- In the *laboratory* frame, the first particle is a projectile, $p_1 = (E_L, \mathbf{p}_L)$
- The second one is a target $p_2 = (m_2, \mathbf{0})$
- In the *center of mass* frame, the total 3-momentum of colliding particles is zero, i.e., $p_1 = (E_1, \mathbf{p})$, $p_2 = (E_2, -\mathbf{p})$
- Since $s = (p_1 + p_2)^2$ is Lorentz invariant, we may write it in both systems
- In particular, in laboratory frame we have

$$s = -(p_1 + p_2)^2 = m_1^2 + m_2^2 - 2(p_1 p_2) = m_1^2 + m_2^2 + 2m_2 E_L$$

- This formula can be also obtained from

$$s = (m_2 + E_L)^2 - \mathbf{p}_L^2$$

- In the center of mass frame, we have

$$s = (E_1 + E_2)^2 \equiv W^2$$

- $W \equiv \sqrt{s}$ is the total c.m. energy. Thus,

$$W^2 = m_1^2 + m_2^2 + 2m_2 E_L$$

Laboratory frame

- To get relation between the values of 3-momenta in these two frames, consider the scalar product $(p_1 p_2)$. Then

$$-(p_1 p_2) = m_2 E_L = \mathbf{p}^2 + E_1 E_2 = \mathbf{p}^2 + \sqrt{(m_1^2 + \mathbf{p}^2)(m_2^2 + \mathbf{p}^2)}$$

or

$$(m_2 E_L - \mathbf{p}^2)^2 = (m_1^2 + \mathbf{p}^2)(m_2^2 + \mathbf{p}^2) \ ,$$

which gives

$$\mathbf{p}^2 (m_1^2 + m_2^2 + 2 m_2 E_L) = m_2^2 (E_L^2 - m_1^2) \ ,$$

or

$$\mathbf{p}^2 W^2 = m_2^2 \mathbf{p}_L^2$$

- Thus, $|\mathbf{p}| = |\mathbf{p}_L| m_2 / W$, and since $\mathbf{p}$ has the same direction as $\mathbf{p}_L$

$$\mathbf{p} = \mathbf{p}_L \frac{m_2}{W}$$

Laboratory and center-of-mass frames, cont.

- Observation: the two initial particles with masses m_1, m_2 combine into a "particle" with mass $W = \sqrt{s}$, which is at rest in the c.m. frame
- Hence, using $|\mathbf{p}| = \sqrt{\lambda(M^2, m_1^2, m_2^2)}/2M$ we get

$$|\mathbf{p}| = \frac{\sqrt{\lambda(s, m_1^2, m_2^2)}}{2\sqrt{s}} = \frac{\sqrt{(s - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2}}{2\sqrt{s}}$$

- In the final state, we have two particles with masses m_3,m_4 which originated from a "particle" with mass $\sqrt{s}$
- Hence, the final particles in c.m. frame have opposite 3-momenta $\mathbf{p}'$, $-\mathbf{p}'$ whose magnitude is given by

$$|\mathbf{p}'| = \frac{\sqrt{\lambda(s, m_3^2, m_4^2)}}{2\sqrt{s}} = \frac{\sqrt{(s - m_3^2 - m_4^2)^2 - 4m_3^2 m_4^2}}{2\sqrt{s}}$$

- In general, there is an angle θ between the directions of $\mathbf{p}$ and $\mathbf{p}'$ (scattering angle in c.m. frame)

Scattering of identical particles

- In particular case of **elastic scattering of identical particles,** when $m_i = m$, all c.m. energies E_i in this case are given by $W/2 = \sqrt{s}/2$, and

$$|\mathbf{p}| = |\mathbf{p}'| = \frac{\sqrt{(s-2m^2)^2 - 4m^4}}{2\sqrt{s}} = \frac{\sqrt{s-4m^2}}{2}$$

- In the laboratory frame, we have

$$s = 2m(E_L + m)$$

and $|\mathbf{p}_L| = |\mathbf{p}|\sqrt{s}/m$ or

$$|\mathbf{p}_L| = \frac{\sqrt{s(s-4m^2)}}{2m} .$$

- The invariants t and u in c.m. variables in this case may be written as

$$t = -(p_1 - p_3)^2 = -(\mathbf{p}_1 - \mathbf{p}_3)^2 = -2\mathbf{p}^2(1-\cos\theta) = -4\mathbf{p}^2\sin^2(\theta/2)$$

and

$$u = -(p_1 - p_4)^2 = -(\mathbf{p}_1 + \mathbf{p}_3)^2 = -2\mathbf{p}^2(1+\cos\theta) = -4\mathbf{p}^2\cos^2(\theta/2)$$