PHYSICS 453

Electromagnetism II
Lecture 12-6

Physics Department
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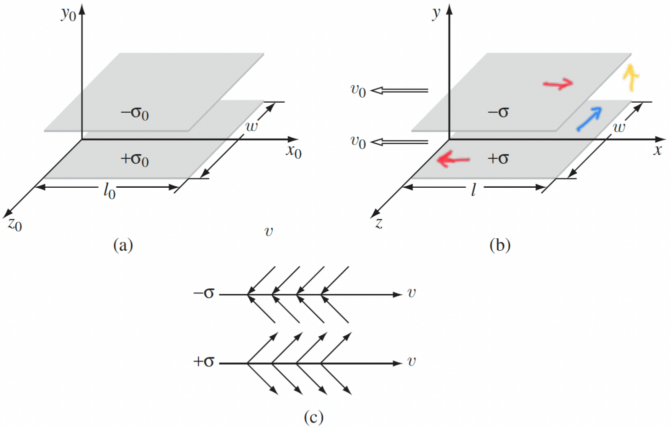
Outline

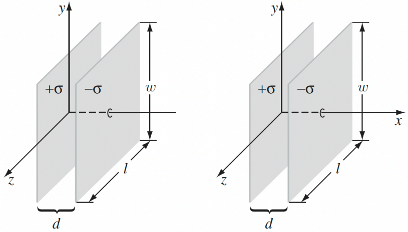
1. Lorentz Transformation of Fields
   - Lorentz Transformation of the electric field
   - Lorentz transformation of $\mathbf{E}$ and $\mathbf{B}$
   - Lorentz transformation of fields
   - Continuity Equation and Four Current
   - Potentials as Four Vectors

Lorentz transformation of the electric field



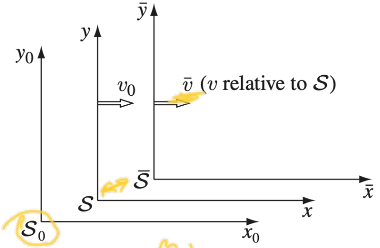
$$E^{(S_0)} = \frac{\sigma_0}{\epsilon_0}, \quad l = \frac{l_0}{\gamma_0} \Rightarrow \sigma^{(S)} = \gamma_0 \sigma_0^{(S_0)} \Rightarrow \mathbf{E}_\perp^{(S)} = \gamma_0 \mathbf{E}_\perp^{(S_0)}, \quad \gamma_0 = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}}$$

Lorentz transformation of the electric field, cont.



$$E^{(S_0)} = \frac{\sigma_0}{\epsilon_0}, \quad E \text{ does not depend on } d \quad \Rightarrow \quad E_{\parallel}^{(S)} = E_{\parallel}^{(S_0)}$$

Lorentz transformation of $\mathbf{E}$ and $\mathbf{B}$



- Start from frame S:

$$E_y^{(S)} = \frac{\sigma^{(S)}}{\epsilon_0}, \quad \mathbf{K}_{\pm}^{(S)} = \mp\sigma^{(S)} v_0 \hat{e}_x \quad \Rightarrow B_z^{(S)} = -\mu_0 \sigma^{(S)} v_0$$

- In the frame $\bar{S}$ moving with speed v relative to S

$$E_y^{(\bar{S})} = \frac{\sigma^{(\bar{S})}}{\epsilon_0}, \quad B_z^{(\bar{S})} = -\mu_0 \sigma^{(\bar{S})} \bar{v}$$

where $\bar{v} = \frac{v+v_0}{1+\frac{v v_0}{c^2}}$ = velocity of $\bar{S}$ with respect to S_0 and

$$\bar{\sigma}^{\bar{S}} = \bar{\gamma}\sigma_0, \qquad \bar{\gamma} = 1/\sqrt{1-\bar{v}^2/c^2}$$

- Transformation of $\mathbf{E}_y$ and $\mathbf{B}_z$

$$\bar{E}_y^{(\bar{S})} = \frac{\bar{\sigma}^{(\bar{S})}}{\epsilon_0} = \frac{\bar{\gamma}\sigma_0^{(S_0)}}{\epsilon_0} = \frac{\bar{\gamma}}{\gamma_0}\frac{\sigma^{(S)}}{\epsilon_0}, \quad B_z^{(\bar{S})} = -\mu_0\sigma^{(\bar{S})}\bar{v} = -\frac{\bar{\gamma}}{\gamma_0}\mu_0\sigma^{(S)}\bar{v}$$

$$\frac{\bar{\gamma}}{\gamma_0} = \frac{\sqrt{1-v_0^2/c^2}}{\sqrt{1-\bar{v}^2/c^2}} = \gamma\left(1+\frac{vv_0}{c^2}\right), \quad \gamma = 1/\sqrt{1-v^2/c^2}$$

Recall $E_y^{(S)} = \frac{\sigma^{(S)}}{\epsilon_0}$, $B_z^{(S)} = -\mu_0\sigma^{(S)}v_0$

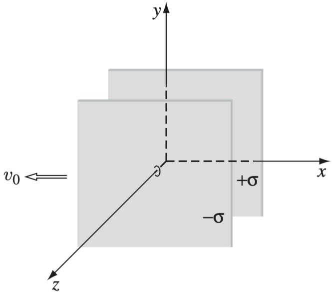
$$\bar{E}_y^{(\bar{S})} = \gamma\left(1+\frac{vv_0}{c^2}\right)\frac{\sigma^{(S)}}{\epsilon_0} = \gamma\left(E_y^{(S)} + vB_z^{(S)}\right),$$

$$\bar{B}_z^{(\bar{S})} = -\gamma\left(1+\frac{vv_0}{c^2}\right)\mu_0\sigma^{(S)}\left(\frac{v+v_0}{1+vv_0/c^2}\right) = \gamma\left(B_z^{(S)} - \frac{vE_y^{(S)}}{c^2}\right)$$

$= -\gamma\left(B + \frac{vE}{c^2}\right)$

Lorentz transformation of $\mathbf{E}$ and $\mathbf{B}$, cont.

- Transformation of $\mathbf{E}_z$ and $\mathbf{B}_y$



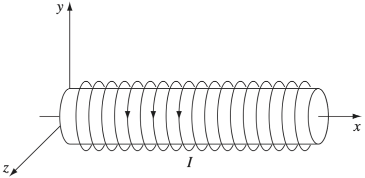
$$E_z^{(S)} = \frac{\sigma^{(S)}}{\epsilon_0}, \quad B_y^{(S)} = \mu_0 \sigma^{(S)} v_0$$

Same derivation with $E_y \to E_z$ and $B_z \to -B_y$

$$\Rightarrow \quad \bar{E}_z^{(\bar{S})} = \gamma(E_z^{(S)} + vB_y^{(S)}), \quad \bar{B}_y^{(\bar{S})} = \gamma\left(B_y^{(S)} + \frac{v}{c^2} B_y^{(S)}\right)$$

Lorentz Transformation of $\mathbf{E}$ and $\mathbf{B}$, cont.

- Transformation of $\mathbf{E}_x$ and $\mathbf{B}_x$
  First, we know that $E_x^{(\bar{S})} = E_x^{(S)}$
  To get transformation of B_x, consider a moving solenoid



- In S frame: $B_x^{(S)} = \mu_0 n^{(S)} I^{(S)}$
- In $\bar{S}$ frame:

$$\bar{n}^{(\bar{S})} = \gamma n^{(S)}, I = \frac{dQ}{dt} \Rightarrow \bar{I}^{(\bar{S})} = \frac{1}{\gamma} I^{(S)} \Rightarrow \bar{B}_x^{(\bar{S})} = B_x^{(S)}$$

Lorentz transformation of fields

- General formulas

$$E'_x = E_x \qquad E'_y = \gamma(E_y - vB_z) \qquad E'_z = \gamma(E_z + vB_y)$$

$$B'_x = B_x \qquad B'_y = \gamma\left(B_y + \tfrac{v}{c^2}E_z\right) \qquad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

- If $\mathbf{B} = 0$ in S

$$\mathbf{B}' = \frac{\gamma v}{c^2}(E_z\hat{e}_y - E_y\hat{e}_z) = \frac{v}{c^2}(E'_z\hat{e}_y - E'_y\hat{e}_z) \;\Leftrightarrow\; \mathbf{B}' = -\frac{1}{c^2}\mathbf{v}\times\mathbf{E}$$

- If $\mathbf{E} = 0$ in S

$$\mathbf{E}' = -\gamma v(B_z\hat{e}_y - B_y\hat{e}_z) = -v(B'_z\hat{e}_y - B'_y\hat{e}_z) \;\Leftrightarrow\; \mathbf{E}' = \mathbf{v}\times\mathbf{B}$$

Continuity Equation and Four Current

- Charge conservation is expressed through the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

- We can write this in a more manifestly covariant form as

$$\frac{1}{c}\frac{\partial}{\partial t}(\rho c) + \nabla \cdot \mathbf{J} = 0$$

- It is therefore tempting to try to introduce a four-current

$$J^\mu = (\rho c, \mathbf{J})$$

- Now the continuity equation can be formally written as

$$\partial_\mu J^\mu = 0$$

- $\partial_\mu J^\mu$ is a scalar $\Rightarrow$ J^μ is a four vector
- Check: consider a frame K' moving with velocity v along the x axis
- If J^μ is indeed a four vector we would have

$$\rho' c = \gamma \left[\rho c - \frac{v}{c} J_x\right]$$

$$J'_x = \gamma \left[J_x - v\rho\right]$$

$$J'_y = J_y$$

$$J'_z = J_z$$

$$\rho' c = \gamma \left[\rho c - \frac{v}{c} J_x \right] \quad , \quad J'_x = \gamma \left[J_x - v\rho \right] \quad , \quad J'_y = J_y \quad , \quad J'_z = J_z$$

- In the non-relativistic limit this gives the expected result

$$\mathbf{J}' = \mathbf{J} - \rho \mathbf{v} \quad , \quad \rho' = \rho$$

- Consider now the case $J_x = 0$. Then we have

$$J'_x = -\gamma v \rho \quad , \quad \rho' = \gamma \rho$$

- It looks like the second equation violates charge conservation
- But: consider what happens to a volume element under this transformation
- In the frame K, we have

$$dV = dx\, dy\, dz$$

- In terms of the K' frame variables

$$dx = \gamma(dx' + v\, dt') \quad , \quad dt = \gamma(dt' + \frac{v}{c^2}\, dx') \quad , \quad dy = dy' \quad , \quad dz = dz'$$

- Thus for measurements made at the same time $(dt' = 0)$

$$dV = dx\, dy\, dz = \gamma dx' dy' dz' = \gamma dV'$$

and the total charge in dV' is

$$\rho' dV' = \rho' \gamma^{-1} dV = \gamma \rho \gamma^{-1} dV = \rho\, dV$$

Continuity Equation and Four Current, cont.

$$\rho' dV' = \rho' \gamma^{-1} dV = \gamma \rho \gamma^{-1} dV = \rho \, dV$$

- Thus both the charge densities and volumes are not separately conserved under this Lorentz transformation, but the charge itself is
- Thus, J^μ is indeed a four vector, and

$$\partial_\mu J^\mu = 0$$

- Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \ , \ \ \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \ , \ \ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \ , \ \ \nabla \cdot \mathbf{B} = 0$$

- They are first-order differential equations expressed in terms of $\mathbf{E}$ and $\mathbf{B}$

Potentials as Four Vectors

- Introduce vector and scalar potentials to satisfy the homogeneous Maxwell equations

$$\mathbf{B} = \nabla \times \mathbf{A} \quad , \quad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

- In a vacuum the inhomogeneous equations become:

$$\nabla^2\phi + \frac{\partial \nabla \cdot \mathbf{A}}{\partial t} = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left[\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right] = -\mu_0 \mathbf{J}$$

- In the **Lorentz gauge**, we have $\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$, and the dynamical equations become

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}$$

- The operator on the l.h.s. of these equations is the four-dimensional Laplacian
- On the r.h.s. we have temporal and spatial components of the current J^μ

Potentials as Four Vectors, cont.

- Let us introduce a notation

$$A^\mu = (\phi, \mathbf{A})$$

so that both equations can be unified in the manifestly covariant form

$$\Box A^\mu = -\mu_0 J^\mu$$

with

$$\Box \equiv \partial^\alpha \partial_\alpha = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

- Furthermore, the Lorentz gauge condition is also manifestly covariant:

$$\partial^\mu A_\mu = 0$$

- $\Box A^\mu$ is a 4-vector, $\partial_\mu A^\mu$ is a scalar (0) $\Rightarrow$ $A^\mu = (\phi, \mathbf{A})$ is a 4-vector