



Lecture 12-7

Covariant Formulation of Maxwell's Equations

Field-Strength
Tensor

Electric and
Magnetic Fields

Levi-Civita Tensor

Dual Field-Strength
Tensor

Maxwell's Equations

Energy and
Momentum Law

PHYSICS 453

Electromagnetism II

Lecture 12-7

Physics Department
Old Dominion University

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Outline

Lecture 12-7

Covariant Formulation of Maxwell's Equations

Field-Strength Tensor

Electric and Magnetic Fields

Levi-Civita Tensor

Dual Field-Strength Tensor

Maxwell's Equations

Energy and Momentum Law

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- Field-Strength Tensor
- Electric and Magnetic Fields
- Levi-Civita Tensor in 4 dimensions
- Dual Field-Strength Tensor
- Maxwell's Equations
- Energy and Momentum Law

Covariant Formulation of Maxwell's Equations

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- In order to formulate the full Maxwell's equations in covariant form, we need to return to the relation between the fields (\mathbf{E} , \mathbf{B}) and the potentials

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla\phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$$

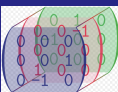
- We need to find a covariant relation between electric and magnetic fields, and the four vector A^μ , and express the fields themselves in covariant form
- Let us write out a couple of these components explicitly

$$\mathbf{B}_x = \frac{\partial \mathbf{A}_z}{\partial y} - \frac{\partial \mathbf{A}_y}{\partial z} = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3} = \frac{\partial A^3}{\partial x_2} - \frac{\partial A^2}{\partial x_3} = \partial^2 A^3 - \partial^3 A^2$$

$$\frac{\mathbf{E}_x}{c} = -\frac{\partial \phi/c}{\partial x} - \frac{1}{c} \frac{\partial \mathbf{A}_x}{\partial t} = -\frac{\partial A^0}{\partial x^1} - \frac{\partial A^1}{\partial x^0} = \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = \partial^0 A^1 - \partial_1 A^0$$

- N.B.: \mathbf{E}_i denotes the i^{th} component of a **three vector**, where we do not need to distinguish between covariant and contravariant vectors
- The equivalent four-vector components are given by

$$E^i = E_i = \mathbf{E}_i$$



Field-Strength Tensor

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- But we should be careful with upper and lower indices for 4-vectors!

$$\mathbf{B}_x = B^1 = \partial^2 A^3 - \partial^3 A^2 \rightarrow F^{23}$$

$$vE_x/c = E^1/c = \partial^0 A^1 - \partial^1 A^0 \rightarrow F^{01}$$

- Hence (\mathbf{E}, \mathbf{B}) are related to a second-rank tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$
- There are six independent components of the two fields.
- For a general second-rank tensor $T^{\mu\nu}$, we can write

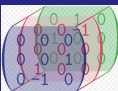
$$T^{\mu\nu} = T_{\text{sym}}^{\mu\nu} + T_{\text{anti-sym}}^{\mu\nu}$$

- The symmetric part has ten components, but the anti-symmetric part has six independent components that we could associate with fields \mathbf{E} and \mathbf{B}
- Thus we introduce the anti-symmetric **Maxwell Field-Strength Tensor**

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

- Writing out the components of $F^{\mu\nu}$ explicitly, we have (μ numbers rows, from 0 to 3, and ν numbers columns)

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$



Electric and Magnetic Fields

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- Electric field ($i = 1, 2, 3$)

$$F^{0i} = E^i/c; F^{i0} = -E^i/c$$

- Magnetic field ($i, j, k = 1, 2, 3$)

$$F^{ij} = \epsilon^{ijk} B^k$$

- ϵ^{ijk} is the 3-dimensional antisymmetric Levi-Civita tensor ($\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = 1, \epsilon^{213} = \epsilon^{132} = \epsilon^{321} = -1$, other components are zeroes)
- We see that \mathbf{E} and \mathbf{B} are not components of four vectors, but rather of an anti-symmetric, second-rank tensor
- Note that we can lower the indices in the usual way

$$F_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta}$$

- The components corresponding to \mathbf{E} change sign, $F_{i0} = -E^i$, whilst those corresponding to \mathbf{B} are unaltered:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}$$



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- Inverting the relation involving B^k gives

$$B^k = \frac{1}{2} \epsilon^{ijk} F_{ij}$$

- Here, the summation over both i and j index is implied
- With a chosen k , there are two possibilities for non-equal i, j in ϵ^{ijk}
- Finally, we will introduce the **dual** field-strength tensor
- But as a precursor we will return to the Levi-Civita tensor
- Levi-Civita Tensor in 4 dimensions is the four-dimensional version of the ϵ_{ijk} encountered in 3-D Euclidean space. It is defined by

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } \mu, \nu, \rho, \sigma \text{ is an } \textit{even} \text{ perm of } 0, 1, 2, 3 \\ -1 & \text{if } \mu, \nu, \rho, \sigma \text{ is an } \textit{odd} \text{ perm of } 0, 1, 2, 3 \\ 0 & \text{if two indices are equal} \end{cases}$$



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- Lowering the indices in the usual way, we immediately see that

$$\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}$$

- Note a very useful relation

$$\epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -2(\delta_{\rho}^{\mu}\delta_{\sigma}^{\nu} - \delta_{\sigma}^{\mu}\delta_{\rho}^{\nu})$$

- If we take $\mu = 0$, then other components of $\epsilon^{\mu\nu\rho\sigma}$ are space-like, and

$$\epsilon^{0ijk} = \epsilon^{ijk}$$



Exercise

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$$\epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -2(\delta_{\rho}^{\mu}\delta_{\sigma}^{\nu} - \delta_{\sigma}^{\mu}\delta_{\rho}^{\nu})$$

- Assume that all indices μ, ν, ρ, σ correspond to space components m, n, r, s
- Then one of the α, β indices corresponds to the time component, i.e. either $\alpha = 0$ or $\beta = 0$, and the remaining one corresponds to a space component
- The left side is $\epsilon^{\alpha\beta mn}\epsilon_{\alpha\beta rs} = \epsilon^{0bmn}\epsilon_{0brs} + \epsilon^{a0mn}\epsilon_{a0rs}$

$$= -\epsilon^{bmn}\epsilon^{brs} - \epsilon^{amn}\epsilon^{ars} = -2\epsilon^{bmn}\epsilon^{brs} = -2\epsilon^{mnb}\epsilon^{brs}$$

- The right side:

$$-2(\delta_{\rho}^{\mu}\delta_{\sigma}^{\nu} - \delta_{\sigma}^{\mu}\delta_{\rho}^{\nu}) \Rightarrow -2(\delta^{mr}\delta^{ns} - \delta^{sm}\delta^{rn})$$

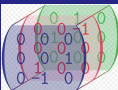
- This gives

$$\epsilon^{mnb}\epsilon^{brs} = \delta^{mr}\delta^{ns} - \delta^{sm}\delta^{rn}$$

- Multiplying by $A^n B^r C^s$ gives

$$\begin{aligned} \epsilon^{mnb} A^n \underbrace{\epsilon^{brs} B^r C^s}_{(\mathbf{B} \times \mathbf{C})^b} &= \epsilon^{mnb} A^n (\mathbf{B} \times \mathbf{C})^b = [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]^m \\ &= [\delta^{mr}\delta^{ns} - \delta^{sm}\delta^{rn}] A^n B^r C^s = B^m (\mathbf{A} \cdot \mathbf{C}) - C^m (\mathbf{A} \cdot \mathbf{B}) \end{aligned}$$

i.e. the $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ formula



Dual Field-Strength Tensor

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- The dual field-strength tensor is defined by

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

- Take $\mu = 0$, then

$$\tilde{F}^{0i} = \frac{1}{2} \epsilon^{0ijk} F_{jk} = \frac{1}{2} \epsilon^{ijk} F_{jk} = B^i$$

or $B^i = \tilde{F}^{i0}$. Similarly, taking both μ and ν space-like, we have

$$\tilde{F}^{ij} = \frac{1}{2} \left[\epsilon^{ij0k} F_{0k} + \epsilon^{ijk0} F_{k0} \right] = \epsilon^{ijk} F_{0k} = -\epsilon^{ijk} E^k / c$$

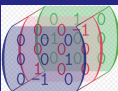
- The elements of $\tilde{F}^{\mu\nu}$ are related to those of $F^{\mu\nu}$ through the substitution

$$\mathbf{E}/c \rightarrow \mathbf{B} \quad , \quad \mathbf{B} \rightarrow -\mathbf{E}/c$$

so that

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

- Transition from $F^{\mu\nu}$ to $\tilde{F}^{\mu\nu}$ reverses the roles of the electric and magnetic fields



Dual Field-Strength Tensor, cont.

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$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

- Lowering the indices yields

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}$$

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- Let us return to Maxwell's equation in a vacuum

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad , \quad \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \quad , \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \quad , \quad \nabla \cdot \mathbf{B} = 0$$

- They are first-order differential equations expressed in terms of \mathbf{E} and \mathbf{B}
- Covariant form of Maxwell's equations will contain terms like $\partial_\mu F_{\nu\rho}$
- To convert equations written for the fields \mathbf{E}, \mathbf{B} into equations for the tensor $F_{\nu\rho}$, we will use the relations $E^i/c = F^{i0} = -F^{0i}$, $F^{ij} = \epsilon^{ijk} B^k$, $B^k = \frac{1}{2} \epsilon^{ijk} F_{ij}$ and $B^i = \tilde{F}^{0i}$, $\tilde{F}^{ij} = -\epsilon^{ijk} E^k/c$
- Looking at $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, we see that it may be written as

$$\frac{\partial}{\partial x^i} \frac{E^i}{c} = \frac{J^0}{c^2 \epsilon_0} = \mu_0 J^0$$

- Using $E^i/c = F^{0i}$, and noting that F^{00} vanishes, we write

$$\partial_\mu F^{0\mu} = \mu_0 J^0$$



Maxwell's Equations, cont.

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- The second inhomogeneous equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$ may be written as

$$\epsilon^{ijk} \frac{\partial}{\partial x^j} B^k = \mu_0 J^i + \frac{1}{c^2} \frac{\partial}{\partial t} E^i$$

- Using $\epsilon^{ijk} B^k = F^{ij}$ and $E^i = cF^{0i}$ gives

$$\frac{\partial}{\partial x^j} F^{ij} = \mu_0 J^i + \frac{1}{c} \frac{\partial}{\partial t} F^{0i}$$

- This can be written as

$$\begin{aligned} \frac{\partial}{\partial x^j} F^{ij} + \frac{\partial}{\partial x^0} F^{i0} &= \mu_0 J^i \\ \implies \partial_\mu F^{i\mu} &= \mu_0 J^i \end{aligned}$$

- Recalling $\partial_\mu F^{0\mu} = \mu_0 J^0$, we see that the two inhomogeneous Maxwell equations can be written in the unified form

$$\partial_\mu F^{\nu\mu} = \mu_0 J^\nu$$



Maxwell's Equations, cont.

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- Turning now to the homogeneous equations, we see that $\nabla \cdot \mathbf{B} = 0$ can be written as

$$\frac{\partial}{\partial x^i} \tilde{F}^{i0} = 0 \quad \implies \partial_\mu \tilde{F}^{\mu 0} = 0$$

- $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$ takes the form

$$\epsilon^{ijk} \frac{\partial}{\partial x^j} E^k + \frac{\partial}{\partial t} B^i = 0$$

- Using $\epsilon^{ijk} E^k / c = -\tilde{F}^{ij} = \tilde{F}^{ji}$ and $B^i = \tilde{F}^{0i}$ we obtain

$$\begin{aligned} \frac{\partial}{\partial x^j} \tilde{F}^{ji} + \frac{\partial}{\partial x^0} \tilde{F}^{0i} &= 0 \\ \implies \partial_\mu \tilde{F}^{\mu i} &= 0 \end{aligned}$$

- The two homogeneous Maxwell equations can be written in the unified form

$$\partial_\nu \tilde{F}^{\mu\nu} = 0$$

- Combining with

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$$

we get covariant formulation of Maxwell's equations



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- Note that we can rewrite $\partial_\mu \tilde{F}^{\mu\nu} = 0$ as

$$\begin{aligned}\frac{1}{2} \partial_\mu \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} &= 0 \\ \implies \epsilon^{\mu\nu\rho\sigma} \partial_\mu F_{\rho\sigma} &= 0,\end{aligned}$$

which we can express as

$$\partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} + \partial_\sigma F_{\mu\rho} = 0$$

- This is known as the **Jacobi Identity**



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- The Lorentz force law in Gaussian units is

$$\frac{d\mathbf{p}}{dt} = q \left\{ \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right\}$$

- To write this in a covariant form, we introduce the proper time $d\tau = \gamma^{-1} dt$

$$\frac{dp^i}{dt} = \frac{dp^i}{d\tau} \frac{d\tau}{dt} = \frac{1}{\gamma} \frac{dp^i}{d\tau}$$

- Thus the force law may be expressed as Minkowski force

$$K^i = \frac{dp^i}{d\tau} = \gamma q \left\{ E^i + \frac{1}{c} \epsilon^{ijk} v^j B^k \right\}$$

- Introduce the four-velocity $\eta^\mu = (\gamma c, \gamma \mathbf{v})$, yielding

$$\begin{aligned} \frac{dp^i}{d\tau} &= \frac{q}{c} \{ \eta^0 F^{i0} + \epsilon^{ijk} \eta^j B^k \} = \frac{q}{c} \{ \eta^0 F^{i0} - \eta^j F^{ij} \} \\ &= \frac{q}{c} \{ \eta_0 F^{i0} + \eta_j F^{ij} \}. \end{aligned}$$

- Thus the Lorentz force law becomes

$$K_i = \frac{dp^i}{d\tau} = \frac{q}{c} \eta_\mu F^{i\mu}$$



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- The analogous equation for the energy is

$$\frac{d}{dt} E^{\text{mech}} = q \mathbf{E} \cdot \mathbf{v}$$

- Thus, writing

$$\frac{d}{dt} E^{\text{mech}} = \frac{d\tau}{dt} \frac{d}{d\tau} E^{\text{mech}} = \frac{1}{\gamma} \frac{d}{d\tau} E^{\text{mech}}$$

we have

$$\frac{dE^{\text{mech}}}{d\tau} = \gamma q F^{i0} v^i = q F^{0i} \eta_i$$

yielding

$$\frac{d}{d\tau} \left(\frac{E^{\text{mech}}}{c} \right) = \frac{q}{c} \eta_\mu F^{0\mu}.$$

- Identifying E^{mech}/c with the component p^0 , we see that both this equation and the Lorentz law can be expressed as

$$K^\mu \equiv \frac{dp^\mu}{d\tau} = \frac{q}{c} \eta_\nu F^{\mu\nu}$$

- Newton's second law is in a manifestly covariant form