Covariant Formulation of Maxwell's Equations
Field-Strength Tensor
Electric and Magnetic Fields
Levi-Civita Tensor
Dual Field-Strength Tensor
Maxwell's Equations
Energy and Momentum Law

PHYSICS 453

Electromagnetism II
Lecture 12-7

Physics Department
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Outline

1. Covariant Formulation of Maxwell's Equations
   - Field-Strength Tensor
   - Electric and Magnetic Fields
   - Levi-Civita Tensor in 4 dimensions
   - Dual Field-Strength Tensor
   - Maxwell's Equations
   - Energy and Momentum Law

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- In order to formulate the full Maxwell's equations in covariant form, we need to return to the relation between the fields $(\mathbf{E}, \mathbf{B})$ and the potentials

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

- We need to find a covariant relation between electric and magnetic fields, and the four vector A^μ, and express the fields themselves in covariant form
- Let us write out a couple of these components explicitly

$$\mathbf{B}_x = \frac{\partial \mathbf{A}_z}{\partial y} - \frac{\partial \mathbf{A}_y}{\partial z} = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3} = \frac{\partial A^3}{\partial x_2} - \frac{\partial A^2}{\partial x_3} = \partial^2 A^3 - \partial^3 A^2$$

$$\frac{\mathbf{E}_x}{c} = -\frac{\partial \phi/c}{\partial x} - \frac{1}{c}\frac{\partial \mathbf{A}_x}{\partial t} = -\frac{\partial A^0}{\partial x^1} - \frac{\partial A^1}{\partial x^0} = \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = \partial^0 A^1 - \partial_1 A^0$$

- N.B.: $\mathbf{E}_i$ denotes the i^{th} component of a **three vector**, where we do not need to distinguish between covariant and contravariant vectors
- The equivalent four-vector components are given by

$$E^i = E_i = \mathbf{E}_i$$

Field-Strength Tensor

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- But we should be careful with upper and lower indices for 4-vectors!

$$\mathbf{B}_x = B^1 = \partial^2 A^3 - \partial^3 A^2 \quad \rightarrow \quad F^{23}$$

$$vE_x/c = E^1/c = \partial^0 A^1 - \partial^1 A^0 \quad \rightarrow \quad F^{01}$$

- Hence $(\mathbf{E}, \mathbf{B})$ are related to a second-rank tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$
- There are six independent components of the two fields.
- For a general second-rank tensor $T^{\mu\nu}$, we can write

$$T^{\mu\nu} = T^{\mu\nu}_{\text{sym}} + T^{\mu\nu}_{\text{anti-sym}}$$

- The symmetric part has ten components, but the anti-symmetric part has six independent components that we could associate with fields $\mathbf{E}$ and $\mathbf{B}$
- Thus we introduce the anti-symmetric **Maxwell Field-Strength Tensor**

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

- Writing out the components of $F^{\mu\nu}$ explicitly, we have (μ numbers rows, from 0 to 3, and ν numbers columns)

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

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Electric and Magnetic Fields

- Electric field ($i = 1, 2, 3$)

$$F^{0i} = E^i/c\,;\ F^{i0} = -E^i/c$$

- Magnetic field ($i, j, k = 1, 2, 3$)

$$F^{ij} = \epsilon^{ijk} B^k$$

- ϵ^{ijk} is the 3-dimensional antisymmetric Levi-Civita tensor ($\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = 1$, $\epsilon^{213} = \epsilon^{132} = \epsilon^{321} = -1$, other components are zeroes)
- We see that $\mathbf{E}$ and $\mathbf{B}$ are not components of four vectors, but rather of an anti-symmetric, second-rank tensor
- Note that we can lower the indices in the usual way

$$F_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta}$$

- The components corresponding to $\mathbf{E}$ change sign, $F_{i0} = -E^i$, whilst those corresponding to $\mathbf{B}$ are unaltered:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Levi-Civita Tensor in 4 dimensions

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- Inverting the relation involving B^k gives

$$B^k = \frac{1}{2}\epsilon^{ijk}F_{ij}$$

- Here, the summation over both i and j index is implied
- With a chosen k, there are two possibilities for non-equal i, j in ϵ^{ijk}
- Finally, we will introduce the **dual** field-strength tensor
- But as a precursor we will return to the Levi-Civita tensor
- Levi-Civita Tensor in 4 dimensions is the four-dimensional version of the ϵ_{ijk} encountered in 3-D Euclidean space. It is defined by

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } \mu,\nu,\rho,\sigma \text{ is an } \textit{even} \text{ perm of } 0,1,2,3 \\ -1 & \text{if } \mu,\nu,\rho,\sigma \text{ is an } \textit{odd} \text{ perm of } 0,1,2,3 \\ 0 & \text{if any two indices are equal} \end{cases}$$

Levi-Civita Tensor in 4 dimensions, cont.

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- Lowering the indices in the usual way, we immediately see that

$$\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}$$

- Note a very useful relation

$$\epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -2\left(\delta^{\mu}_{\ \rho}\delta^{\nu}_{\ \sigma} - \delta_{\sigma}^{\ \mu}\delta_{\rho}^{\ \nu}\right)$$

- If we take $\mu = 0$, then other components of $\epsilon^{\mu\nu\rho\sigma}$ are space-like, and

$$\epsilon^{0ijk} = \epsilon^{ijk}$$

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Exercize

$$\epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -2\left(\delta^{\mu}_{\ \rho}\delta^{\nu}_{\ \sigma} - \delta_{\sigma}^{\ \mu}\delta_{\rho}^{\ \nu}\right)$$

- Assume that all indices μ, ν, ρ, σ correspond to space components m, n, r, s
- Then one of the α, β indices corresponds to the time component, i.e. either $\alpha = 0$ or $\beta = 0$, and the remaining one corresponds to a space component
- The left side is $\epsilon^{\alpha\beta mn}\epsilon_{\alpha\beta rs} = \epsilon^{0bmn}\epsilon_{0brs} + \epsilon^{a0mn}\epsilon_{a0rs}$

$$= -\epsilon^{bmn}\epsilon^{brs} - \epsilon^{amn}\epsilon^{ars} = -2\epsilon^{bmn}\epsilon^{brs} = -2\epsilon^{mnb}\epsilon^{brs}$$

- The right side:

$$-2\left(\delta^{\mu}_{\ \rho}\delta^{\nu}_{\ \sigma} - \delta_{\sigma}^{\ \mu}\delta_{\rho}^{\ \nu}\right) \Rightarrow -2\left(\delta^{mr}\delta^{ns} - \delta^{sm}\delta^{rn}\right)$$

- This gives

$$\epsilon^{mnb}\epsilon^{brs} = \delta^{mr}\delta^{ns} - \delta^{sm}\delta^{rn}$$

- Multiplying by $A^n B^r C^s$ gives

$$\epsilon^{mnb}A^n \underbrace{\epsilon^{brs}B^r C^s}_{(\mathbf{B}\times\mathbf{C})^b} = \epsilon^{mnb}A^n(\mathbf{B}\times\mathbf{C})^b = [\mathbf{A}\times(\mathbf{B}\times\mathbf{C}]^m$$

$$= [\delta^{mr}\delta^{ns} - \delta^{sm}\delta^{rn}]A^n B^r C^s = B^m(\mathbf{A}\cdot\mathbf{C}) - C^m(\mathbf{A}\cdot\mathbf{B})$$

i.e. the $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$ formula

Dual Field-Strength Tensor

- The dual field-strength tensor is defined by

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

- Take $\mu = 0$, then

$$\tilde{F}^{0i} = \frac{1}{2}\epsilon^{0ijk}F_{jk} = \frac{1}{2}\epsilon^{ijk}F_{jk} = B^i$$

or $B^i = \tilde{F}^{i0}$. Similarly, taking both μ and ν space-like, we have

$$\tilde{F}^{ij} = \frac{1}{2}\left[\epsilon^{ij0k}F_{0k} + \epsilon^{ijk0}F_{k0}\right] = \epsilon^{ijk}F_{0k} = -\epsilon^{ijk}E^k/c$$

- The elements of $\tilde{F}^{\mu\nu}$ are related to those of $F^{\mu\nu}$ through the substitution

$$\mathbf{E}/c \to \mathbf{B} \;\;,\;\; \mathbf{B} \to -\mathbf{E}/c$$

so that

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

- Transition from $F^{\mu\nu}$ to $\tilde{F}^{\mu\nu}$ reverses the roles of the electric and magnetic fields

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Dual Field-Strength Tensor, cont.

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

- Lowering the indices yields

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}$$

Maxwell's Equations

- Let us return to Maxwell's equation in a vacuum

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \;,\; \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \;,\; \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \;,\; \nabla \cdot \mathbf{B} = 0$$

- They are first-order differential equations expressed in terms of $\mathbf{E}$ and $\mathbf{B}$
- Covariant form of Maxwell's equations will contain terms like $\partial_\mu F_{\nu\rho}$
- To convert equations written for the fields $\mathbf{E}, \mathbf{B}$ into equations for the tensor $F_{\nu\rho}$, we will use the relations $E^i/c = F^{0i} = -F^{i0}$, $F^{ij} = \epsilon^{ijk} B^k$, $B^k = \frac{1}{2}\epsilon^{ijk} F_{ij}$ and $B^i = \tilde{F}^{0i}$, $\tilde{F}^{ij} = -\epsilon^{ijk} E^k/c$
- Looking at $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, we see that it may be written as

$$\frac{\partial}{\partial x^i}\frac{E^i}{c} = \frac{J^0}{c^2\epsilon_0} = \mu_0 J^0$$

- Using $E^i/c = F^{0i}$, and noting that F^{00} vanishes, we write

$$\partial_\mu F^{0\mu} = \mu_0 J^0$$

Maxwell's Equations, cont.

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- The second inhomogeneous equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$ may be written as

$$\epsilon^{ijk} \frac{\partial}{\partial x^j} B^k = \mu_0 J^i + \frac{1}{c^2} \frac{\partial}{\partial t} E^i$$

- Using $\epsilon^{ijk} B^k = F^{ij}$ and $E^i = cF^{0i}$ gives

$$\frac{\partial}{\partial x^j} F^{ij} = \mu_0 J^i + \frac{1}{c} \frac{\partial}{\partial t} F^{0i}$$

- This can be written as

$$\frac{\partial}{\partial x^j} F^{ij} + \frac{\partial}{\partial x^0} F^{i0} = \mu_0 J^i$$
$$\implies \partial_\mu F^{i\mu} = \mu_0 J^i$$

- Recalling $\partial_\mu F^{0\mu} = \mu_0 J^0$, we see that the two inhomogeneous Maxwell equations can be written in the unified form

$$\partial_\mu F^{\nu\mu} = \mu_0 J^\nu$$

Maxwell's Equations, cont.

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- Turning now to the homogeneous equations, we see that $\nabla \cdot \mathbf{B} = 0$ can be written as

$$\frac{\partial}{\partial x^i} \tilde{F}^{i0} = 0 \qquad \Longrightarrow \partial_\mu \tilde{F}^{\mu 0} = 0$$

- $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$ takes the form

$$\epsilon^{ijk} \frac{\partial}{\partial x^j} E^k + \frac{\partial}{\partial t} B^i = 0$$

- Using $\epsilon^{ijk} E^k / c = -\tilde{F}^{ij} = \tilde{F}^{ji}$ and $B^i = \tilde{F}^{0i}$ we obtain

$$\frac{\partial}{\partial x^j} \tilde{F}^{ji} + \frac{\partial}{\partial x^0} \tilde{F}^{0i} = 0$$
$$\Longrightarrow \partial_\mu \tilde{F}^{\mu i} = 0$$

- The two homogeneous Maxwell equations can be written in the unified form

$$\partial_\nu \tilde{F}^{\mu\nu} = 0$$

- Combining with

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$$

we get covariant formulation of Maxwell's equations

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Maxwell's Equations, cont.

- Note that we can rewrite $\partial_\mu \tilde{F}^{\mu\nu} = 0$ as

$$\frac{1}{2}\partial_\mu \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0$$
$$\implies \epsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} = 0,$$

which we can express as

$$\partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} + \partial_\sigma F_{\mu\rho} = 0$$

- This is known as the **Jacobi Identity**

Energy and Momentum Law

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- The Lorentz force law in Gaussian units is

$$\frac{d\mathbf{p}}{dt} = q\left\{\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right\}$$

- To write this in a covariant form, we introduce the proper time $d\tau = \gamma^{-1}dt$

$$\frac{dp^i}{dt} = \frac{dp^i}{d\tau}\frac{d\tau}{dt} = \frac{1}{\gamma}\frac{dp^i}{d\tau}$$

- Thus the force law may be expressed as Minkowski force

$$K^i = \frac{dp^i}{d\tau} = \gamma q\left\{E^i + \frac{1}{c}\epsilon^{ijk}v^j B^k\right\}$$

- Introduce the four-velocity $\eta^\mu = (\gamma c, \gamma\mathbf{v})$, yielding

$$\begin{aligned}\frac{dp^i}{d\tau} &= \frac{q}{c}\{\eta^0 F^{i0} + \epsilon^{ijk}\eta^j B^k\} = \frac{q}{c}\{\eta^0 F^{i0} - \eta^j F^{ij}\}\\ &= \frac{q}{c}\{\eta_0 F^{i0} + \eta_j F^{ij}\}\,.\end{aligned}$$

- Thus the Lorentz force law becomes

$$K_i = \frac{dp^i}{d\tau} = \frac{q}{c}\,\eta_\mu F^{i\mu}$$

Energy and Momentum Law, cont.

- The analogous equation for the energy is

$$\frac{d}{dt}E^{\rm mech} = q\mathbf{E}\cdot\mathbf{v}$$

- Thus, writing

$$\frac{d}{dt}E^{\rm mech} = \frac{d\tau}{dt}\frac{d}{d\tau}E^{\rm mech} = \frac{1}{\gamma}\frac{d}{d\tau}E^{\rm mech}$$

we have

$$\frac{dE^{\rm mech}}{d\tau} = \gamma q F^{i0}v^i = qF^{0i}\eta_i$$

yielding

$$\frac{d}{d\tau}\left(\frac{E^{\rm mech}}{c}\right) = \frac{q}{c}\eta_\mu F^{0\mu}.$$

- Identifying $E^{\rm mech}/c$ with the component p^0, we see that both this equation and the Lorentz law can be expressed as

$$K^\mu \equiv \frac{dp^\mu}{d\tau} = \frac{q}{c}\eta_\nu F^{\mu\nu}$$

- Newton's second law is in a manifestly covariant form