804 Final Exam (34 points). 05/03/20, 12:30-15:30 p.m.

Problem 1.

An electric field pulse in vacuum is described by

$$\vec{E}(\vec{r},t) \;=\; E_0\hat{e}_3 \exp\Big(-\frac{(y-ct)^2}{l^2}\Big)$$

1. Find the curl of the magnetic field $\vec{B}$ associated with the pulse.
2. Find the magnetic field.

Solution

Magnetic field can be obtained from Faraday's Law

$$\frac{\partial\vec{B}}{\partial t} \;=\; -\vec{\nabla}\times\vec{E} \;=\; 2\hat{e}_1 E_0\frac{y-ct}{l^2}\exp\Big(-\frac{(y-ct)^2}{l^2}\Big)$$
$$\Rightarrow\;\vec{B} \;=\; \frac{E_0}{c}\hat{e}_1\exp\Big(-\frac{(y-ct)^2}{l^2}\Big)$$

Problem 2

Do the electric dipole and the magnetic dipole interact when they are

(a) at rest - NO

(b) moving with respect to each other? - YES

Problem 3

A monochromatic plane wave, linearly polarized in the XZ plane, is incident from above on the conducting XY plane. The angle of incidence is 45°. Find the electric and magnetic fields of the reflected wave.

Extra credit - 3 points

Find the surface charge density and the surface current density

Solution

The electric field of the incident wave is

$$\vec{E}_I \;=\; E_0\frac{\hat{e}_1+\hat{e}_3}{\sqrt{2}}e^{-i\omega t+ik\frac{\hat{e}_1-\hat{e}_3}{\sqrt{2}}\cdot\vec{r}} \;=\; E_0\frac{\hat{e}_1+\hat{e}_3}{\sqrt{2}}e^{-i\omega t+ik\frac{x-z}{\sqrt{2}}}$$

The reflected wave moves along the $\frac{\hat{e}_1+\hat{e}_3}{\sqrt{2}}$ direction and polarization should be also in the XZ plane so the reflected wave is

$$\vec{E}_R \;=\; E_R\frac{\hat{e}_1-\hat{e}_3}{\sqrt{2}}e^{-i\omega t+ik\frac{\hat{e}_1+\hat{e}_3}{\sqrt{2}}\cdot\vec{r}} \;=\; E_R\frac{\hat{e}_1-\hat{e}_3}{\sqrt{2}}e^{-i\omega t+ik\frac{x+z}{\sqrt{2}}}$$

Boundary condition $E_{\parallel}^{\text{above}} = 0 \Rightarrow (\vec{E}_I + \vec{E}_R)\Big|_{z=0} \cdot \hat{e}_1 = 0$ yields $E_R = -E_0$ so the reflected wave is

$$\vec{E}_R = -E_0 \frac{\hat{e}_1 - \hat{e}_3}{\sqrt{2}} e^{-i\omega t + ik\frac{x+z}{\sqrt{2}}}$$

$$\vec{B}_R = \frac{\hat{k}_R}{c} \times \vec{E}_R = \frac{\hat{e}_1 + \hat{e}_3}{c\sqrt{2}} \times \vec{E}_R = -E_0 \hat{e}_2 e^{-i\omega t + ik\frac{x+z}{\sqrt{2}}}$$

Extra credit solution

The magnetic field of the incident wave has the form

$$\vec{B}_I = \frac{\hat{k}_I}{c} \times \vec{E}_I = \frac{\hat{e}_1 - \hat{e}_3}{c\sqrt{2}} \times \vec{E}_I = E_0 \frac{\hat{e}_1 - \hat{e}_3}{\sqrt{2}} \times \frac{\hat{e}_1 + \hat{e}_3}{c\sqrt{2}} e^{-i\omega t + ik\frac{x-z}{\sqrt{2}}} = -E_0 \hat{e}_2 e^{-i\omega t + ik\frac{x-z}{\sqrt{2}}}$$

The surface charge density is

$$\sigma = \frac{E_{\perp}^{\text{above}} - E_{\perp}^{\text{below}}}{\epsilon_0} = \frac{1}{\epsilon_0} (\vec{E}_I + \vec{E}_R) \cdot \hat{e}_3 \Big|_{z=0} = 0$$

The surface current density is

$$\vec{K} = \frac{1}{\mu_0} \hat{n}_{1\to 2} \times (\vec{B}^{(2)} - \vec{B}^{(1)}) = \frac{1}{\mu_0} \hat{e}_3 \times (\vec{B}_I + \vec{B}_R)\Big|_{z=0} = -\frac{2E_0}{\mu_0} \hat{e}_2 e^{-i\omega t + ikx}$$

Problem 4.

The uniformly charged disc (charge Q, radius R) is spinning with constant angular velocity ω.

(1) Find the magnetic moment of the disc.

(2) Find the flux of the electromagnetic energy (Poynting vector) far away from the disc.

(3) Does the disc radiate?

Solution

The magnitude of the magnetic dipole moment of the disc is

$$m = \sum \pi s^2 \Delta I(s) = \int_0^R ds (\sigma \omega s) \pi s^2 = \frac{\pi}{4} \sigma \omega R^4$$

The direction is $\hat{e}_3$ so

$$\vec{m} = \frac{\hat{e}_3}{4} Q \omega R^2$$

The electric and magnetic fields at large distances are

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2}, \qquad \vec{B} = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3}$$

and therefore

$$\vec{S} \;=\; \frac{1}{\mu_0}\vec{E}\times\vec{B} \;=\; \frac{Q}{16\pi^2\epsilon_0}\frac{\vec{m}\times\hat{r}}{r^5}$$

In spherical coordinates

$$\vec{S} \;=\; \frac{\omega Q^2 R^2}{64\pi\epsilon_0}\frac{\sin\theta}{r^5}\hat{e}_\phi$$

Disc does not radiate since $|\vec{S}| \sim \frac{1}{r^5}$

Problem 5.

The highest energy cosmic rays are thought to be protons. In principle, a cosmic ray proton can strike a proton in a hydrogen atom in the upper atmosphere and make a W$^+$-boson in the process

$$p + p \;\rightarrow\; = \; p + n + W^+$$

What is the minimum energy for the cosmic ray proton in order for this process to be allowed? The rest masses are: $m_{\text{proton}} = 938\frac{\text{MeV}}{c^2}$, $m_{\text{neutron}} = 940\frac{\text{MeV}}{c^2}$, and $m_{W^+-\text{boson}} = 80.4\frac{\text{GeV}}{c^2}$

Solution

At the threshold, the momenta of three final particles in the c.m. frame are negligible so

$$s \;=\; (m_p + m_n + m_W)^2$$

For the two incoming protons in the lab frame

$$s \;=\; (p_1 + p_2)^2 \;=\; 2m_p^2 + 2m_p E$$

and therefore

$$E_{\min} \;=\; \frac{(m_p + m_n + m_W)^2 - 2m_p^2}{2m_p} \;\simeq\; 3608 \text{ GeV}$$

Problem 6.

In a certain frame K the electric field $\vec{E}$ and the magnetic filed $\vec{B}$ are such that $\vec{E} = \vec{B}c$.
(1) Is there a frame moving in the direction orthogonal to $\vec{E}$ (and $\vec{B}$) where the the angle between the fields $\vec{E}'$ and $\vec{B}'$ is

(a) 60°, (b) 90°, c) 120° ?

(2) With what velocity should that frame(s) move with respect to K?

Solution

First, from the Lorentz invariant $\vec{E}\cdot\vec{B} = E^2 > 0$ we see that the angles 120° and 90° are not possible (since $\vec{E}'\cdot\vec{B}' < 0$ for 120° and $\vec{E}'\cdot\vec{B}' = 0$ for 90°. The angle 60° is possible. Let us find the relevant boost velocity. After the Lorentz boost we have

$$\vec{E}' = \gamma(\vec{E} + \vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E})$$
$$\vec{B}' = \gamma(\vec{B} - \vec{\beta}\times\vec{E}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B})$$

For the boost in the direction orthogonal to $\vec{E} = \vec{B}$ the last terms in the above equation drop so we have

$$\vec{E}' = \gamma(\vec{E} + \vec{\beta}\times\vec{B}) \;=\; \gamma(E\hat{e}_3 - \beta E\hat{e}_2)$$
$$\vec{B}' = \gamma(\vec{B} - \vec{\beta}\times\vec{E}) \;=\; \gamma(E\hat{e}_3 + \beta E\hat{e}_2)$$

where we've chosen $\vec{E} = \vec{B} \parallel \hat{e}_3$ and $\beta \parallel \hat{e}_1$. The angle 60° between $\vec{E}'$ and $\vec{B}'$ means that

$$\frac{\vec{E}'\cdot\vec{B}'}{|\vec{E}'||\vec{B}'|} = \cos 60^\circ = \frac{1}{2} \quad\Leftrightarrow\quad \frac{1-\beta^2}{1+\beta^2} = \frac{1}{2} \quad\Rightarrow\quad v = \frac{c}{\sqrt{3}}$$