

**Problem 1.**

A wire loop of radius  $a$  and resistance  $R$  lies in the  $XY$  plane. There is a uniform magnetic field  $\vec{B} = B\hat{z}$  filling the whole space. What total charge passes a given point in the loop when it is rotated by  $90^\circ$  around the  $x$  axis?

**Solution:**

The induced current is

$$I(t) = \frac{1}{R}\mathcal{E}(t) = -\frac{1}{R}\frac{d\Phi}{dt}$$

so the total charge passing a given point has the form

$$Q = \int_0^\infty I(t)dt = -\frac{1}{R}\int_0^\infty \frac{d\Phi}{dt}dt = \frac{1}{R}(\Phi_{\text{initial}} - \Phi_{\text{final}}) = \frac{\pi a^2 B}{R}$$

**Problem 2.**

A pure magnetic dipole  $\vec{m}$  sits in the center of uniformly charged spherical shell of charge  $Q$  and radius  $R$ . Find:

- Momentum stored in the fields of this configuration, and
- Angular momentum

**Solution:**

Let us work in spherical polars with  $\vec{m} = m\hat{z}$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} \text{ at } r > R, \quad \vec{E} = 0 \text{ at } r < R$$

$$\vec{B} = \frac{\mu_0}{4\pi r^3} [2\hat{r} \cos \theta + \hat{\theta} \sin \theta]$$

(a)

The momentum density is  $\vec{g} =$

$$\vec{g} = \epsilon_0 \vec{E} \times \vec{B} = \frac{\mu_0 m Q}{16\pi^2 r^5} \hat{r} \times [2\hat{r} \cos \theta + \hat{\theta} \sin \theta] = \frac{\mu_0 m Q}{16\pi^2 r^5} \sin \theta \hat{\phi}$$

By asimuthal symmetry, the total momentum is zero.

(b)

The angular momentum density is

$$\vec{l} = \vec{r} \times \vec{g} = \frac{\mu_0 m Q}{16\pi^2 r^4} \sin \theta \hat{r} \times \hat{\phi} = -\frac{\mu_0 m Q}{16\pi^2 r^4} \sin \theta \hat{\theta}$$

By symmetry, the total angular momentum is collinear to  $\hat{z}$  so we need to integrate

$$l_z = \frac{\mu_0 m Q}{16\pi^2 r^4} \sin^2 \theta$$

$$L_z = \int_R^\infty dr r^2 \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi l_z = \frac{\mu_0 m Q}{8\pi} \int_R^\infty dr r^{-2} \int_0^\pi d\theta \sin^3 \theta = \frac{\mu_0 m Q}{6\pi R}$$

### Problem 3:

A circularly polarized electromagnetic plane wave is normally incident on an infinitely large plane made from a perfect conductor. Find the charge and current densities induced on the conducting plane.

### Solution

On p. 401 of the textbook the circularly polarized wave is defined as the sum of two waves polarized in the orthogonal directions with phase shift  $90^\circ$ . Let us first solve the problem for a linearly polarized wave and then use the superposition principle.

Let us choose the  $z$  axis in the direction normal to the plane and the wave coming from above the plane. Suppose the incident wave is linearly polarized in  $X$  direction. It has the form

$$\vec{E}_I(z, t) = \hat{e}_1 E_0 e^{-i\omega t + ikz}, \quad \vec{B}_I(z, t) = \frac{\hat{n}}{c} \times \vec{E}(z, t) = \frac{\hat{e}_3}{c} \times \vec{E}(z, t) = \hat{e}_2 \frac{E_0}{c} e^{-i\omega t + ikz}$$

The reflected wave is

$$\vec{E}_R(z, t) = E_R \hat{e}_1 e^{-i\omega t - ikz}, \quad \vec{B}_R(z, t) = \frac{\hat{n}}{c} \times \vec{E}(z, t) = -\frac{\hat{e}_3}{c} \times \vec{E}(z, t) = -\frac{\hat{e}_2}{c} \times E_R e^{-i\omega t - ikz}$$

The boundary condition for the electric field is  $E_{\parallel}^{\text{above}} = \vec{E}_I + \vec{E}_R = E_{\parallel}^{\text{below}} = 0$  so  $E_R = -E_0$  and the reflected wave is

$$\vec{E}_R(z, t) = -\hat{e}_1 E_0 e^{-i\omega t - ikz}, \quad \vec{B}_R(z, t) = \frac{\hat{e}_2}{c} \times E_I e^{-i\omega t - ikz} \quad (*)$$

The sum of the reflected and incident waves takes the form:

$$\vec{E}(z > 0, t) = 2\hat{e}_1 E_0 \sin kz e^{-i\omega t}, \quad \vec{B}(z > 0, t) = 2\hat{e}_2 \frac{E_0}{c} \cos kz e^{-i\omega t}$$

Now,  $\sigma$  and  $K$  can be found from the remaining boundary conditions

$$\epsilon_1 \vec{E}_{\perp}^{\text{above}} - \epsilon_2 \vec{E}_{\perp}^{\text{below}} = \sigma, \quad \frac{1}{\mu_1} \vec{B}_{\perp}^{\text{above}} - \frac{1}{\mu_2} \vec{B}_{\perp}^{\text{below}} = \vec{B}(z = 0+, t) = \vec{K}$$

We get  $\sigma = 0$  and  $\vec{K} = \frac{2}{\mu_0 c} \hat{e}_2 E_0 e^{-i\omega t}$ . Taking the real part we get

$$K_x = 0, \quad K_y = 2 \frac{E_0}{c} \cos \omega t,$$

Next, let us consider wave polarized in  $Y$  direction with the same  $E_0$  and phase shift  $\frac{\pi}{2}$ . It is clear that  $\sigma$  is still zero. Moreover, it is easy to see that  $\vec{B}$  is obtained by replacement  $\hat{e}_3 \times \hat{e}_1 = \hat{e}_2$  in Eq. (\*) by  $\hat{e}_3 \times \hat{e}_2 = -\hat{e}_1$  so  $\vec{K} = -\frac{2}{\mu_0 c} \hat{e}_1 E_0 e^{-i\omega t - \frac{\pi}{2}}$

$$K_y = 0, \quad K_x = -2 \hat{e}_1 \frac{E_0}{c} \cos \left( \omega t - \frac{\pi}{2} \right) = 2 \hat{e}_1 \frac{E_0}{c} \sin \omega t$$

Finally, due to the superposition principle

$$K_x = \frac{2}{\mu_0 c} E_0 \sin \omega t, \quad K_y = \frac{2}{\mu_0 c} E_0 \cos \omega t$$