453 Midterm II (16 points). April 1, 4:30 –6:00 p.m.

Problem 1 (4 points).

1. Consider the vector potential $A(\vec{r}) = \vec{r} \times \hat{e}_3$.
(a) What is the magnetic field?
(b) Does this potential satisfy Coulomb gauge condition $\vec{\nabla} \cdot \vec{A} = 0$?
2. Repeat the same steps for the potential $A(\vec{r}) = \vec{r} \times (\vec{r} \times \hat{e}_3)$.
3 (Extra credit - 3 points). For the potential with $\vec{\nabla} \cdot \vec{A} \neq 0$ modify $\vec{A}$ such that the magnetic field remains the same but the Coulomb condition is satisfied.

Solution:
1. $A(\vec{r}) = \vec{r} \times \hat{e}_3 = -x\hat{e}_2 + y\hat{e}_1 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A} \;=\; -2\hat{e}_3$ and $\vec{\nabla} \cdot \vec{A} \;=\; 0$

2. $A(\vec{r}) = \vec{r} \times (\vec{r} \times \hat{e}_3) = xz\hat{e}_1 + yz\hat{e}_2 - (x^2+y^2)\hat{e}_3 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A} \;=\; 3x\hat{e}_2 - 3y\hat{e}_1$ and $\vec{\nabla} \cdot \vec{A} \;=\; 2z$

3(Extra credit). For the second case $\vec{\nabla} \cdot \vec{A} \;=\; 2z \;\neq\; 0$. We need a gauge transformation $\vec{A}' \;=\; \vec{A} + \vec{\nabla}\Lambda$ such that the Coulomb gauge condition $\vec{\nabla} \cdot \vec{A}' \;=\; \vec{\nabla} \cdot \vec{A} + \nabla^2\Lambda \;=\; 0$ is satisfied $\Rightarrow \nabla^2\Lambda \;=\; -2z$. It is easy to guess that the one of the solutions of this differential equation is $\Lambda(\vec{r}) \;=\; -\frac{z^3}{3}$ so $\vec{A}' \;= xz\hat{e}_1 + yz\hat{e}_2 - (x^2+y^2+z^2)\hat{e}_3$. This solution is not unique - there is an (infinite) number of other Λ's.

Problem 2 (6 points).

A particle of charge q moves in a circle of radius a at a constant angular velocity ω. Assume that the circle lies in the x, y plane, centered at the origin and at time $t = 0$, the charge is at $(a, 0)$ on the positive x axis. For points on the z axis, find
(a) the Lienard-Wiechert potentials and
(b) the time-averaged electric field.

Solution
(a). The Lienard-Wiechert potentials are given by

$$\begin{aligned}
\phi(\vec{r},t) &= \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{w}(t_r)| - \frac{1}{c}\vec{v}(t_r)\cdot(\vec{r}-\vec{w}(t_r))} \\
\vec{A}(\vec{r},t) &= \frac{\vec{v}(t_r)}{c^2}\phi(\vec{r},t)
\end{aligned}$$

In our case $|\vec{r}-\vec{w}(t)| = \sqrt{z^2+a^2}$ and $\vec{v}(t)\cdot(\vec{r}-\vec{w}(t)) = 0$ for any t so we get

$$\begin{aligned}
\phi(\vec{r},t) &= \frac{q}{4\pi\epsilon_0\sqrt{z^2+a^2}} \\
\vec{A}(\vec{r},t) &= \frac{\vec{v}(t_r)}{c^2}\phi(\vec{r},t) \;=\; \frac{\mu_0 q}{4\pi\sqrt{z^2+a^2}}[-\sin(t-t_r)\hat{e}_1 + \cos(t-t_r)\hat{e}_2]
\end{aligned}$$

where $t_r = \frac{1}{c}\sqrt{z^2+a^2}$ is the retarded time.
(b). By symmetry, the time-averaged electric field is collinear to the z axis so it is sufficient to find $E_z(z,0,0)$

$$\langle E_z(z,0,0)\rangle \;=\; -\langle\frac{\partial}{\partial z}\phi(z,0,0)\rangle - \langle\frac{\partial A_z}{\partial t}\rangle \;=\; -\frac{\partial}{\partial z}\phi(z,0,0) \;=\; \frac{qz}{4\pi\epsilon_0(z^2+a^2)^{3/2}}$$

so

$$\langle\vec{E}(z,0,0)\rangle \;=\; \frac{qz}{4\pi\epsilon_0(z^2+a^2)^{3/2}}\hat{e}_3$$

Problem 3 (6 points).

A particle of charge q and mass m moves through an empty space with the velocity $\vec{v} = v\hat{e}_1$ ($v \ll c$). At time $t = 0$, a uniform magnetic field B is applied along the z-axis. How long it will take for the particle to lose half of its kinetic energy? (Assume that the magnetic field is sufficiently weak, so that the particle loses half of its energy after many revolutions).

Solution

In the uniform magnetic field, the particle moves around the circle with the radius $R = \frac{v}{\omega_B}$ where $\omega_B = \frac{qB}{m}$ is the cyclotron frequency. Our particle will radiate so v (and R) will slowly decrease with time. The radiated power is

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{\mu_0 q^2 \omega_B^2 v^2}{6\pi c} = \frac{\mu_0 q^2 \omega_B^2}{3\pi m c} \frac{m v^2}{2}$$

so we have the differential equation

$$P = -\frac{dE_{\text{kin}}(t)}{dt} = -\frac{\mu_0 q^2 \omega_B^2}{3\pi m c} E_{\text{kin}}(t)$$

The solution is

$$E_{\text{kin}}(t) = E_0 e^{-\frac{\mu_0 q^2 \omega_B^2}{3\pi m c} t}$$

so the particle will lose half of its kinetic energy after time

$$t = \frac{3\pi m c}{\mu_0 q^2 \omega_B^2} \ln 2 = \frac{3\pi m^3 c}{\mu_0 q^4 B^2} \ln 2$$