453 Midterm II (16 points). April 7, 4:30 −6:00 p.m.

Problem 1. (*5 points*)

1. Consider the vector potential $A(\vec{r}) = \vec{r} \times \hat{e}_3$.
(a) What is the magnetic field?
(b) Does this potential satisfy Coulomb gauge condition $\vec{\nabla} \cdot \vec{A} = 0$?
2. Repeat the same steps for the potential $A(\vec{r}) = \vec{r} \times (\vec{r} \times \hat{e}_3)$.
3. (Extra credit - 2 points). For the potential with $\vec{\nabla} \cdot \vec{A} \neq 0$ modify $\vec{A}$ such that the magnetic field remains the same but the Coulomb condition is satisfied.

Solution:
1. $A(\vec{r}) = \vec{r} \times \hat{e}_3 = -x\hat{e}_2 + y\hat{e}_1 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A} \; = \; -2\hat{e}_3$ and $\vec{\nabla} \cdot \vec{A} \; = \; 0$

2. $A(\vec{r}) = \vec{r} \times (\vec{r} \times \hat{e}_3) = xz\hat{e}_1 + yz\hat{e}_2 - (x^2 + y^2)\hat{e}_3 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A} \; = \; 3x\hat{e}_2 - 3y\hat{e}_1$ and $\vec{\nabla} \cdot \vec{A} \; = \; 2z$

3. (Extra credit). For the second case $\vec{\nabla} \cdot \vec{A} \; = \; 2z \; \neq \; 0$. We need a gauge transformation $\vec{A}' \; = \; \vec{A} + \vec{\nabla}\Lambda$ such that the Coulomb gauge condition $\vec{\nabla} \cdot \vec{A}' \; = \; \vec{\nabla} \cdot \vec{A} + \nabla^2\Lambda \; = \; 0$ is satisfied $\Rightarrow \nabla^2\Lambda \; = \; -2z$. It is easy to guess that the one of the solutions of this differential equation is $\Lambda(\vec{r}) \; = \; -\frac{z^3}{3}$ so $\vec{A}' \; = xz\hat{e}_1 + yz\hat{e}_2 - (x^2 + y^2 + z^2)\hat{e}_3$. This solution is not unique - there is an (infinite) number of other Λ's.

Problem 2. (*5 points*)

A point charge q of mass m is released at a distance d above infinite conducting plane located at $z = 0$. Find the radiated power at a function of the distance z from the plane. (Assume no gravity forces)

Solution

The dipole moment of the charge plus image $\bar{q} = -q$ charge is

$$\vec{p} \; = \; 2qz\hat{e}_z$$

so the power radiated by the system "charge plus image charge" is

$$P_{q+\bar{q}} \; = \; \frac{\mu_0}{6\pi c}|\ddot{p}|^2 \; = \; \frac{4q^2\mu_0}{6\pi c}|\ddot{z}|^2 \; = \; \frac{\mu_0 q^6}{384\pi^3 m^2 z^4 c\epsilon_0^2}$$

where I used $m\ddot{z} \; = \; \frac{q^2}{16\pi\epsilon_0 z^2}$. Half of this power does not exist (no fields in the conductor) so the radiated power is

$$P \; = \; \frac{\mu_0 q^6}{768\pi^3 m^2 z^4 \epsilon_0^2 c}$$

The Larmor formula used blindly gives only half of the result

$$P_{\text{Larmor}} \; = \; \frac{\mu_0 q^2 |\ddot{z}|^2}{6\pi c} \; = \; \frac{\mu_0 q^6}{1536\pi^3 m^2 z^4 \epsilon_0^2 c}$$

The rest comes from the radiation of surface charges moving on the conductor's surface.

Problem 3. (*6 points*)

A particle of charge q and mass m moves through an empty space with the velocity $\vec{v} = v\hat{e}_1$ ($v \ll c$). At time $t = 0$, a uniform magnetic field B is applied along the z-axis. The particle start to move in a circular orbit, but due to radiation the radius of the orbit decreases so the orbit looks like a helix. Assume that the magnetic field is sufficiently weak, so that the change of radius is gradual.
(a) *3 points*: What fraction of the energy is lost after one revolution?
(b) *3 points* Using method of Problem 11.14, find the time when the particle loses half of its initial energy.

Solution

(a) In the uniform magnetic field, the particle moves around the circle with the radius $R = \frac{v}{\omega_B}$ where $\omega_B = \frac{qB}{m}$ is the cyclotron frequency. Our particle will radiate so v (and R) will slowly decrease with time. The radiated power is

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{\mu_0 q^2 \omega_B^2 v^2}{6\pi c} = \frac{\mu_0 q^2 \omega_B^2}{3\pi mc} \frac{mv^2}{2}$$

so after one revolution

$$\frac{\Delta E}{E} = P(\frac{2\pi}{\omega})\frac{1}{mv^2/2} = \frac{2\mu_0 q^2 \omega_B}{3mc}$$

(b) The differential equation is $\frac{dE}{dt} = -P$ so

$$P = -\frac{dE_{\text{kin}}(t)}{dt} = -\frac{\mu_0 q^2 \omega_B^2}{3\pi mc} E_{\text{kin}}(t)$$

The solution is

$$E_{\text{kin}}(t) = E_0 e^{-\frac{\mu_0 q^2 \omega_B^2}{3\pi mc} t}$$

so the particle will lose half of its kinetic energy after time

$$t = \frac{3\pi mc}{\mu_0 q^2 \omega_B^2} \ln 2 = \frac{3\pi m^3 c}{\mu_0 q^4 B^2} \ln 2$$