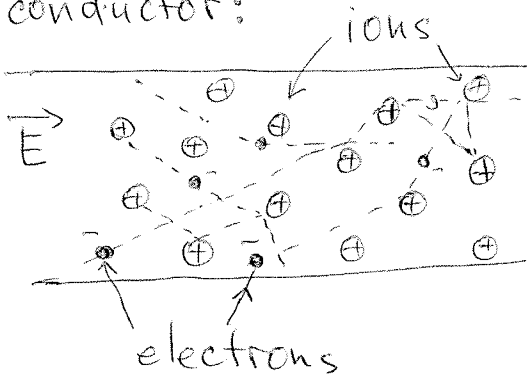
Electrodynamics

1. Ohm's law

conductor:



$V_{thermal} = \text{const}$
direction is random

$\Rightarrow \vec{V}^{ave}_{drift} = \dfrac{e\lambda_{ave}}{2m_e V_{ther}} \vec{E}$

electron bounces off the ions

$\Rightarrow t_{ave} = \dfrac{\lambda_{ave}}{V_{thermal}}$ $\quad$ λ_{ave} - average distance between ions

average drift velocity:

$$\left.\begin{aligned} \vec{V}^{ave}_{drift} &= \frac{1}{2}\vec{a}\,t_{ave} = \frac{\vec{a}\lambda_{ave}}{2V_{ther}} \\ \vec{a} &= \frac{\vec{F}}{m_e} = \frac{e}{m_e}(\vec{E} + \underbrace{\cancel{\vec{v}\times\vec{B}}}_{v \ll c}) \end{aligned}\right\} \Rightarrow$$

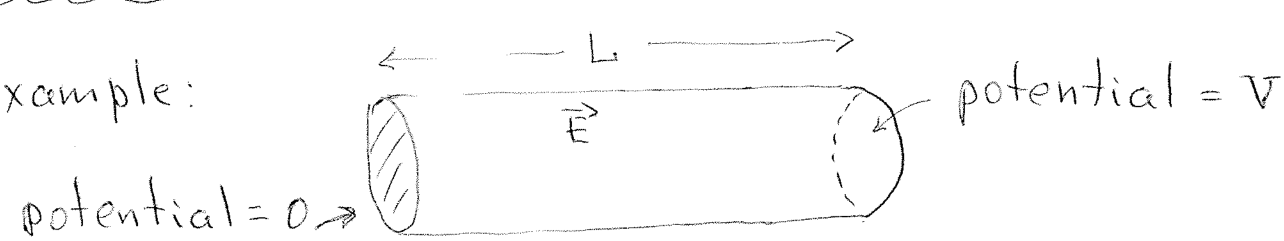
If we have N molecules per unit volume and f free electrons per molecule

$$\vec{J} = Nfe\vec{V}_{ave} = \underbrace{\left(\frac{Nfe^2\lambda_{ave}}{2m_e V_{ther}}\right)}_{\sigma}\vec{E}$$

σ - "conductivity" $\quad$ ($\rho = \frac{1}{\sigma}$ "resistivity")

$\boxed{\vec{J} = \sigma\vec{E}}$ Ohm's law

Example:



The field inside the cylinder is uniform (may be proved) $\Rightarrow E = \dfrac{V}{L}$ $\quad$ $\vec{E} \uparrow\uparrow$ axis of the cylinder

$\Rightarrow J = \sigma\dfrac{V}{L}$ $\qquad$ $I = JA = \sigma A\dfrac{V}{L}$ $\Rightarrow$ $V = I\underbrace{\dfrac{L}{\sigma A}}_{\text{resistance}}$

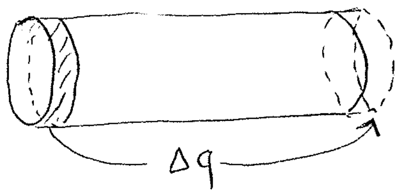
A – area of the cross section

$\Rightarrow V = IR$ traditional form of Ohm's law

Work done by the electric force is converted into heat in the resistor

$\Delta W = \Delta q\, V$

$\Delta q = I \Delta t$

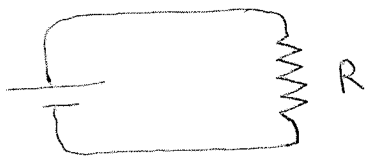


$\Delta W = V I \Delta t$

$P = \frac{\Delta W}{\Delta t} = V I = I^2 R$

"Joule heating law"

Electromotive force (emf)



Why $I = \text{const}$ throughout the circuit?

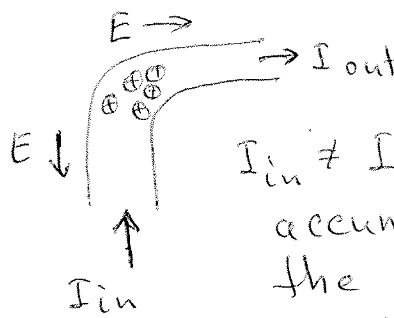
$J = \sigma \vec{f} =$

force exerted per unit charge

$= \sigma (\vec{f}_s + \vec{E})$

due to the source

electrostatic force which smoothes out the flow of charge



$I_{in} \neq I_{out} \Rightarrow$ there will be accumulation of charges in the bend $\Rightarrow$ the charges will provide $\vec{E}$ which will correct the mismatch between I_{in} and I_{out}

Definition:

$\mathcal{E} = \oint \vec{f} \cdot d\vec{l}$ "emf"

Since $\oint \vec{E} \cdot d\vec{l} = 0$ $\quad \mathcal{E} = \oint \vec{f}_s \cdot d\vec{l}$ = work done by the source per unit charge

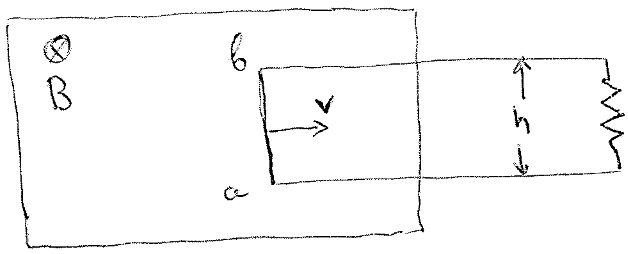
$\mathcal{E} = \oint \vec{f} \cdot d\vec{l} = \oint \frac{1}{\sigma} \vec{J} \cdot d\vec{l} =$

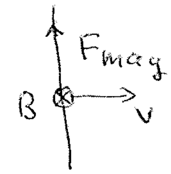
$= \oint \frac{I}{a\sigma} dl = I \underbrace{\oint \frac{1}{a\sigma} dl}_{R} = IR$

emf determines the current in the circuit

Motional emf

Elementary generator





$\vec{F}_{mag} = e\vec{v}\times\vec{B} \Rightarrow$

$\vec{f}_{mag} = \vec{v}\times\vec{B}$

$|f_{mag}| = vB$

$\mathcal{E} = \oint \vec{f}_{mag}\cdot d\vec{\ell} = vBh$

Power converted into heat in the resistor is $P = I^2R = \mathcal{E}^2/R$

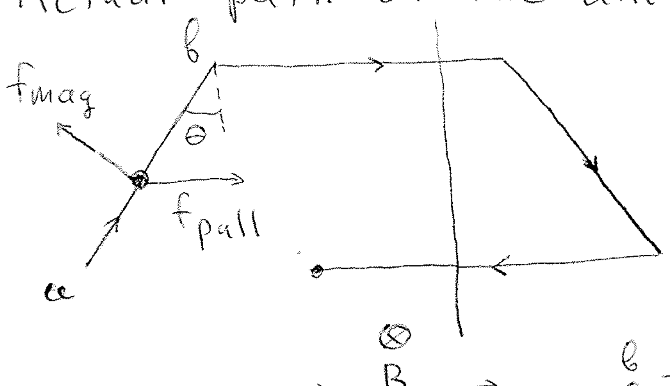
Q: What force is doing this work?

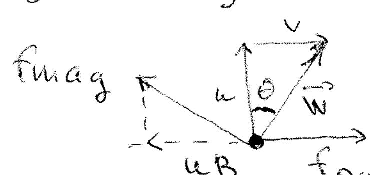
A (wrong): Magnetic force since it is responsible for the e.m.f.

magnetic forces never do work!

A (right): Force exerted by the person who pulls the wire

Actual path of the unit of charge





$f_{pull} = uB = f_{mag}\cos\theta$

$$\text{Work} = \oint \vec{f}_{pull}\cdot d\vec{\ell} = \int_a^b \vec{f}_{pull}\cdot d\vec{\ell} = f_{pull}\,\overset{h/\cos\theta}{\overset{\shortparallel}{\ell}}\sin\theta = uBh(\tan\theta) =$$
$$= \underline{vBh} \Rightarrow$$

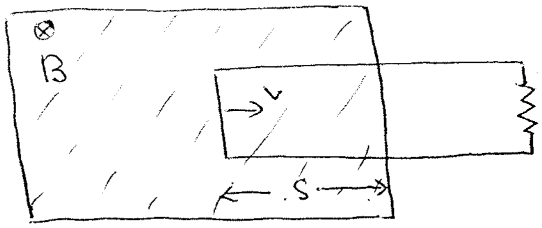
$\Rightarrow$ work done on unit charge is equal to the e.m.f.

However, this work is done by f_{pull} since f_{mag} does no work. On the other hand, f_{pull} contributes nothing to e.m.f. since it is $\perp$ to the wire.

$$\text{e.m.f.} = \oint \vec{f}\cdot d\vec{\ell}$$

"snapshot" of the contour (not the actual path of the charge)

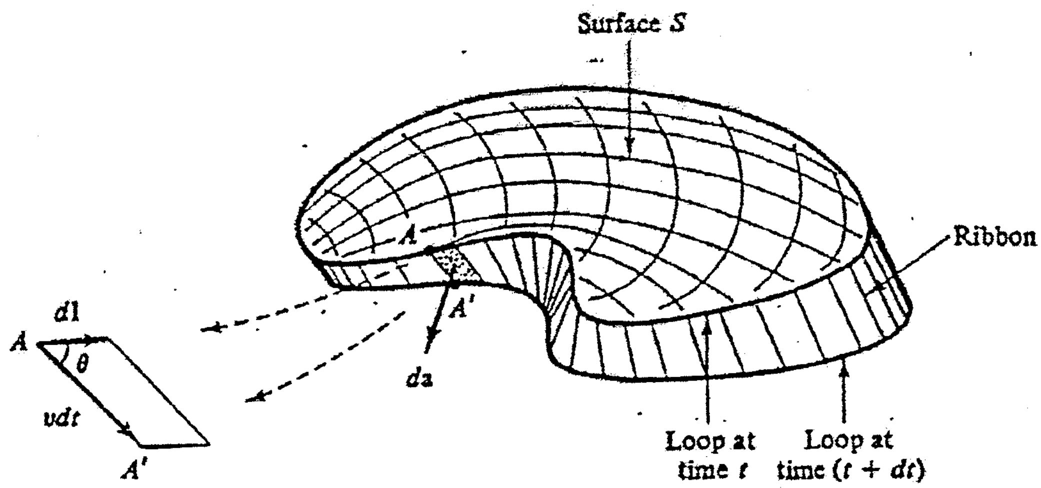
Flux rule



$$\phi = \int \vec{B}\cdot d\vec{a} = Bhs \qquad \frac{d\phi}{dt} = Bh\frac{ds}{dt} = -Bhv \Rightarrow \boxed{\mathcal{E} = -\frac{d\phi}{dt}}$$

flux rule for motional emf

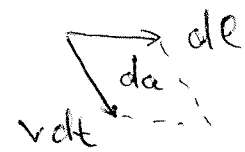
One can prove that $\mathcal{E} = -\frac{d\phi}{dt}$ holds true for any loop moving in a static magnetic field



Enlargement of da

Change of flux = flux through the ribbon

$$d\phi = \int_{\text{ribbon}} \vec{B}\cdot d\vec{a} \qquad d\vec{a} = (\vec{v}\times\vec{\ell})\,dt$$



$$\Rightarrow \frac{d\phi}{dt} = \oint \vec{B}\cdot(\vec{v}\times d\vec{\ell})$$

Velocity of the charge $\vec{w} = \vec{v} + \vec{u}$

$\vec{u}$: velocity of the charge down the wire

$$\vec{v}\times d\vec{\ell} = (\vec{w} - \cancel{\vec{u}})\times d\vec{\ell} = \vec{w}\times d\vec{\ell}$$

$\vec{u}\times d\vec{\ell} = \emptyset$ since $\vec{u} \uparrow\uparrow d\vec{\ell}$

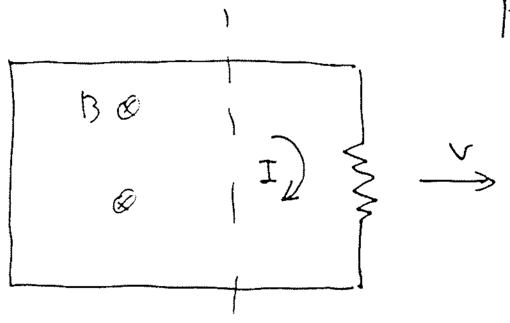
$$\Rightarrow \frac{d\phi}{dt} = \oint \vec{B}\cdot(\vec{w}\times d\vec{\ell}) = -\oint (\vec{w}\times\vec{B})\cdot d\vec{\ell}$$

$\vec{f}_{mag}$ = magnetic force per unit charge

$$\Rightarrow \frac{d\phi}{dt} = -\oint \vec{f}_{mag}\cdot d\vec{\ell} = -\mathcal{E}$$

Faraday's Law

Consider familiar setup



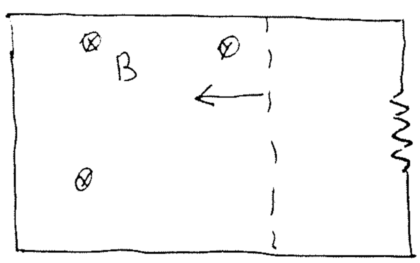
We pull the wire $\Rightarrow$

$\Rightarrow$ current $I = \frac{1}{R}\frac{\partial\phi}{dt}$ flows

$\left(\mathcal{E} = -\frac{\partial\phi}{\partial t}\right)$.

Let us now go the frame where the wire is at rest.

What do we see



The region of magnetic field shrinks ($\Leftrightarrow$ magnetic field is changing). Otherwise, nothing happens

But

the current still flows. What force is driving the charges?

Magnetic forces cannot move the charge since $F_{mag} \sim v$

$\Rightarrow$ should be the electric force (other than the electrostatic forces which cancel out).

Faraday's guess:

A changing magnetic field induces an electric field

In modern terms, it is a direct consequence of relativity.

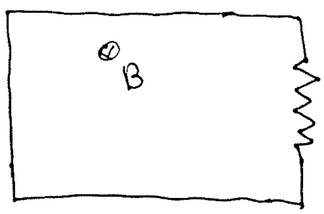
Universal flux rule

Whenever the magnetic flux thru a loop changes, an e.m.f.

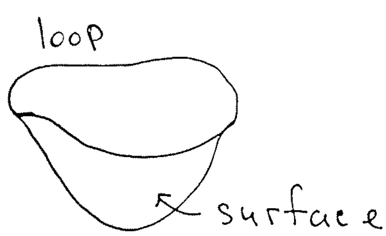
$$\mathcal{E} = -\frac{d\phi}{dt}$$

appears in the loop

Faraday's experiment



strength of the magnetic field changes $\Rightarrow$ current flows.



$$\mathcal{E} = \int_{\text{loop}} \vec{E}\cdot d\vec{l} = -\frac{d\phi}{dt} = -\frac{d}{dt}\int_{\text{surface}} \vec{B}\cdot d\vec{a}$$

$$\parallel$$

Stokes' theorem

$$\parallel$$

$$\int_{\text{surface}} (\vec{\nabla}\times\vec{E})\cdot d\vec{a}$$

$$\Rightarrow \int_{\text{surface}} (\vec{\nabla}\times\vec{E})\cdot da = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a} = -\frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

This is true for any loop $\Rightarrow$ for any surface $\Rightarrow$

$$\Rightarrow \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

– Faraday's law in the differentia form (one of the 4 Maxwell's eqns. for electrodynamics)

Lenz law: how to remember signs in Faraday's law

Lenz law: the motional e.m.f. produces such current that the magnetic field due to this produced current tends to counteract the change of flux that induces the e.m.f. (Nature abhors a change in flux)

How to calculate the induced electric field

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \xLeftrightarrow[\text{to}]{\text{similar}} \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

$$\vec{\nabla}\cdot\vec{E} = 0 \ \text{(no charges)} \qquad\qquad \vec{\nabla}\cdot\vec{B} = 0.$$

But we know the solution to the second pair of equations:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3x'\, \vec{J}(r')\times\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

$\Rightarrow$ the solution of the first pair of equations is obtained by the replacement $\mu_0\vec{J} \longrightarrow -\frac{\partial\vec{B}}{\partial t}$:

$$\vec{E}(\vec{r}) = -\frac{1}{4\pi}\int d^3x'\,\frac{\partial\vec{B}(r',t)}{\partial t}\times\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = \frac{d}{dt}\Big\{\underbrace{-\frac{1}{4\pi}\int d^3x'\,\vec{B}(\vec{r}\,',t)\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}}_{\vec{A}(\vec{r})}\Big\}$$

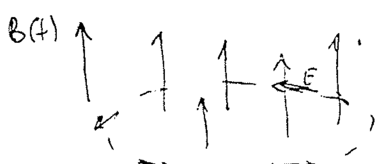
$$\Rightarrow \vec{E}(\vec{r},t) = -\frac{d\vec{A}(\vec{r},t)}{\partial t}$$

Check: $\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = -\frac{\partial\vec{B}}{\partial t}$

In the integral form

$$\oint\vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc} \quad \xLeftrightarrow[\text{to}]{\text{similar}} \quad \oint\vec{E}\cdot d\vec{\ell} = -\frac{d\phi}{dt}$$

Example:
Time-dependent uniform magnetic field



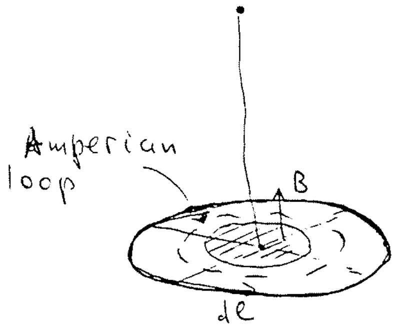
$\leftarrow$ Amperian loop

Situation here is similar to the magnetic field of the uniform current in $\vec{e}_3$ direction $\Rightarrow$
$\Rightarrow$ induced electric field is circumferential

$$\oint\vec{E}\cdot d\vec{\ell} = E\,2\pi r = -\frac{d\phi}{dt} = -\frac{d}{dt}\pi r^2 B = -\pi r^2\frac{dB}{dt} \quad\Rightarrow\quad E = -\frac{r}{2}\frac{dB}{dt}$$

If B_0 is decreasing, $\vec{E}$ is counterclockwise $\Rightarrow$ it tries to induce current which will support the decreasing flux.

Another example



Charged rim is suspended so that it is free to rotate. (Line charge density is λ)

$$\oint \vec{E}\cdot d\vec{\ell} = -\frac{d\phi}{dt} = -\pi a^2 \frac{dB}{dt} \Rightarrow \vec{E} = -\frac{a^2}{2s}\frac{\partial B}{\partial t}\hat{\phi}$$

Torque on segment $d\vec{\ell}$ is
$\vec{R}\times d\vec{F}$ $\qquad d\vec{F} = \lambda d\ell\cdot\vec{E} \Rightarrow$

$$\Rightarrow d\vec{N} = \vec{R}\times\vec{E}\,\lambda d\ell \Rightarrow d\vec{N} = RE\lambda d\ell\,\vec{e}_3$$

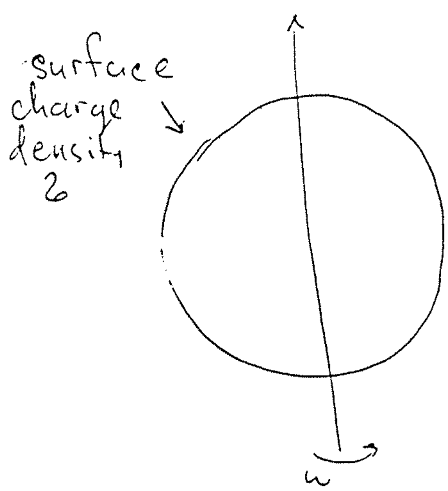
The total torque is

$$N = R\lambda\oint E\,d\ell = -R\lambda\pi a^2\frac{\partial B}{\partial t}$$

and the direction is $\uparrow\uparrow \vec{e}_3$

NB: Quasistatic approximation: we use Faraday's law for changing magnetic fields but use magnetostatics (Bio-Savart law etc) to calculate these magnetic fields. It is OK in most practical cases (unless we have the strong radiation of electromagnetic waves at very high frequencies).

Example: spinning spherical shell with time-dependent angular velocity



$$\vec{E} = \vec{E}_c + \vec{E}_f$$

$\vec{E}_c$ = ordinary Coulomb field =

$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{r^3}\vec{r}\,\Theta(r-R)$$

$\vec{E}_f$ = Faraday field due to changing magnetic field =

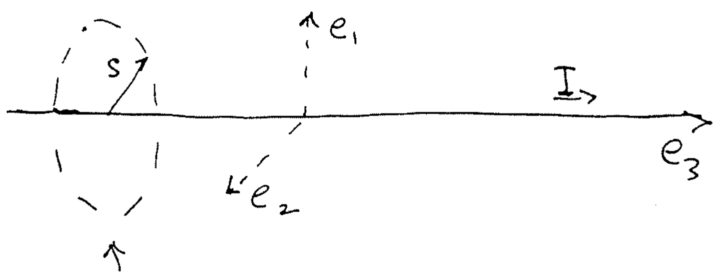
$$= -\frac{\partial\vec{A}}{\partial t}$$

$$\vec{A} = \frac{\mu_0 R\sigma}{3}\omega(t)\, r\sin\theta\,\hat{\varphi}\,\Theta(R-r) + \frac{\mu_0 R^4\sigma}{3}\omega(t)\frac{\sin\theta}{r^2}\hat{\varphi}\,\Theta(r-R) \Rightarrow$$

$$\Rightarrow \vec{E}_f = -\frac{\mu_0 R\sigma}{3}\frac{d\omega}{dt} r\sin\theta\,\hat{\varphi}\,\Theta(R-r) - \frac{\mu_0 R^4\sigma}{3}\frac{d\omega}{dt}\frac{\sin\theta}{r^2}\hat{\varphi}\,\Theta(r-R)$$

Provided that $\omega(t)$ changes slowly

Example 7.9:
Induced electric field due to the infinite straight wire with current $I(t)$



Amperian loop $\quad B(s,t)\cdot 2\pi s = \mu_0 I(t) \Rightarrow \vec{B}(s,t) = \frac{\mu_0}{2\pi s} I(t)\,\hat{\varphi}$

From magnetostatics $\Rightarrow \vec{A}(s,t) = -\frac{\mu_0 I(t)}{2\pi} \ln s\, \hat{e}_3$

$\Rightarrow \vec{E}(s,t) = -\frac{\partial}{\partial t}\vec{A}(s,t) = -\frac{\mu_0}{2\pi} \ln s \frac{dI(t)}{dt}\hat{e}_3$

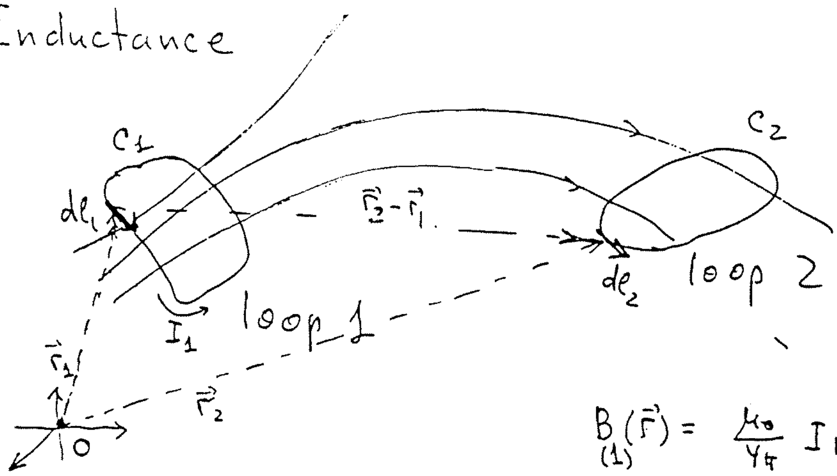
Actually, it is a little bit more tricky since the correct formula for the vector potential of a long straight wire of length L is

$\vec{A}(s) = \frac{\mu_0}{2\pi} I \hat{e}_3 \ln\frac{L}{s} \Rightarrow$

$\Rightarrow \vec{E}(s,t) = \frac{\mu_0}{2\pi}\frac{dI(t)}{dt}\ln\frac{L}{s}\hat{e}_3 = \frac{\mu_0}{2\pi}\frac{dI(t)}{dt}(\ln L - \ln s)\hat{e}_3$

The term $\frac{\mu_0}{2\pi} I \ln L\, \hat{e}_3$ was an unobservable additional constant term in the vector potential $\vec{A}(\vec{s})$ in the case of a time-independent current. Now it is perfectly observable electric field so one should be careful when using $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$ with calculated in magnetostat. A better way is to solve the eqn $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ anew with the appropriate boundary condition $\vec{E}\to\phi$ at infinity.

Inductance



$$B_{(1)}(\vec{r}) = \frac{\mu_0}{4\pi} I_1 \oint d\vec{\ell}_1 \times \frac{(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3}$$

$\Rightarrow B_{(1)}$ is proportional to I_1 $\Rightarrow$ the flux of $B_{(1)}$ through loop 2 is also proportional to I_1

$\Phi_2 = M_{21} I_1$ $\qquad$ M_{21} – „mutual inductance" of the two loops.

Why „mutual"

$$\Phi_2 = \int \vec{B}_{(1)} \cdot da_2 = \int (\vec{\nabla} \times \vec{A}_{(1)}) \cdot d\vec{a}_2 \overset{\text{Stokes' theorem}}{=} \oint_{C_2} \vec{A}_{(1)} d\vec{\ell}_2$$

$$\left.\begin{array}{l}\vec{A}_{(1)}(\vec{r}) = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} d\vec{\ell}_1 \frac{1}{|\vec{r}-\vec{r}_1|} \Rightarrow \vec{A}_{(1)}(r_2) = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} d\vec{\ell}_1 \frac{1}{|\vec{r}_2-\vec{r}_1|}\end{array}\right\} \Rightarrow$$

$$\Rightarrow \Phi_2 = \oint_{C_2} \frac{\mu_0 I_1}{4\pi} \oint_{C_1} d\vec{\ell}_1 \frac{1}{|\vec{r}_2-\vec{r}_1|} \cdot d\vec{\ell}_2 = \frac{\mu_0}{4\pi} I_1 \oint_{C_1} \oint_{C_2} \frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{|\vec{r}_1-\vec{r}_2|}$$

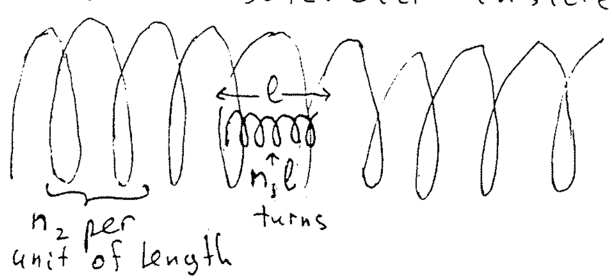
$$\Rightarrow M_{21} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{|\vec{r}_1-\vec{r}_2|}$$ „Neumann formula"

This double integral is rather difficult for practical purposes, but two important properties are seen immediately:

1. M_{21} is purely geometrical quantity
2. $M_{21} = M_{12}$ $\Rightarrow$ we will call it M – the mutual inductance of the two loops

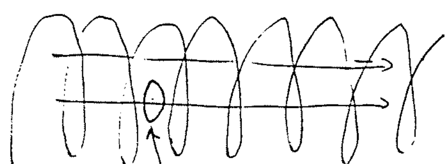
Corollary of (2): For any two loops the flux through loop 2 when we run a current I in the loop 1 is the same as the flux through loop 1 when we run the same current I around loop 2.

Example : small solenoid inside the large one.



Suppose we run a current I in the short solenoid. What will be the flux through the long one?

To solve this problem by direct computation of the flux through the large solenoid is technically impossible. On the other hand, to calculate the flux through small solenoid when I is flowing in the large one is very simple



$B = \mu_0 n_2 I$

a single turn from small solenoid. The flux is $B \cdot \pi R^2$ (R is the radius of small solenoid)

$\Rightarrow$ Flux through one turn $= \mu_0 n_2 I \pi R^2$

Flux through $N_1 \ell$ turns $= \mu_0 n_1 n_2 \ell I \pi R^2 =$

$\Rightarrow \Phi = \mu_0 n_1 n_2 \ell I \pi R^2$ is also the flux through large solenoid when we run current I in the small one

$$M_{12} = \mu_0 n_1 n_2 \pi R^2 \ell.$$

If we vary current in loop 1

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}$$ – e.m.f. in the loop 2.

But the flux through loop 1 also varies $\Rightarrow$ there will be an e.m.f. in the loop 1 itself

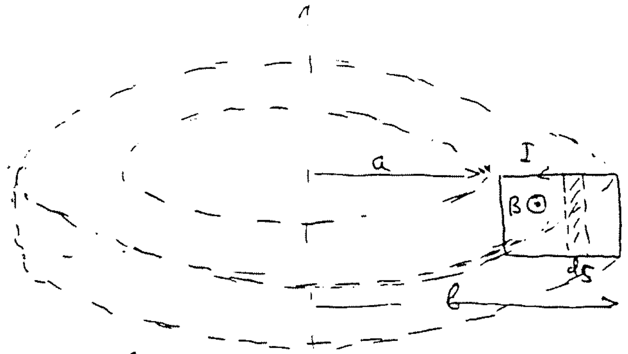
$$\mathcal{E}_1 = -\frac{d\Phi_1}{dt} \qquad \Phi_1 = L I$$

„self-inductance"

$$\Rightarrow \mathcal{E} = -L\frac{dI}{dt}$$

Self-inductance L is also determined by the geometry of the loop.

Example : self - inductance of a toroidal coil



one turn of the coil

$B(r) = \frac{\mu_0}{2\pi s} N I$ N - # of turns.

Flux through a single turn

$\Phi = \int \vec{B} \cdot d\vec{a} = h \int_a^b B(r)\, dr =$

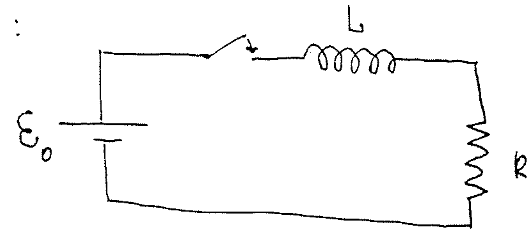
$= \frac{\mu_0 h}{2\pi} N I \int_a^b \frac{1}{s} ds = \frac{\mu_0 N I}{2\pi} h \ln b/a$

$\Rightarrow$ The total flux is n times bigger, so

$L = \frac{\mu_0}{2\pi} N^2 h \ln b/a$

Due to Lenz law , inductance is always positive

Example :



What happens if we turn on the switch

Ohm's law $\mathcal{E}_0 + \left(-L \frac{dI}{dt}\right) = I R$

$\uparrow$ e.m.f. due to self-inductance

$\Rightarrow \quad L \frac{dI}{dt} = \mathcal{E}_0 - I R$ $\leftarrow$ first - order differential equation

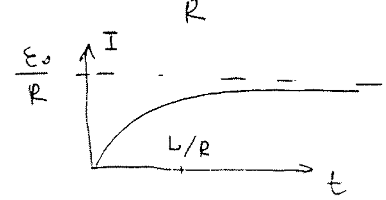
$\frac{dI}{dt} = \frac{\mathcal{E}_0}{L} - I \frac{R}{L} \Rightarrow \frac{dI}{\mathcal{E}_0/L - I/L\, R} = dt \Rightarrow \int \frac{dI}{\mathcal{E}_0/L - I/L\, R} = \frac{t}{L} R \Rightarrow$

$\Rightarrow \ln\left(\frac{\mathcal{E}_0}{L} - \frac{I}{L} R\right) = -\frac{tR}{L} + c \Rightarrow -\frac{I}{L} R + \frac{\mathcal{E}_0}{L} = e^c e^{-\frac{tR}{L}} \Rightarrow I = \frac{\mathcal{E}_0}{R} - k e^{-t\frac{R}{L}}$

$(k = e^c \frac{L}{R})$

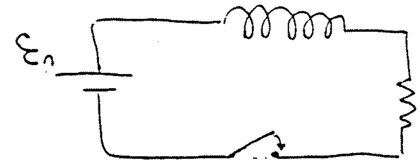
the constant k is to be determined from initial condition $I(t)\big|_{t=0} = 0 \Rightarrow k = \frac{\mathcal{E}_0}{R}$

$\Rightarrow I(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-t\frac{R}{L}}\right)$



$\frac{L}{R}$ = „time constant"

Energy in magnetic fields



What is the energy stored in the inductance L?

Work done on a unit charge against the back emf. in one trip around the circuit is $-\mathcal{E}$. The amount of charge per unit of time is $I \Rightarrow$

work per unit of time $= -\mathcal{E} I = LI\frac{dI}{dt} \quad \Rightarrow \frac{dw}{dt} = LI\frac{dI}{dt}$

The total work is

$$W = \int_0^\infty \frac{dw}{dt}\,dt = \int_0^\infty LI\frac{dI}{dt}\,dt = L\int_0^I I\,dI = \frac{LI^2}{2} \quad \leftarrow \text{energy stored in the inductance}$$

(This formula is similar to $W = \frac{1}{2}CV^2$ - electrostatic energy stored in the capacitor with capacitance C).

One can rewrite $W = \frac{LI^2}{2}$ in a very nice way

$$LI = \Phi = \int_S \vec{B}\cdot d\vec{a} = \int_S (\vec{\nabla}\times\vec{A})\cdot da = \oint_C \vec{A}\cdot d\vec{l}$$

C - boundary of the surface S

$\vec{A}$ ← vector potential

$$\Rightarrow W = \frac{1}{2} I \oint_C \vec{A}\cdot d\vec{l} = \frac{1}{2}\oint_C (\vec{A}\cdot\vec{I})\,dl$$

For the volume currents we must replace $\vec{I}\,dl$ by $\vec{J}\,d\tau$

$$\Rightarrow W = \frac{1}{2}\int_{\text{volume}} (\vec{A}\cdot\vec{J})\,d\tau = \frac{1}{2\mu_0}\int_{\text{volume}} \vec{A}\cdot(\vec{\nabla}\times\vec{B})\,d\tau$$

$$A\cdot(\vec{\nabla}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}^2 - \vec{\nabla}\cdot(\vec{A}\times\vec{B})$$

$$\Rightarrow W = \frac{1}{2\mu_0}\int_{\text{volume}} d\tau\, B^2 - \underbrace{\frac{1}{2\mu_0}\int_{\text{volume}} d\tau\, \nabla\cdot(\vec{A}\times\vec{B})}_{}$$

$$\frac{1}{2\mu_0}\int_{\text{surface}} (\vec{A}\times\vec{B})\cdot d\vec{a} = 0$$

if the surface lies on infinity and the field decreases

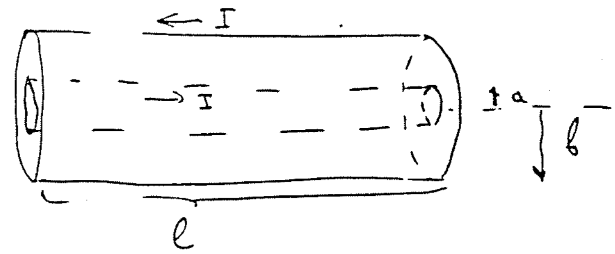
$$\Rightarrow W = \frac{1}{2\mu_0}\int_{\text{all space}} d\tau\, B^2$$

← „energy stored in the magnetic field"

The total energy stored in the electromagnetic field is

$$W = \frac{1}{2}\int_{\text{all space}} d\tau \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$$

Example: coaxial cable



Q: Find the energy stored in a section of length l

Between the cylinders $\vec{B}(r) = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$.
Elsewhere, $\vec{B} = 0$

$$\Rightarrow \quad W = \frac{1}{2\mu_0}\underset{\substack{\text{between}\\\text{cylinders}}}{\int d\tau\, B^2} = \frac{l}{2\mu_0}\int_a^b dr\,d\varphi\, r B^2 = \frac{l\pi}{\mu_0}\int_a^b dr\, r B^2 =$$

$$= \frac{l\pi}{\mu_0}\int_a^b dr\, r\,\frac{\mu_0^2 I^2}{4\pi^2 r^2} = \frac{\mu_0 I^2}{4\pi} l \int_a^b \frac{dr}{r} = \frac{\mu_0 I^2}{4\pi} l \ln\frac{b}{a}$$

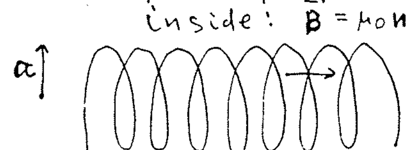
$$\Rightarrow \quad W = \frac{\mu_0 I^2}{4\pi} l \ln\frac{b}{a}$$

Comparing this to the formula $W = \frac{1}{2}LI^2$ we can find the inductance:

$$L = \frac{\mu_0}{2\pi} l \ln\frac{b}{a}$$

Example: solenoid

inside: $\vec{B} = \mu_0 n I \hat{\varphi}$



outside: $B = \emptyset$

$\Phi_{\text{one loop}} = B\pi a^2 = \mu_0 n I \pi a^2 \Rightarrow$

$\Rightarrow \Phi = \mu_0 n^2 I \pi a^2 l \Rightarrow L = \mu_0 \pi a^2 n^2 l$

1) $W = \frac{1}{2}LI^2 \qquad L = \mu_0\pi a^2 n^2 l \Rightarrow W = \frac{1}{2}\mu_0 \pi a^2 n^2 I^2 l$

2) $W = \underset{\text{all space}}{\int} \frac{B^2}{2\mu_0} d^3x' = \frac{1}{2\mu_0}\underset{\text{cylinder}}{\int} B^2 d^3x' = \frac{1}{2\mu_0}(\mu_0 n I)^2 \underbrace{\int_{\text{cylinder}} d^3x'}_{\pi a^2 l} = \frac{\mu_0}{2}\pi a^2 l n^2 I^2$

Maxwell's equations.

Before Maxwell

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \qquad \text{Gauss law}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \text{Faraday's law}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \qquad \text{Ampere's law}$$

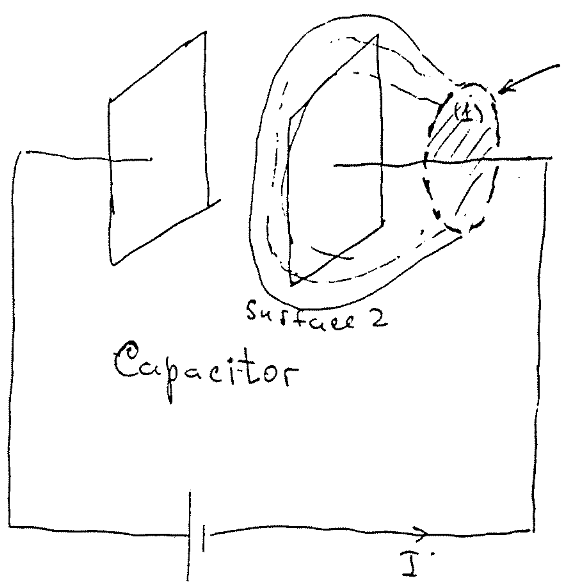
Maxwell realized that there is a problem with Ampere's law if one wants to apply it outside magnetostatics.

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0(\vec{\nabla}\cdot\vec{J})$. On the other hand,

$\vec{\nabla}\cdot(\vec{\nabla}\times \overrightarrow{\text{whatever}}) = 0$ $\Rightarrow$ contradiction if $\vec{\nabla}\cdot\vec{J}\neq 0$ (for non-steady currents)

Less formally: consider an example

Amperian loop



Ampere's law in the integral form reads:

$$\oint \vec{B}\cdot d\vec{l} = I_{enc}\,\mu_0$$

But if we draw surface 1 (plane circle)

$I_{enc} = I$

whereas if we draw surface 2

$I_{enc} = 0$ $\Rightarrow$

For non-steady currents the current enclosed by the loop is an ill-defined notion

How to fix this problem

Maxwell's solution : suppose we have

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + (\overrightarrow{\text{smth else}}),$$

then

$$\vec{\nabla} \cdot (\mu_0 \vec{J}) + \vec{\nabla} \cdot (\overrightarrow{\text{smth else}}) = 0 \Rightarrow$$

$$\Rightarrow \vec{\nabla} \cdot (\overrightarrow{\text{smth else}}) = - \vec{\nabla} \cdot \vec{J} \mu_0$$

But, due to continuity equation (conservation of charges

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \vec{\nabla} \cdot \vec{E}) = -\vec{\nabla} \cdot (\epsilon_0 \frac{\partial \vec{E}}{\partial t}) \Rightarrow$$

$$\Rightarrow \vec{\nabla}(\overrightarrow{\text{smth else}}) = \vec{\nabla}(\epsilon_0 \frac{\partial \vec{E}}{\partial t})$$

the first guess that comes to mind is that $\overrightarrow{\text{smth else}} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$, and experiments confirm that it is true

$$\Rightarrow \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \underbrace{\mu_0 \epsilon_0}_{= \frac{1}{c^2}} \frac{\partial \vec{E}}{\partial t}$$

· Ampere's law with Maxwell's correction

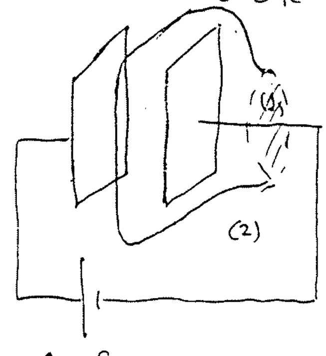
This new term is very small numerically (essentially, it is a relativistic effect since $\mu_0 \epsilon_0 = 1/c^2$) and this is the reason that it was never observed by Faraday and others.

Maxwell's form

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \vec{J}_d) \qquad \vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

„displacement current"

Let us check again the example with the capacitor



$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}$$

1. Surface (1) $\vec{E} = 0$ outside the capacitor $\Rightarrow \frac{\partial E}{\partial t} = 0 \Rightarrow$

$$\Rightarrow \oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc} = \mu_0 I$$

2. Surface (2) $E_{inside} = \frac{\sigma}{\epsilon_0} \Rightarrow \left(\frac{\partial E}{\partial t}\right)_{inside} = \frac{1}{\epsilon_0} \frac{\partial \sigma}{\partial t} = \frac{1}{\epsilon_0} \frac{1}{A} \underbrace{\frac{dQ}{dt}}_{I(t)}$

$$\Rightarrow \oint \vec{B} \cdot d\vec{\ell} = \underbrace{\cancel{\mu_0 I_{enc}}}_{=0} + \frac{\mu_0 \epsilon_0}{A} \int \frac{1}{\epsilon_0} I \, da = \mu_0 I \Rightarrow \text{same result}$$

Example: problem 7.34 $\qquad \theta(x) = 1$ if $x \geq 0 \qquad \theta(x) = 0$ if $x < 0$

$$\vec{E}(\vec{r},t) = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(vt-r)\hat{r}$$

$$\vec{B}(\vec{r},t) = \emptyset$$

1 Check $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = 0$ $\qquad \vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\vec{\nabla}f)$

$$\vec{\nabla}\times\left(\theta(vt-r)\frac{-q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}\right) = \theta(vt-r)\underbrace{\vec{\nabla}\times\frac{\vec{r}}{r^2}}_{=\emptyset}\left(\frac{-q}{4\pi\epsilon_0}\right) - \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}\times\underbrace{\vec{\nabla}\theta(vt-r)}_{?}$$

$$\vec{\nabla}\theta(vt-r) = \sum \hat{e}_i \frac{\partial}{\partial x_i}\theta(vt-\sqrt{x_1^2+x_2^2+x_3^2}) = \text{chain rule} =$$

$$= -\sum \hat{e}_i \frac{\partial\sqrt{\sum x_k^2}}{\partial x_i}\cdot\left.\frac{\partial\theta(z)}{\partial z}\right|_{z=vt-\sqrt{\sum x_i^2}} = -\sum\hat{e}_i\frac{x_i}{\sqrt{\sum x_i^2}}\delta(vt-\sqrt{\sum x_i^2}) = -\frac{\vec{r}}{r}\delta(vt-r)$$

$$\boxed{\Rightarrow \vec{\nabla}\theta(vt-r) = -\hat{r}\,\delta(vt-r)}$$

$$\Rightarrow \frac{\hat{r}}{r^2}\times\vec{\nabla}\theta(vt-r) = \emptyset$$

2. Find ρ and J

$$\rho = \epsilon_0\vec{\nabla}\cdot\vec{E} = -\frac{q}{4\pi}\vec{\nabla}\cdot\left(\theta(vt-r)\frac{\hat{r}}{r^2}\right) \qquad \vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)$$

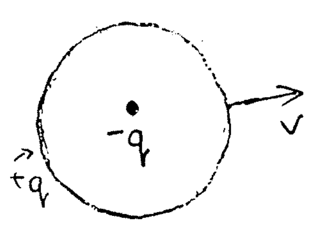
$$\Rightarrow \rho = -\frac{q}{4\pi}\left(\theta(vt-r)\underbrace{\vec{\nabla}\cdot\frac{\hat{r}}{r^2}}_{4\pi\delta^{(3)}(\vec{r})} + \frac{\hat{r}}{r^2}\underbrace{\vec{\nabla}\theta(vt-r)}_{-\hat{r}\delta(vt-r)}\right) = -q\,\theta(vt)\delta^3(\vec{r}) + \frac{q}{4\pi r^2}\delta(vt-r)$$

$$\mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} = \vec{\nabla}\times\vec{B} = \emptyset \quad\Rightarrow\quad \vec{J} = -\epsilon_0\frac{\partial\vec{E}}{\partial t} = \frac{1}{4\pi}\frac{q\hat{r}}{r^2}\frac{\partial}{\partial t}\theta(vt-r) =$$

$$= \frac{v}{4\pi}\frac{q\hat{r}}{r^2}\delta(vt-r)$$

$$\Rightarrow \rho = -q\,\theta(t)\delta^3(\vec{r}) + \frac{q}{4\pi r^2}\delta(vt-r)$$

$$\vec{J} = \frac{1}{4\pi}\frac{qv}{r^2}\delta(vt-r)\hat{r}$$



$$\int d^3x \frac{q}{4\pi r^2}\delta(vt-\ldots$$
$$= \frac{q\,4\pi}{4\pi r^2}\int r^2 dr\,\delta(\ldots$$
$$= q$$

Check continuity eqn

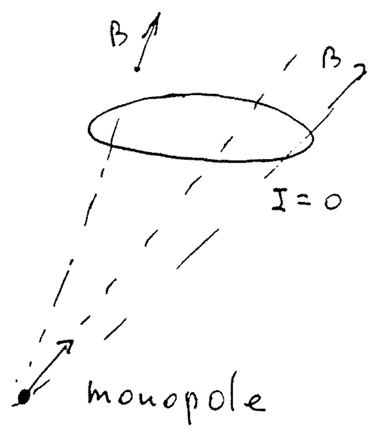
$$\frac{\partial\rho}{\partial t} = -q\delta^3(\vec{r})\frac{\partial}{\partial t}\theta(t) + \frac{q}{4\pi r^2}\frac{\partial}{\partial t}\delta(vt-r) = -q\delta^3(\vec{r})\delta(t) + \frac{qv}{4\pi r^2}\delta'(vt-r)$$

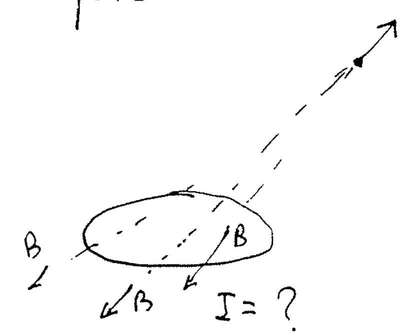
$$\vec{\nabla}\cdot\vec{J} = \frac{qv}{4\pi}\vec{\nabla}\cdot\left(\delta(vt-r)\frac{\hat{r}}{r^2}\right) = \frac{qv}{4\pi}\left(\delta(vt-r)\underbrace{\vec{\nabla}\cdot\frac{\hat{r}}{r^2}}_{\delta^3(\vec{r})} + \underbrace{\frac{\hat{r}}{r^2}\cdot\vec{\nabla}\delta(vt-r)}_{-\hat{r}\delta'(vt-r)}\right)$$

$$= qv\,\delta(vt)\delta^3(\vec{r}) - \frac{qv}{4\pi r^2}\delta'(vt-r)$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = \emptyset$$

Problem 7.36
Detection of a magnetic monopole

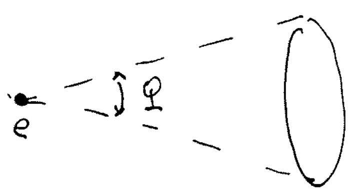




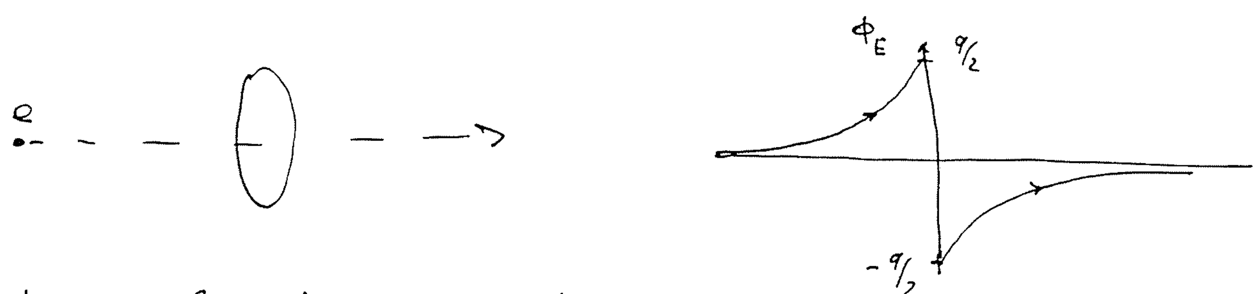
Self-inductance is L;
the resistance is ϕ

$$\text{Flux}(t=\infty) - \text{Flux}(t=-\infty) = L(I(t=\infty) - I(t=-\infty)) = LI$$

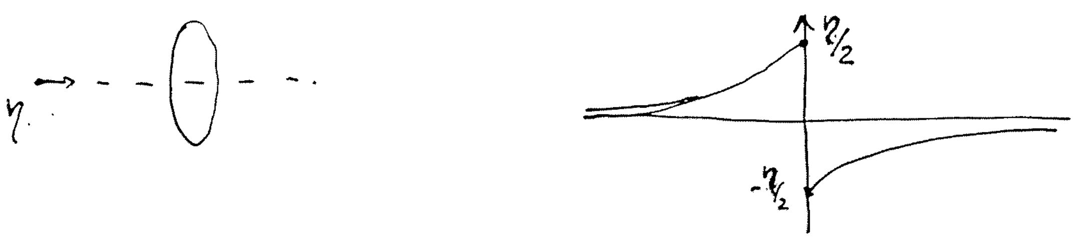
To be on familiar grounds, let us calculate the charge in electric flux when the electric charge passes thru a circle



$\frac{q}{4\pi\epsilon_0}\Omega$ ← solid angle

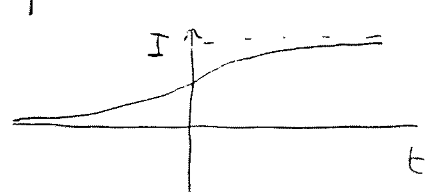


Similarly, for the magnetic monopole



$\Rightarrow$

$$\Rightarrow \quad I(t=\infty) - I(t=-\infty) = \frac{?}{L}$$



Four Maxwell's equations

$$\vec\nabla\cdot\vec E = \frac{\rho}{\epsilon_0}$$

$$(*)\qquad \vec\nabla\cdot\vec B = 0$$

$$\vec\nabla\times\vec E = -\frac{\partial B}{\partial t}$$

$$\vec\nabla\times\vec B = \mu_0\vec J + \mu_0\epsilon_0\frac{\partial\vec B}{\partial t}$$

They are almost symmetrical with respect to $\vec E$ and $\vec B$. If there were the magnetic monopoles, the equations would be

$$\vec\nabla\cdot\vec E = \frac{\rho}{\epsilon_0} \qquad \vec\nabla\times\vec E = -\mu_0\vec L - \frac{\partial\vec B}{\partial t}$$

$$\nabla\cdot\vec B = \mu_0\eta \qquad \nabla\times\vec B = \mu_0\vec J + \mu_0\epsilon_0\frac{\partial\vec E}{\partial t}$$

η - „density" of magnetic monopoles and L is their „current"

We do not see them in the experiment ⇒ probably they do not exist for some unknown reason.

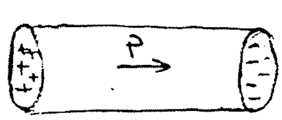
Maxwell's equations in matter

In matter, the original Maxwell's eqs (*) are unconvenient due to "bound" charges and currents in the r.h.s.'s. Let us rewrite them in terms of free charges only.

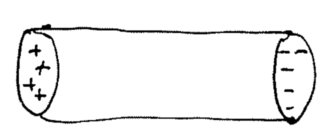
Recall $\rho_b = -\nabla\cdot\vec P$ - bound charge

$\vec J_b = \vec\nabla\times\vec M$ - bound current

$\rho_b = -\vec\nabla\cdot\vec P$ + conservation of charge ⇒ any change in $\vec P$ involves a flow of (bound) charge J_p



$\sigma_b = \vec P\cdot\vec n = P \qquad \sigma_b = -P$



P increases ⇒ current

$$dI = \frac{\partial\sigma_b}{\partial t}da_\perp = \frac{\partial P}{\partial t}da_\perp \Rightarrow$$

$$\Rightarrow \vec J_p = \frac{\partial\vec P}{\partial t}$$

NB: $\vec J_p$ has nothing to do with $\vec J_b$

J_p : 

J_b : 

Self-consistency check of continuity eqn

$$\vec{\nabla}\cdot\vec{J}_p = \vec{\nabla}\cdot\frac{\partial\vec{P}}{\partial t} = \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{P}) = -\frac{\partial\rho_b}{\partial t}$$

thus,

$$\left.\begin{aligned} \rho &= \rho_f + \rho_b = \rho_f - \vec{\nabla}\cdot\vec{P} = \text{free} + \text{bound} \\ \vec{J} &= \vec{J}_f + \vec{J}_b + \vec{J}_p = \vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t} = \text{free} + 2\ \text{bound} \end{aligned}\right\} \Rightarrow$$

Gauss law

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}(\rho_f - \vec{\nabla}\cdot\vec{P}) \iff \vec{\nabla}\cdot\vec{D} = \rho_f \qquad \vec{D} \equiv \epsilon_0\vec{E} + \vec{P}$$

Ampere's law

$$\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}\right) + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times(\vec{B} - \mu_0\vec{M}) = \mu_0\left(\vec{J}_f + \frac{\partial}{\partial t}(\epsilon_0\vec{E} + \vec{P})\right) \iff \vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$$

Maxwell's eqns in terms of free charges and currents are

$$\vec{\nabla}\cdot\vec{D} = \rho_f \qquad \vec{\nabla}\cdot\vec{B} = \phi$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$$

For linear media

$$\vec{P} = \epsilon_0\chi_e\vec{E} \iff \vec{D} = \epsilon\vec{E} \qquad \epsilon = \epsilon_0(1+\chi_e)$$

$$\vec{M} = \chi_m\vec{H} \iff \vec{H} = \frac{1}{\mu}\vec{B} \qquad \mu = \mu_0(1+\chi_m)$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_f + \underbrace{\frac{\partial\vec{D}}{\partial t}}_{\text{"displacement current"}}$$

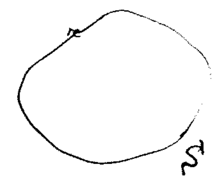
Boundary conditions for Maxwell's eqns in matter

Maxwell's eqns in the integral form

$$\oint_{S'} \vec{D}\cdot d\vec{a} = (Q_{free})_{enc}$$

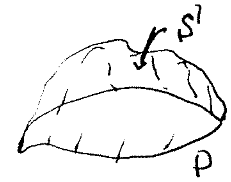
$$\oint_{S'} \vec{B}\cdot d\vec{a} = 0$$

↑ any closed surface



$$\oint_P \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a}$$

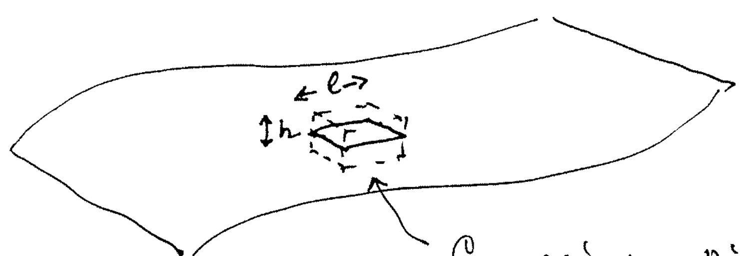
$$\oint_P \vec{H}\cdot d\vec{\ell} = (I_{free})_{enc} + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{a}$$



Math notation

$P \equiv \partial S \equiv$ boundary of S

1. Discontinuity of $\vec{D}$



Gaussian pillbox with $h \ll \ell \to 0$

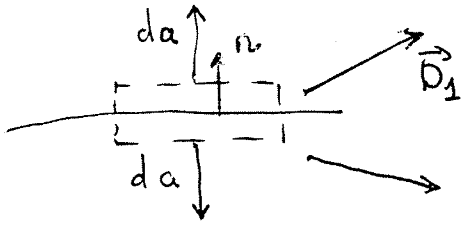
$$\int_{\text{pill box}} \vec{D}\cdot d\vec{a} = \int_{\text{pillbox}} da\,(\sigma)_{free}$$

$h \ll \ell \Rightarrow$ Flux thru the sides is small (in comparison to the flux thru the top or bottom)

$$\Rightarrow \oint \vec{D}\cdot d\vec{a} \simeq (\vec{D}_1\cdot\hat{n} - \vec{D}_2\cdot\hat{n})\,\Delta a = \sigma_f\,\Delta a \Rightarrow \vec{D}_1\cdot\hat{n} - \vec{D}_2\cdot\hat{n} = (\sigma)_{free}$$

↑ area of the lid of pillbox

side view →

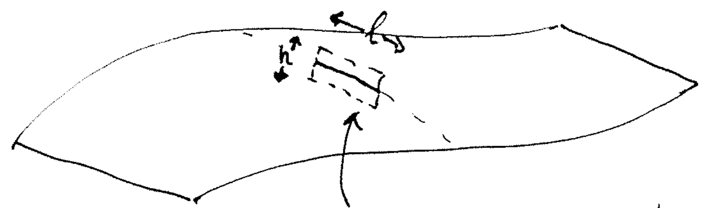


$$D_1^\perp - D_2^\perp = \sigma_f$$

2. Repeat the same steps for $\vec{B} \Rightarrow$

$\Rightarrow B_1^\perp = B_2^\perp$ is continuous

3. E is continuous



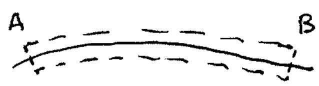
Amperian rectange with $h \ll l$

$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\oint \vec{B}\cdot d\vec{a} = 0$ since flux $\sim Bhl \to \emptyset$

$\vec{E}_1^{\parallel}\cdot\vec{l} - \vec{E}_2\cdot\vec{l} + \underbrace{\cancel{\vec{h}\cdot\vec{E}_\perp}}_{h \ll l} = (\vec{E}_1 - \vec{E}_2)\cdot\vec{l} = 0 \Rightarrow \vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = \emptyset \Rightarrow$

$\Rightarrow E^{\parallel}$ is continuous

(For the rigorous proof, we must take at first the limit $h \to 0$

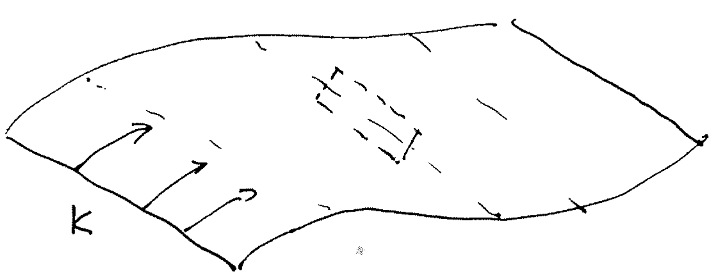


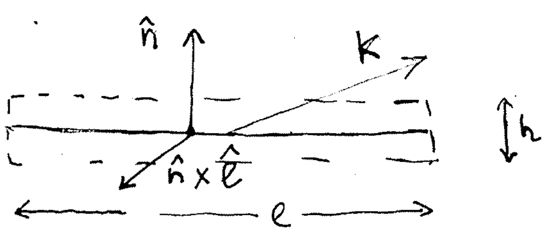
$\Rightarrow \left(\int_A^B \vec{E}\cdot d\vec{l}\right)_{above} - \left(\int_A^B \vec{E}\cdot d\vec{l}\right)_{below} \xrightarrow[h \to 0]{} 0$

$\Rightarrow \int_A^B (\vec{E}_1 - \vec{E}_2)\cdot d\vec{l} = 0$

After that, we take limit $l \to 0$ and get $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$)

4. Discontinuity of H





(Possible) volume current $\vec{J}$ does not contribute in the limit $h \to 0$, but $\vec{K}$ does.

$\oint \vec{H}\cdot d\vec{l} = (I_{free})_{enc} + \underbrace{\frac{d}{dt}\int \vec{D}\cdot d\vec{a}}_{\to 0 \text{ as } h \to 0} \Rightarrow (\vec{H}_1 - \vec{H}_2)\cdot\vec{l} = \vec{K}_{free}\cdot(\hat{n}\times\hat{l})$

$\Rightarrow (\vec{H}_1 - \vec{H}_2)\cdot\vec{l} = (\vec{K}_f \times \hat{n})\cdot\vec{l} \Rightarrow \vec{H}_1^{\parallel} - \vec{H}_2^{\parallel} = \vec{K}_f \times \hat{n}$

Summary

$$D_1^\perp - D_2^\perp = \sigma_f \qquad \vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = \emptyset$$

$$B_1^\perp - B_2^\perp = \emptyset \qquad \vec{H}_1^{\parallel} - \vec{H}_2^{\parallel} = \vec{K}_f \times \hat{n}$$

In the case of linear media ($D = \epsilon E$ and $H = \frac{1}{\mu} B$)

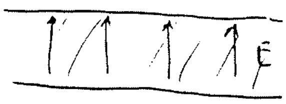
$$\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = \sigma_f \qquad \vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = \emptyset$$

$$B_1^\perp - B_2^\perp = \emptyset \qquad \frac{1}{\mu_1}\vec{B}_1^{\parallel} - \frac{1}{\mu_2}\vec{B}_2^{\parallel} = \vec{K}_f \cdot \hat{n}$$

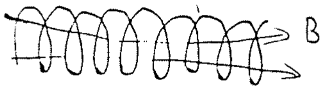
Without any charge or current at the surface

$$\left.\begin{array}{ll} \epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp & \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} \\ B_1^\perp = B_2^\perp & \frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel} \end{array}\right\} \text{basic eqns. for optics}$$

Conservation of energy in electrodynamics



$W_e = \frac{1}{2} C V^2 = \frac{\epsilon_0}{2} \int E^2 d^3x$ ← work done to charge a capacitor

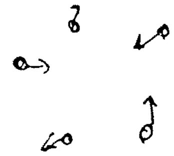


$W_m = \frac{1}{2} L I^2 = \frac{1}{2\mu_0} \int B^2 d^3x$ ← work done to build up a current

⇒ a guess for the energy stored in an electromagnetic field is

$$U_{em} = \frac{1}{2} \int d^3x \left(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0} \vec{B}^2 \right) \qquad (*)$$

Proof of the guess (*)

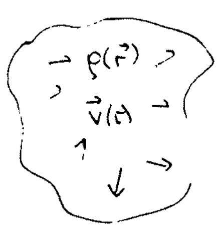


$$dW = \sum \vec{F}_{(i)} \cdot d\vec{\ell}_{(i)} = \sum q_{(i)} (\vec{E}(r_i) + \cancel{\vec{v}_{(i)} \times \vec{B}(r_i)}) \cdot \vec{v}_{(i)}\, dt$$
$$= \sum q_{(i)} \vec{E}(r_i) \cdot \vec{v}_{(i)}\, dt$$

set of (moving) charges q_i

For the continuous distribution of charges we get

$q_{(i)} \to \rho(\vec{r})\, d^3x$ and $q_i \vec{v}_i \to \rho(\vec{r}) \vec{v}(\vec{r})\, d^3x = J(\vec{r})\, d^3x$ ⇒



$$\frac{dW}{dt} = \int_V \vec{E} \cdot \vec{J}\, d^3x$$

Let us rewrite this eqn in terms of $\vec{E}$ and $\vec{B}$ only

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \Rightarrow \vec{J} = \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{E} \cdot \vec{J} = \frac{1}{\mu_0} \vec{E} \cdot (\vec{\nabla} \times \vec{B}) - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \Rightarrow \vec{E} \cdot (\vec{\nabla} \times \vec{B}) = \vec{B} \cdot \underbrace{(\vec{\nabla} \times \vec{E})}_{-\frac{\partial \vec{B}}{\partial t}} - \vec{\nabla} \cdot (\vec{E} \times \vec{B}) = -\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

$$\vec{E} \cdot \vec{J} = -\frac{1}{\mu_0} \vec{\nabla} \cdot (\vec{E} \times \vec{B}) - \frac{1}{\mu_0} \underbrace{\vec{B} \cdot \frac{\partial \vec{B}}{\partial t}}_{\frac{1}{2} \frac{d}{dt} \vec{B}^2} - \epsilon_0 \underbrace{\vec{E} \cdot \frac{\partial \vec{E}}{\partial t}}_{\frac{1}{2} \frac{\partial}{\partial t} \vec{E}^2} = -\frac{1}{2} \frac{\partial}{\partial t} \left(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0} \vec{B}^2 \right) - \frac{1}{\mu_0} \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

$$\Rightarrow \frac{dW}{dt} = \int_V d^3x \left\{ -\frac{1}{2} \frac{\partial}{\partial t} \left[\epsilon_0 \vec{E}^2(\vec{r},t) + \frac{1}{\mu_0} \vec{B}^2(\vec{r},t) \right] - \frac{1}{\mu_0} \underbrace{\vec{\nabla} \cdot \left(\vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t) \right)}_{\oint_{\text{surface}} (\vec{E} \times \vec{B}) \cdot d\vec{a}} \right\}$$

$$\Rightarrow \frac{dW}{dt} = -\frac{d}{dt}\int_V d^3x\ \frac{1}{2}\left(\epsilon_0 \vec{E}^2(\vec{r},t) + \frac{1}{\mu_0}\vec{B}^2(\vec{r},t)\right) - \frac{1}{\mu_0}\oint_S \left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)\cdot d\vec{a}$$

Poynting's theorem

$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$ Poynting vector: "energy flux density"

(the energy per unit time, per unit area, transported by the electromagnetic fields)

In terms of $\vec{S}$

$$\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S \vec{S}\cdot d\vec{a} \qquad U_{em} = \int d^3x\ u_{em}$$

$u_{em} = \frac{1}{2}\left(\epsilon_0 \vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^2\right)$ ← energy density of the e.m. fields

$$\frac{dW}{dt} = \frac{d}{dt}\int d^3x\ u_{mech}$$

mechanical energy density

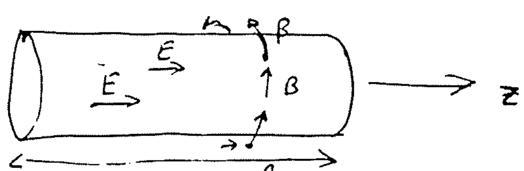
$$\Rightarrow \frac{d}{dt}\int_V d^3x\ (u_{mech} + u_{em}) = -\oint_S \vec{S}\cdot d\vec{a} \underset{\text{Gauss theorem}}{=} -\int_V \vec{\nabla}\cdot\vec{S}\ d^3x \Rightarrow$$

$$\Rightarrow \frac{d}{dt}(u_{mech} + u_{e.m.}) = -\vec{\nabla}\cdot\vec{S}$$ ← conservation of energy in the differential form

(cf. to $\frac{d\rho}{dt} = -\vec{\nabla}\cdot\vec{J}$ - conservation of charge)

Example 8.1

Joule heating of the wire



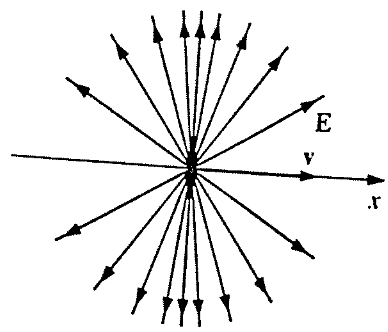
Energy (per unit time) delivered to the wire is $\oint \vec{S}\cdot d\vec{a}$
surface of the cylinder

$$\left.\begin{aligned}\vec{E} &= \frac{V}{L}\hat{e}_3 \\ \vec{B} &= \frac{\mu_0 I}{2\pi a}\hat{\varphi}\end{aligned}\right\} \vec{S} = \frac{\mu_0 I V}{2\pi a l \cancel{\mu_0}}\overbrace{\hat{e}_3\times\hat{\varphi}}^{-\hat{r}} = -\frac{IV}{2\pi a l}\hat{r}$$ ← inward flow

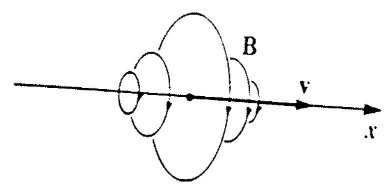
$$\Rightarrow \int_{\substack{\text{surface}\\ \text{of the}\\ \text{cylinder}}} \vec{S}\cdot d\vec{a} = \int S\,da = S\int da = S\,2\pi a l = VI \Rightarrow \frac{dW}{dt} = VI$$ - Ohm's law

Conservation of momentum in electrodynamics

Electric and magnetic fields of a moving charge:



(a)



(b)

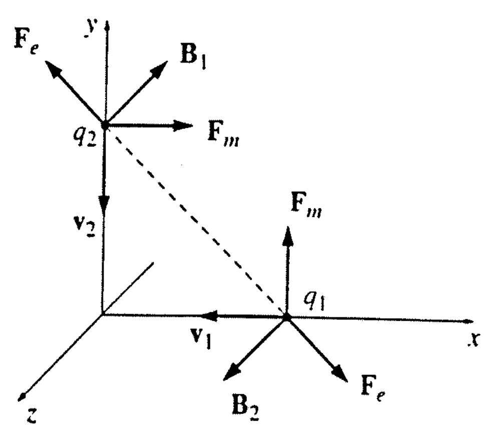
Test of Newton's Third Law:

$F_{electric}$ - OK

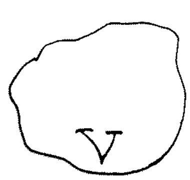
F_{mag} - not OK

Q: How?

A: Fields themselves carry momentum



Maxwell's stress tensor.



$$\vec{F} = \int_V d^3x \, \rho(r) \left(\vec{E}(\vec{r}) + \vec{v} \times \vec{B}(r)\right)$$

← Lorentz force acting on charges distributed over volume V

$f = \rho(E + \vec{v} \times \vec{B})$ -

- force per unit volume

A couple of mathematical tricks:

$$\vec{f} = \rho\vec{E} + \rho\vec{v}\times\vec{B} = \rho\vec{E} + \vec{J}\times\vec{B} = \epsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} + \left(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}$$

$$\frac{\partial\vec{E}}{\partial t}\times\vec{B} = \frac{d}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times\underbrace{\frac{\partial\vec{B}}{\partial t}}_{-\vec{\nabla}\times\vec{E}} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times(\vec{\nabla}\times\vec{E}) \Rightarrow$$

$$\Rightarrow \vec{f} = \epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E})\right] - \frac{1}{\mu_0}\vec{B}\times(\vec{\nabla}\times\vec{B}) - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} \Rightarrow$$

$$\Rightarrow \vec{\nabla}(\vec{E}^2) = 2(\vec{E}\cdot\vec{\nabla})\vec{E} + 2\vec{E}\times(\vec{\nabla}\times\vec{E}) \Rightarrow \vec{E}\times(\vec{\nabla}\times\vec{E}) = \frac{1}{2}\vec{\nabla}(E^2) - (\vec{E}\cdot\vec{\nabla})\vec{E}$$

similarly, $\vec{B}\times(\vec{\nabla}\times\vec{B}) = \frac{1}{2}\vec{\nabla}(B^2) - (\vec{B}\cdot\vec{\nabla})\vec{B}$

$$\Rightarrow \vec{f} = \epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E}\right] + \frac{1}{\mu_0}\Big[(\vec{B}\cdot\vec{\nabla})\vec{B} + \underbrace{(\vec{\nabla}\cdot\vec{B})\vec{B}}_{=\emptyset}\Big] - \frac{1}{2}\vec{\nabla}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\cdot\vec{B})$$

we add this for symmetry

Maxwell strength tensor

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right) \qquad \overleftrightarrow{T} \equiv \{T_{ij}\}$$

$$\vec{f} = \vec{\nabla}\cdot\overleftrightarrow{T} - \mu_0\epsilon_0\frac{\partial\vec{S}}{\partial t}$$

About tensors.

Let us at first recall the definition of a vector (for simplicity, in 2 dimensions)

vector ≡ "magnitude + direction"

Mathematically, vector is defined as a pair of numbers ("components" of the vector)

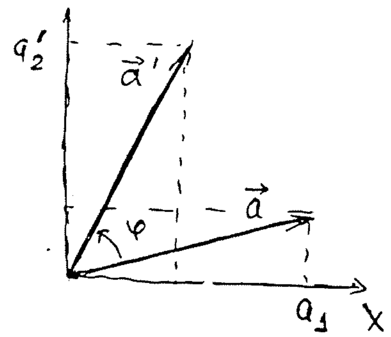
Not any pair of numbers constitutes a vector; this pair must behave in a specific way under rotations (and shifts)

How does a vector behave under translations and rotations

Under translations - trivial behavior

$a_1 \to a_1$
$a_2 \to a_2$

Under rotations - non-trivial transformation



(a_1', a_2') - components of the new vector $\vec{a}'$

$$a_1' = a_1 \cos\varphi - a_2 \sin\varphi$$
$$a_2' = a_1 \sin\varphi + a_2 \cos\varphi$$

(check: $a_1'^2 + a_2'^2 = a_1^2 + a_2^2 = \vec{a}^2$)

Mathematical notation

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \iff a_i' = \sum_{k=1}^{2} R_{ik} a_k$$

$R(\varphi)$ - matrix of the rotation on angle φ

$R_{11} = R_{22} = \cos\varphi$
$R_{21} = -R_{12} = \sin\varphi$

Definition: a vector is a pair of numbers (a_1, a_2) which behaves like $(a_1, a_2) \to (a_1', a_2')$ $\quad a_i' = R_{ik}(\varphi) a_k$ under the rotations.

Let us now take two vectors, $\vec{a}$ and $\vec{b}$, and consider the products of their components

$a_1 b_1$, $a_1 b_2$, $a_2 b_1$, $a_2 b_2$

These 4 numbers behave under rotations as follows:

$$a_1' b_1' = (a_1 \cos\varphi - a_2 \sin\varphi)(b_1 \cos\varphi - b_2 \sin\varphi)$$
$$a_1' b_2' = (a_1 \cos\varphi - a_2 \sin\varphi)(b_1 \sin\varphi + b_2 \cos\varphi)$$
$$a_2' b_1' = (a_1 \sin\varphi + a_2 \cos\varphi)(b_1 \cos\varphi - b_2 \sin\varphi)$$
$$a_2' b_2' = (a_1 \sin\varphi + a_2 \cos\varphi)(b_1 \sin\varphi + b_2 \cos\varphi)$$

Formally,

$$a_i' b_j' = \sum_{k,l} R_{ik} R_{jl} a_k b_l \quad (*)$$

Definition of a (2-dim) tensor:

4 numbers $t_{11}, t_{12}, t_{21}, t_{22}$ form a tensor if they behave as (*) under the rotations (and do not change under shifts)

$t'_{11}, t'_{12}, t'_{21}, t'_{22}$ - components of the rotated tensor

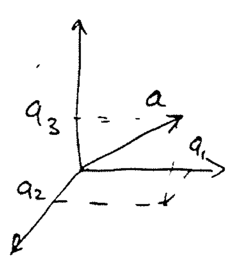
$$t'_{11} = t_{11}\cos^2\varphi - (t_{12}+t_{21})\sin\varphi\cos\varphi + t_{22}\sin^2\varphi$$
$$t'_{12} = t_{11}\cos\varphi\sin\varphi + t_{12}\cos^2\varphi - t_{21}\sin^2\varphi - t_{22}\cos\varphi\sin\varphi$$
$$t'_{21} = t_{11}\cos\varphi\sin\varphi - t_{12}\sin^2\varphi + t_{21}\cos^2\varphi - t_{22}\cos\varphi\sin\varphi$$
$$t'_{22} = t_{11}\sin^2\varphi + (t_{12}+t_{21})\sin\varphi\cos\varphi + t_{22}\cos^2\varphi$$

In matrix notations $t'_{ij} = \sum_{k,l} R_{ik} R_{l} t_{jl}$

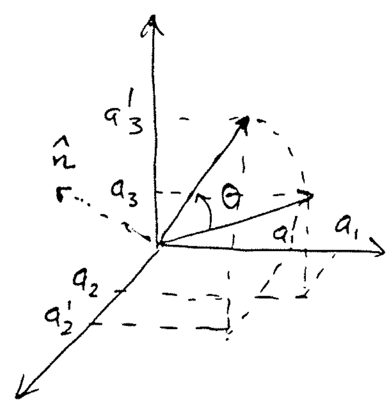
t_{12} is not necessarily equal to t_{21}; if $t_{12} = t_{21}$ → "symmetric tensor"

For 3-dimensional tensors - similar definition

3 d vector: a_1, a_2, a_3



Under rotations



$a'_i = \sum R_{ik} a_k$

matrix of 3d rotations

(can be parametrized by $\hat{n}, \theta$ or by 3 Euler angles)

⇒ 3d vector is a set of 3 numbers $\underbrace{}_{(a_1 a_2 a_3)}$ which transform as $a'_i = \sum_k R_{ik} a_k$ under the rotation

Similarly to the 2d case, a 3d tensor is defined as a set of 9 numbers $t_{12}, t_{13}, t_{21}, t_{22}, t_{23}, t_{31}, t_{32}, t_{33}$ which behave as products of components of two vectors under the rotations

$$t'_{ij} = \sum_{k,l=1}^{3} R_{ik} R_{jl} t_{kl}$$

Let us prove that δ_{ij} is a tensor (for simplicity, in the 2d case)

$$\delta_{11}' = \overset{=1}{\delta_{11}}\cos^2\varphi - (\underset{=\emptyset}{\delta_{12}} + \underset{=\emptyset}{\delta_{21}})\sin\varphi\cos\varphi + \overset{=1}{\delta_{22}}\sin^2\varphi = \cos^2\varphi + \sin^2\varphi = 1 = \delta_{11}$$

$$\delta_{12}' = \delta_{11}\cos\varphi\sin\varphi + \underset{=\emptyset}{\delta_{12}}\cos^2\varphi - \underset{=\emptyset}{\delta_{21}}\sin^2\varphi - \delta_{22}\cos\varphi\sin\varphi = \emptyset$$

$$\delta_{21}' = (\delta_{11} - \delta_{22})\sin\varphi\cos\varphi - {}_{12}\sin^2\varphi + {}_{21}\cos^2\varphi = \emptyset$$

$$\delta_{22}' = \delta_{11}\sin^2\varphi + (\delta_{12} + \delta_{21})\sin\varphi\cos\varphi + \delta_{22}\cos^2\varphi = 1 = \delta_{22}$$

$\Rightarrow$ δ_{ij} is a tensor. It can be proved for 3d case in a similar way

We will need two mathematical properties

1. A sum of two tensors is also a tensor $\left((A+B)_{ij} \overset{def}{\equiv} A_{ij} + B_{ij}\right)$

Proof: $A_{ij}' + B_{ij}' = \sum_{k,l} R_{ik} R_{jl} A + \sum_{k,l} R_{ik} R_{jl} B_{kl} = \sum_{k,l} R_{ik} R_{jl} (A_{kl} + B_{kl})$

$\Rightarrow (A+B)_{ij}' = \sum_{k,l} R_{ik} R_{jl} (A+B)_{kl}$

2. If a_i is a vector and t_{ij} is a tensor, the sums $\sum a_k t_{ki}$ form a vector (denoted by $(\vec{a}\cdot\overleftrightarrow{t})$)

Proof: $(\vec{a}\cdot\overleftrightarrow{t})_i' \equiv (\vec{a}'\cdot\overleftrightarrow{t}')_i = \sum_k a_k' t_{ki}' = \sum_k \sum_{l,m,n} R_{kl} a_l R_{km} R_{in} t_{mn}$

Property of the rotation matrix

$\sum_k R_{kl} R_{km} = \delta_{mn}$ (Proof: for any vector a $\vec{a}'^2 = a^2 \Rightarrow$
$\Rightarrow \sum_k a_k' a_k' = \sum_{k,l,m} \underbrace{R_{kl} R_{km}} a_l a_m = \sum_l a_l a_l = \sum_{l,m} \underbrace{\delta_{lm}} a_l a_m$)

$\Rightarrow (\vec{a}\cdot\overleftrightarrow{t})_i = \sum_{m,l,n} \delta_{ml} R_{in} a_l t_{mn} = \sum_{l,n} R_{in} a_l t_{ln} = \sum_n R_{in} (\sum a_l t_{ln}) = \sum_n R_{in} (\vec{a}\cdot\overleftrightarrow{t})_n$

$\Rightarrow$ $\vec{a}\cdot\overleftrightarrow{t}$ is a vector

Maxwell stress tensor

$$T_{ij} = \epsilon_0 \left(E_i E_j - \tfrac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \tfrac{1}{2}\delta_{ij} B^2\right)$$

tensor tensor tensor tensor $\Rightarrow$ T_{ij} is a tensor

$(\vec{\nabla}\cdot\overleftrightarrow{T})_i = \sum_k \partial_k T_{ki} = \sum_k \epsilon_0 \left(\partial_k(E_k E_i) - \tfrac{1}{2}\partial_i(E^2)\right) + \frac{1}{\mu_0}(E \leftrightarrow B) = \sum_k \epsilon_0 \left((\partial_k E_k) E_i + E_k \partial_k E_i - \tfrac{1}{2}\partial_i E^2\right) + \frac{1}{\mu_0}(E \leftrightarrow B) \Rightarrow (\vec{\nabla}\cdot\overleftrightarrow{T})_i = \epsilon_0 \left[(\vec{\nabla}\cdot\vec{E})\vec{E}_i + (\vec{E}\cdot\vec{\nabla})\vec{E}_i - \tfrac{1}{2}\partial_i E^2\right] + \frac{1}{\mu_0}(E \leftrightarrow B) \Rightarrow \vec{\nabla}\cdot\overleftrightarrow{T} = \epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} - \tfrac{1}{2}\vec{\nabla}E^2\right] + \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B} - \tfrac{1}{2}\vec{\nabla}B^2\right]$

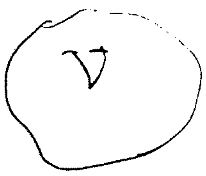
Maxwell stress tensor

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{\delta_{ij}}{2} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{\delta_{ij}}{2} B^2\right)$$

force (per unit volume) acting on a charged body

$$f = \vec{\nabla}\cdot\overleftrightarrow{T} - \epsilon_0\mu_0 \frac{\partial \vec{S}}{\partial t} \qquad \left(\text{in components } f_i = \sum_j \partial_j T_{ji} - \epsilon_0\mu_0 \frac{\partial S_i}{\partial t}\right)$$

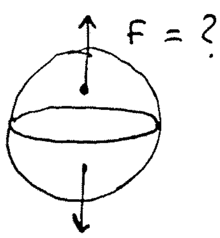
Total force on the charges in $\mathcal{V}$



$$\vec{F} = \oint_S \overleftrightarrow{T}\cdot d\vec{a} - \mu_0\epsilon_0 \frac{d}{dt}\int_{\mathcal{V}} \vec{S}\, d^3x$$

↑ force per unit area (stress) acting on the surfa

Example:



Uniformly charged sphere
The (electric) field is static ⇒

$$\Rightarrow \vec{F} = \int_{\text{Surface}} \overleftrightarrow{T}\cdot d\vec{a} = \int_{\text{"bowl"}} \overleftrightarrow{T}\cdot d\vec{a} + \int_{\text{"disk"}} \overleftrightarrow{T}\cdot d\vec{a}$$

1. $\int_{\text{bowl}} \overleftrightarrow{T}\cdot d\vec{a}$ From symmetry it is clear that $\vec{F} \uparrow\uparrow e_3 \Rightarrow$ it is enough to calculate $\int (\overleftrightarrow{T}\cdot d\vec{a})_3$

$$F_3 = \int_{\text{bowl}} \sum_{j=1}^{3} \epsilon_0 \left(E_3 E_j - \frac{\delta_{3j}}{2} E^2\right) da_j \qquad d\vec{a} = \hat{r}\, da \Rightarrow da_j = \hat{r}_j\, da$$

$$\vec{E} = \frac{Q}{4\pi R^2 \epsilon_0}\hat{r} \Rightarrow E_j = \frac{Q}{4\pi R^2\epsilon_0}\hat{r}_j$$

$$\Rightarrow \sum_j \left(E_3 E_j - \frac{\delta_{3j}}{2}E^2\right) da_j = \left(\frac{Q}{4\pi R^2 \epsilon_0}\right)^2 \underbrace{\sum_j \left(\hat{r}_3 \hat{r}_j - \frac{\delta_{3j}}{2}\right)\hat{r}_j}_{\hat{r}_3 - \frac{\hat{r}_3}{2} = \frac{\hat{r}_3}{2}} da =$$

$$= \frac{Q^2}{32\pi^2 R^4 \epsilon_0^2}\int_{\text{bowl}} da\, \hat{r}_3 =$$

$$= \frac{Q^2}{32\pi^2 R^4\epsilon_0^2}\int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi\, R^2 \sin\theta\cos\theta = \frac{Q^2}{16\pi R^2 \epsilon_0}\underbrace{\int_0^{\pi/2} d\theta \sin\theta\cos\theta}_{1/2} = \frac{Q^2}{32\pi R^2 \epsilon_0}$$

2. $\int \overleftrightarrow{T}\cdot d\vec{a}$ Again,

$$F_3 = \int_{\text{disc}} \sum_j \epsilon_0 \left(E_3 E_j - \frac{\delta_{3j}}{2} E^2\right) da_j$$

For the disc $d\vec{a} = -\hat{e}_3\, da$

The field inside the uniformly charged sphere is

$$\vec{E} = \frac{Qr}{4\pi\epsilon_0 R^3}\hat{r}$$

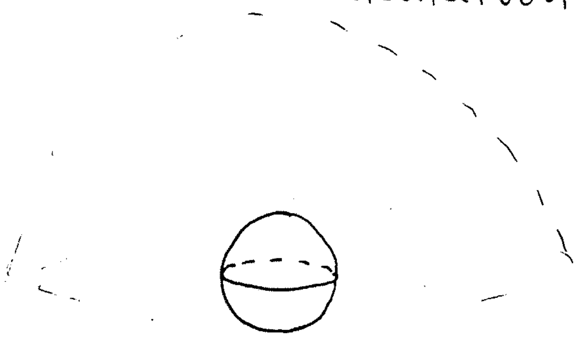
$$\Rightarrow F_3 = -\int \sum_j \epsilon_0 \left(\hat{r}_3\hat{r}_3 - \frac{\delta_{33}}{2}\right)\left(\frac{Qr}{4\pi R^3\epsilon_0}\right)^2 da =$$

Inside the disc $\vec{r} \perp \hat{e}_3 \Rightarrow r_3 = \emptyset$

$$= \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi R^3\epsilon_0}\right)^2 \int r^2 da = \frac{Q^2\, 2\pi}{32\pi^2 R^6\epsilon_0}\int_0^R r^3 dr =$$

$$= \frac{Q^2}{64\pi\epsilon_0 R^2} \quad \Rightarrow \quad \vec{F}_{total} = \vec{F}_{bowl} + \vec{F}_{disc} = \frac{3Q^4}{64\pi\epsilon_0 R^2}\hat{e}_3$$

Alternative calculation



surface = xy plane + ∞ hemisphere

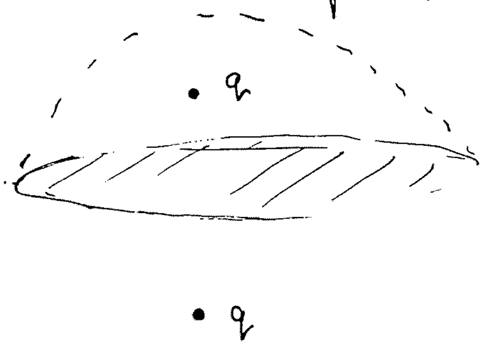
$$\int_{\text{surface}} \overleftrightarrow{T}\cdot d\vec{a} = \int_{\text{xy plane}} \overleftrightarrow{T}\cdot d\vec{a} + \cancel{\int_{\infty\text{ hemisphere}} \overleftrightarrow{T}\cdot d\vec{a}}$$

$$\Rightarrow F_3 = -\int_{\text{all plane}} \sum_j \epsilon_0 \left(\hat{r}_3\hat{r}_3 - \frac{\delta_{33}}{2}\right) E^2(r)\, da \Rightarrow$$

$$\Rightarrow F_3 = \frac{1}{2}\int_{\text{all plane}} \epsilon_0 E^2(r)\, da = \frac{\cancel{2}\pi}{\cancel{2}}\int_0^R r\,dr\, \epsilon_0 \left(\frac{Qr}{4\pi\epsilon_0 R^3}\right)^2 + \frac{\cancel{2}\pi}{\cancel{2}}\int_R^\infty r\,dr\, \epsilon_0\left(\frac{Q}{4\pi\epsilon_0 r}\right)^2$$

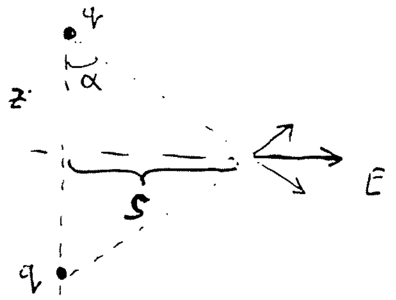
$$= \frac{Q^2}{16\pi\epsilon_0}\left(\int_0^R \frac{1}{R^6} r^3 dr + \int_R^\infty \frac{dr}{r^3}\right) = \frac{Q^2}{16\pi\epsilon_0}\left(\frac{1}{4R^2} + \frac{1}{2R^2}\right) = \frac{3Q^2}{64\pi\epsilon_0 R^2}$$

Another example: Coulomb law



$$F_3 = \int_{\text{xy plane}} \overleftrightarrow{T}\cdot d\vec{a} =$$

$$= -\int_{\text{plane}} da \left(E_3E_3 - \frac{1}{2}E^2\right)\epsilon_0$$



$$\vec{E} = 2\,\frac{q}{(z^2+r^2)}\,\frac{\sin\alpha}{4\pi\epsilon_0}\,\hat{s} \Rightarrow E_3 = \emptyset$$

$$E^2 = \frac{q^2}{4\pi^2\epsilon_0^2}\,\frac{z^2}{(z^2+s^2)^3}$$

$$\Rightarrow F_3 = \frac{\epsilon_0}{2}\int_0^\infty s\,ds\int_0^{2\pi} d\varphi\,\frac{q^2}{4\pi^2\epsilon_0^2}\,\frac{z^2}{(z^2+s^2)^3} = \frac{z^2q^2}{4\pi\epsilon_0}\int_0^\infty ds\,\frac{s}{(z^2+s^2)^3} = \frac{z^2q^2}{4\pi\epsilon_0}\,\frac{1}{4z^4}$$

$$\Rightarrow F_3 = \frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{2z}\right)^2 \equiv \text{Coulomb law.}$$

Conservation of momentum

$$\vec{F} = \frac{d\vec{p}_{mech}}{dt} \Rightarrow \frac{d\vec{p}_{mech}}{dt} = -\mu_0\epsilon_0\frac{d}{dt}\int_{\mathcal{V}}\vec{S}\,d^3x + \oint_{S}\overleftrightarrow{T}\cdot d\vec{a}$$

$\vec{P}_{em} = \mu_0\epsilon_0\int_{\mathcal{V}}\vec{S}\,d^3x$ → momentum stored in the electromagnetic fields themselves

$$\Rightarrow \frac{d}{dt}(\vec{P}_{mech}+\vec{P}_{em}) = \oint_{S}\overleftrightarrow{T}\cdot d\vec{a} \Rightarrow \frac{d}{dt}(\vec{P}_{mech}+\vec{P}_{em})_{\text{all space}} = \emptyset$$

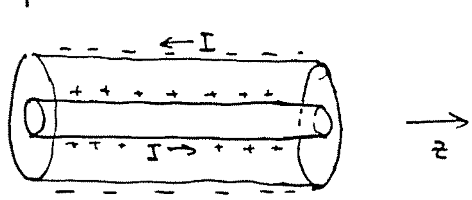
Conservation of momentum in a differential form

$$\vec{P}_{mech} = \int_{\mathcal{V}} d^3x\,\vec{\mathcal{P}}_{mech} \qquad \vec{P}_{em} = \int_{\mathcal{V}} d^3x\,\vec{\mathcal{P}}_{em} \qquad \vec{\mathcal{P}}_{em} = \mu_0\epsilon_0\vec{S}$$

$$\Rightarrow \frac{d}{dt}\int_{\mathcal{V}} d^3x\,(\vec{\mathcal{P}}_{mech}+\vec{\mathcal{P}}_{em}) = \oint_{S}\overleftrightarrow{T}\cdot d\vec{a} = \int\vec{\nabla}\cdot\overleftrightarrow{T}\,d^3x \Rightarrow \frac{d}{dt}(\mathcal{P}_{mech}+\mathcal{P}_{em})\ \vec{\nabla}\cdot\overleftrightarrow{T}$$

$\overleftrightarrow{T}$ is the momentum flux density

Example 8.3



$E = B = \emptyset$ outside the cable

$E = \frac{\lambda}{2\pi\epsilon_0 s}\,\hat{s} \qquad B = \frac{\mu_0 I}{2\pi s}\,\hat{\varphi}$ inside

$$\vec{S} = \frac{1}{\mu_0}\,\vec{E}\times\vec{B} = \frac{\lambda I}{4\pi^2\epsilon_0 s^2}\,\hat{e}_3$$

$$\Rightarrow \vec{P}_{e.m.} = \mu_0\epsilon_0\int\vec{S}\,d^3x = \frac{\mu_0\lambda I\hat{e}_3}{4\pi^2}\int_0^\ell dz\int_a^b\frac{s\,ds}{s^2}\,2\pi = \frac{\mu_0\lambda I\ell}{2\pi}\ln\frac{b}{a}\,\hat{e}_3$$

This momentum is compensated by the "hidden" mechanical momentum (a relativistic effect)

Angular momentum of electromagnetic fields

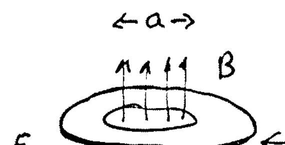
$u_{em} = \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2)$ – energy (per unit volume)

$\vec{\mathcal{P}}_{em} = \mu_0 \epsilon_0 \vec{S} = \epsilon_0 \vec{E} \times \vec{B}$ – momentum

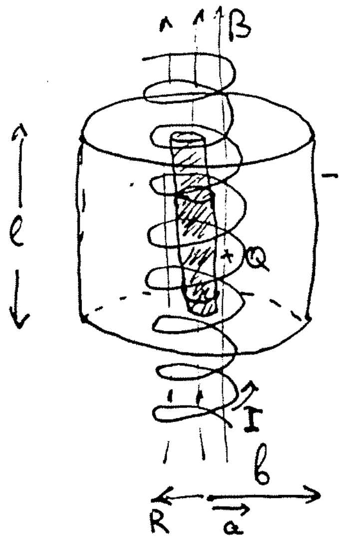
$\Rightarrow \ \ell_{em} = \vec{r} \times \vec{\mathcal{P}} = \epsilon_0 \vec{r} \times (\vec{E} \times \vec{B})$ – angular momentum of electromagnetic fields

Example 8.4

Example 7.8 :



B is switched off $\Rightarrow$ electric field $\Rightarrow$ torque $N = -B\lambda a^2 \frac{dB}{dt} \Rightarrow$
$\Rightarrow$ total angular momentum is $\int N dt = \lambda \pi a^2 b B_0$



$$\left.\begin{array}{ll} E = \dfrac{Q/\ell}{2\pi\epsilon_0 s}\hat{s} & a<s<b \\ E = \emptyset & \text{otherwise} \end{array}\right\} \text{ before switch off the current}$$

$$B = \mu_0 n I \hat{e}_3 \, \Theta(R-s)$$

$= \emptyset$ outside the solenoid

$$\Rightarrow \vec{\mathcal{P}}_{em} = \epsilon_0 \vec{E} \times \vec{B} = -\frac{\mu_0 n I Q}{2\pi \ell s}\hat{\varphi}$$

$$\Rightarrow \vec{\ell}_{em} = \vec{r} \times \vec{\mathcal{P}}_{em} = (\vec{s} + z\hat{e}_3) \times \left(-\frac{\mu_0 n I Q}{2\pi \ell s}\right)\hat{\varphi}$$

$$= -\frac{\mu_0 n I Q}{2\pi \ell}\hat{e}_3 - \underbrace{z\frac{\mu_0 n I}{2\pi \ell s}\hat{r}}_{\text{will average to } \emptyset \text{ due to cylindrical symmetry}} = -\frac{\mu_0 n I Q}{2\pi \ell}\hat{e}_3$$

$$\Rightarrow \vec{L}_{em} = \ell_{em} \cdot \overbrace{\text{volume}}^{\pi(R^2-a^2)\ell} = -\frac{1}{2}\mu_0 n I Q (R^2 - a^2)\ell$$

When the current is turned off

$$\vec{E} = \begin{cases} -\dfrac{\mu_0 n}{2}\dfrac{dI}{dt}\dfrac{R^2}{s}\hat{\varphi} & s>R \\ -\dfrac{\mu_0 n}{2}\dfrac{dI}{dt}\, s\hat{\varphi} & s<R \end{cases}$$

$\Rightarrow$ The torque on the outer cylider is

$$\vec{N}_b = \hat{r} \times (-Q\vec{E}) = \frac{1}{2}\mu_0 n Q R^2 \frac{dI}{dt}\vec{e}_3$$

$\Rightarrow$ The outer cylinder picks up the angular momentum

$$\vec{L}_b = \int N_b \, dt = \tfrac{1}{2}\mu_0 n Q R^2 \hat{e}_3 \int_0^\infty \frac{dI}{dt} dt = -\frac{\mu_0 n}{2} I Q R^2 \hat{e}_3$$

Similarly, the torque on the inner cylinder is

$$\vec{N}_a = -\tfrac{1}{2}\mu_0 n Q a^2 \frac{dI}{dt} \hat{e}_3$$

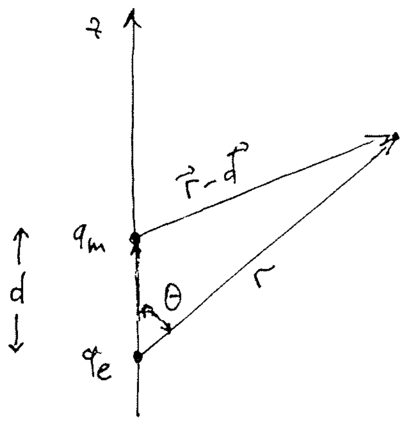
$\Rightarrow L_a = \tfrac{1}{2}\mu_0 n I Q a^2 \hat{e}_3$ Check : $L_b + L_a = -\tfrac{1}{2}\mu_0 n I Q^2 (R^2 - a^2)\, \ell$

Thomson's dipole

q_e (electric charge) $\qquad q_m$ ← magnetic monopole

Let us calculate the angular momentum stored in the electromagnetic field of this "dipole"

$$\ell = \epsilon_0\, \vec r \times (\vec E(r) \times \vec B(\vec r))$$



$$\ell(\vec r) = \epsilon_0\, \vec r \times \left(\frac{q_e}{4\pi\epsilon_0} \frac{\vec r}{r^3} \times \frac{q_m \mu_0}{4\pi} \frac{\vec r - \vec d}{|\vec r - \vec d|^3} \right)$$

$$\left(\text{NB: } \widehat{a-b} \neq \hat a - \hat b \Rightarrow \frac{\vec r - \vec r'}{|\vec r - \vec r'|^3} \neq \frac{\hat r - \hat r'}{|\vec r - \vec r'|^2} \right)$$

$$\ell(\vec r) = \frac{q_e q_m}{16\pi^2} \mu_0 \frac{-\vec r \times (\vec r \times \vec d)}{r^3 (r^2+d^2-2dr\cos\theta)^{3/2}}$$

$$\vec r \times (\vec r \times \vec d) = \vec r(\vec d \cdot \vec r) - \vec d r^2 = -d r^2 (1-\cos\theta)\, \hat e_3$$

$$\Rightarrow \vec L = \int d^3x\, \ell(\vec r) = \int_0^\infty r^2 dr \int_0^\pi d\theta \sin\theta \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \frac{q_e q_m}{16\pi^2} \mu_0 \frac{d(1-\cos\theta)\, \hat e_3}{r (d^2+r^2-2dr\cos\theta)^{3/2}}$$

$$= \frac{q_e q_m}{8\pi} \mu_0 d \int_0^\infty dr \int_0^2 dt \frac{t r\, \hat e_3}{((d-r)^2 + 2drt)^{3/2}} = \qquad t \equiv 1-\cos\theta$$

$$= \frac{q_e q_m}{8\pi} \mu_0 \underbrace{\int_0^\infty d\lambda \int_0^2 dt \frac{t\lambda\, \hat e_3}{((\lambda-1)^2 + 2\lambda t)^{3/2}}}_{=2} = \frac{\mu_0}{4\pi} q_e q_m\, \hat e_3$$

Calculation: $\displaystyle \int_0^\infty d\lambda \int_0^2 dt \frac{\lambda t}{((\lambda-1)^2+2\lambda t)^{3/2}} = \int_0^\infty d\lambda \int_0^2 dt \frac{\lambda}{|\lambda-1|^3} \frac{t}{\left(1+\frac{2\lambda}{(\lambda-1)^2} t\right)^{3/2}} =$

$$\boxed{\frac{2\lambda}{(\lambda-1)^2} t \equiv y} = \int_0^\infty d\lambda \frac{|\lambda-1|}{4\lambda} \int_0^{2\frac{2\lambda}{(\lambda-1)^2}} \frac{y}{(1+y)^{3/2}} dy = \int_0^\infty d\lambda \frac{|\lambda-1|}{4\lambda} \int_1^{\frac{(\lambda+1)^2}{(\lambda-1)^2}} dy \frac{y-1}{y^{3/2}} =$$

$$\int_0^\infty d\lambda \frac{|\lambda-1|}{4\lambda} \cdot 2\left(\frac{\lambda+1}{|\lambda-1|} - 1 - 1 + \frac{|\lambda-1|}{\lambda+1} \right) = \int_0^\infty d\lambda \frac{1}{2\lambda}\left(\lambda+1-2|\lambda-1| + \frac{(\lambda-1)^2}{\lambda+1}\right) = \int_0^\infty \frac{d\lambda}{\lambda}\left(\frac{\lambda^2+1}{\lambda+1} - |\lambda-1|\right) =$$

$$\int_0^1 \frac{d\lambda}{\lambda}\left(\frac{\lambda^2+1}{\lambda+1} - 1 + \lambda\right) + \int_1^\infty \frac{d\lambda}{\lambda}\left(\frac{\lambda^2+1}{\lambda+1} - \lambda + 1\right) = \int_0^1 d\lambda \frac{2\lambda}{\lambda+1} + \int_1^\infty \frac{d\lambda}{\lambda} \frac{2}{\lambda+1} =$$

$$= 2\int_0^1 d\lambda \frac{\lambda}{\lambda+1} + 2\int_0^1 d\lambda' \frac{1}{1+\lambda'} = 2 \Big)$$

$\Rightarrow L = \frac{\mu_0}{4\pi} q_e q_m$ ← does not depend on separation

Quant. mech: $\quad L = n\frac{\hbar}{2} \qquad n = 0, 1, 2 \ldots$

$\Rightarrow \frac{\mu_0}{4\pi} q_e q_m = n\frac{\hbar}{2}$ if the magnetic monopole would exist somewhere