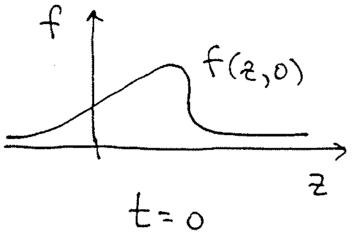
Waves

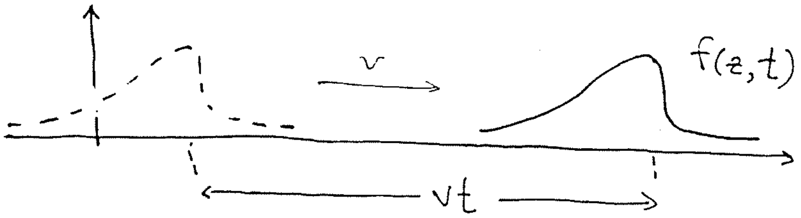
I. Mechanical waves in one dimension

A function

$f(z,t) = g(z - vt)$

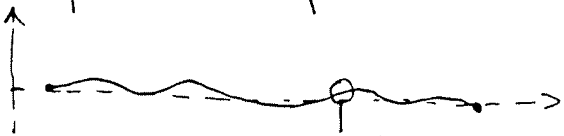
is called a (1-dim) wave



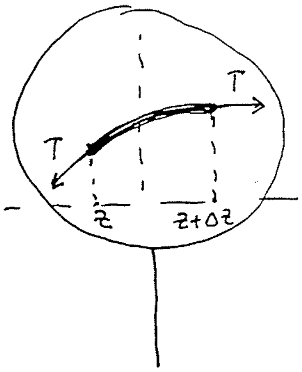


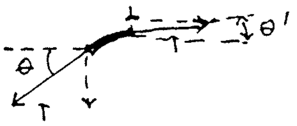
A "fixed" shape is traveling to the right at speed v.

Typical example - waves in strings



string under constant tension T





$\Delta F_z = T\sin\theta' - T\sin\theta$

θ, θ' - small $\Rightarrow$

$\Rightarrow \Delta F_z \simeq T(\tan\theta' - \tan\theta)$ $\qquad \sin\theta \simeq \theta \simeq \tan\theta$ at $\theta \ll 1$

$\tan\theta = \frac{\partial f}{\partial z} \Rightarrow \Delta f_z = T\left(\frac{\partial f}{\partial z}\Big|_{z+\Delta z} - \frac{\partial f}{\partial z}\Big|_{z}\right) \simeq T\frac{\partial^2 f}{\partial z^2}\Delta z$

Newton's 2nd law

$\underset{\uparrow}{\mu}\Delta z \frac{\partial^2 f}{\partial t^2} = \Delta F_z \Rightarrow \mu\frac{\partial^2 f}{\partial t^2} = T\frac{\partial^2 f}{\partial z^2} \Rightarrow \frac{\partial^2 f(z,t)}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 f(z,t)}{\partial t^2} \Rightarrow$

mass per unit length

$$(*) \qquad \frac{\partial^2 f(z,t)}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f(z,t)}{\partial t^2}$$

"wave equation"

$v = \sqrt{T/\mu}$

A solution of wave equation is

$f(z,t) = g(z - vt)$

where $g(u)$ is an arbitrary function of $u = z - vt$

Proof

$$\frac{\partial}{\partial z} g(z-vt) = g'(u)\Big|_{u=z-vt} \qquad \frac{\partial^2}{\partial z^2} g(z-vt) = g''(u)\Big|_{u=z-vt}$$

On the other hand,

$$\frac{\partial}{\partial t} g(z-vt) \underset{\substack{\uparrow \\ \text{chain rule}}}{=} \frac{\partial(z-vt)}{\partial t} g'(u)\Big|_{u=z-vt} = -vg'(u)\Big|_{u=z-vt}$$

$$\frac{\partial^2}{\partial t^2} g(z-vt) = \frac{\partial}{\partial t}\left(-vg'(u)\Big|_{u=z-vt}\right) = -v\frac{\partial}{\partial t}\left(g'(u)\Big|_{u=z-vt}\right) = -v\frac{\partial(z-vt)}{\partial t} g''(u)\Big|_{u=z-}$$

$$= v^2 g''(u)\Big|_{u=z-vt} \quad \Rightarrow \quad \frac{\partial^2}{\partial z^2} g(z-vt) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2} g(z-vt)$$

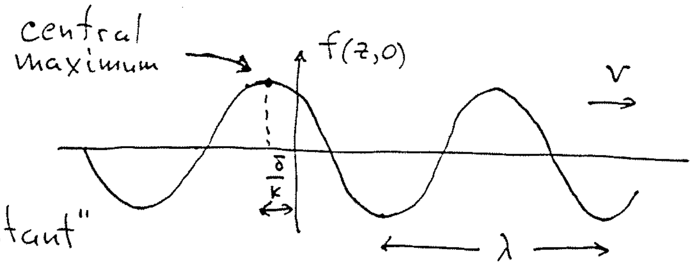
Similarly, $f(z,t) = h(z+vt)$ is also a solution of wave eqn. ($v \leftrightarrow -v$ does not change the eqn)

It can be proved that the most general solution of the wave eqn (*) is the sum of the two solutions

$$f(z,t) = g(z-vt) + h(z+vt) \qquad \left.\begin{matrix} g(u) \\ h(u) \end{matrix}\right\} \text{arbitrary functions}$$

Sinusoidal waves

$$f(z,t) = \underset{\substack{\uparrow \\ \text{"amplitude"}}}{A} \cos\{\overbrace{\underset{\substack{\uparrow \\ \text{"wave number"}}}{k}(z-vt) + \underset{\substack{\uparrow \\ \text{"phase constant"}}}{\delta}}^{\text{"phase"}}\}$$



Wavelength : $\{k(z+\lambda-vt)+\delta\} - \{k(z-vt)+\delta\} = 2\pi \Rightarrow k\lambda = 2\pi \Rightarrow \lambda = \frac{2\pi}{k}$

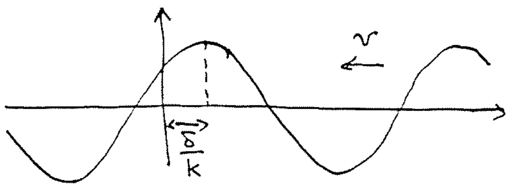
Period $T = \frac{\lambda}{v} = \frac{2\pi}{kv} \Rightarrow$ frequency $\nu = \frac{1}{T} \Rightarrow$

$\omega = \frac{2\pi}{T} = kv$ - angular frequency

$$f(z,t) = A\cos(kz-\omega t+\delta)$$

Similarly, the sinusoidal wave traveling to the left is

$$f(z,t) = A\cos(kz+\omega t-\delta)$$
$$\parallel$$
$$A\cos(-kz-\omega t-\delta)$$



change of sign of k gives the same wave going in the opposite direction

Complex notation

$\cos x = Re[e^{ix}]$ $\qquad$ $e^{ix} = \cos x + i \sin x$

$\Rightarrow f(z,t) = Re\, A e^{i(kz-\omega t+\delta)}$

Complex wave function

$\tilde{f}(z,t) = \tilde{A}\, e^{i(kz-\omega t)}$ $\qquad$ $\tilde{A} \equiv A e^{i\delta}$ $\qquad$ $f(z,t) = Re\,\tilde{f}(z,t)$

Exersize 9.1

$f_1(z,t) = A_1 \cos(kx-\omega t+\delta_1)$

$f_2(z,t) = A_2 \cos(kx-\omega t+\delta_2)$

$\Big\}\ (f_1+f_2)(z,t) = A_3 \cos(kx-\omega t+\delta_3)$

$A_3, \delta_3 = ?$

1) Using the cos notations

$$A_1\cos(kx-\omega t+\delta_1) + A_2\cos(kx-\omega t+\delta_2) = \overbrace{(A_1\cos\delta_1 + A_2\cos\delta_2)}^{A_3\cos\delta_3}\cos(kx-\omega t) -$$

$$-\underbrace{(A_1\sin\delta_1 + A_2\sin\delta_2)}_{A_3\sin\delta_3}\sin(kx-\omega t) = A_3\cos\delta_3\cos(kx-\omega t) - A_3\sin\delta_3\sin(kx-\omega t)$$

$$\Rightarrow A_3^2 = (A_1\cos\delta_1 + A_2\cos\delta_2)^2 + (A_1\sin\delta_1 + A_2\sin\delta_2)^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\delta_1-\delta_2)$$

$$\Rightarrow A_3 = \sqrt{A_1^2+A_2^2+2A_1A_2\cos(\delta_1-\delta_2)} \qquad \delta_3 = \arctan\frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2}$$

2) Using the exp notations

$$\tilde{f}_3 = \tilde{A}_1 e^{i(kx-\omega t)} + \tilde{A}_2 e^{i(kx-\omega t)} = (\tilde{A}_1+\tilde{A}_2)e^{i(kx-\omega t)} = \tilde{A}_3 e^{i(kx-\omega t)}$$

$$\Rightarrow \tilde{A}_3 = \tilde{A}_1 + \tilde{A}_2 \Rightarrow \begin{matrix} A_3 e^{i\delta_3} = A_1 e^{i\delta_1} + A_2 e^{i\delta_2} \\ A_3 e^{-i\delta_3} = A_1 e^{-i\delta_1} + A_2 e^{-i\delta_2} \end{matrix} \Big\} \Rightarrow A_3^2 = A_1^2 + A_2^2 + A_1A_2(e^{i(\delta_1-\delta_2)} + e^{i(\delta_2-\delta_1)}) = A_1^2+A_2^2+2A_1A_2\cos\delta_{12}$$

$$e^{i\delta_3} = \frac{A_1e^{i\delta_1}+A_2e^{i\delta_2}}{\sqrt{A_1^2+A_2^2+2A_1A_2\cos(\delta_1-\delta_2)}} \Rightarrow \tan\delta_3 = \frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2}$$

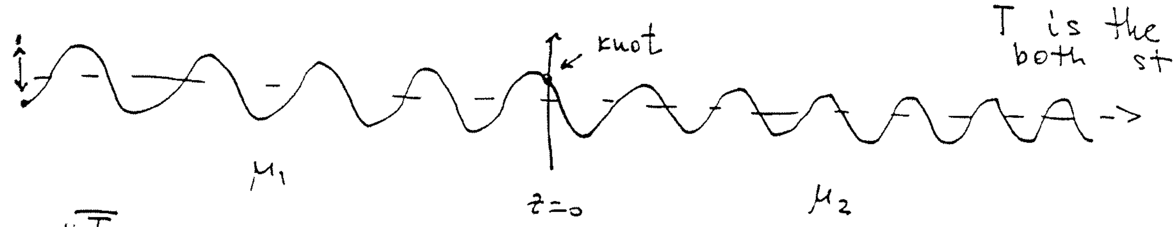
Linear combination of sinusoidal waves

Any wave can be expressed as

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} dk\ \tilde{A}(k) e^{i(kz-\omega t)} \qquad \omega = \omega(k)$$

("Fourier transform" of $\tilde{f}(\omega)$)

Reflection and transmission



T is the same for both strings

$v = \sqrt{\frac{T}{\mu}} \quad \Rightarrow v_1 \neq v_2$

$\left.\begin{matrix} \\ \omega_1 = \omega_2 \text{ (can be proved)} \end{matrix}\right\} \Rightarrow \frac{\lambda_1}{\lambda_2} = \frac{k_2}{k_1} = \frac{v_1}{v_2}$ (recall $\omega = kv$)

incident wave, reflected wave (travels in the opposite directi ⇒ $k_1 \to -k_1$)

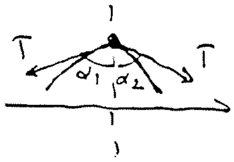
$$\tilde{f}(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z - \omega t)} + \tilde{A}_R e^{i(-k_1 z - \omega t)} & z<0 \\ \tilde{A}_T e^{i(k_2 z - \omega t)} & z>0 \end{cases}$$

↑ transmitted wave

the string is unbroken at $z=0 \Rightarrow \tilde{f}(0^-,t) = \tilde{f}(0^+,t)$

function f is continuous

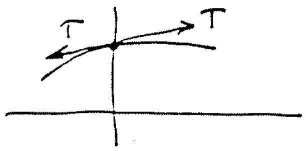
Moreover, if the knot is massless, $\frac{\partial \tilde{f}}{\partial z}$ is also continuous



$T(\cos\alpha_1 + \cos\alpha_2) = ma$ ⇒ if $m=0$

mass of the knot

$\alpha_1 + \alpha_2 = \pi$



⇓ $\frac{\partial f}{\partial z}$ is continuous at $z=0$

$$\left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^-} = \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^+}$$

$$\left.\begin{matrix} \tilde{f}(0^-,t) = \tilde{f}(0^+,t) \\ \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^-} = \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^+} \end{matrix}\right\}$$

two eqns for 2 complex numbers ⇔ 4 eqn for 4 real numbers $A_R, \delta_R, A_T, \delta_T$

$$\tilde{A}_I + \tilde{A}_R = \tilde{A}_T$$

$$k_1(\tilde{A}_I - \tilde{A}_R) = k_2 \tilde{A}_T \quad \Leftarrow \quad \frac{\partial f(z,t)}{\partial z} = \begin{cases} k_1 \tilde{A}_I e^{i(k_1 z - \omega t)} - k_1 \tilde{A}_R e^{i(-k_1 z - \omega t)} & z<0 \\ k_2 \tilde{A}_T e^{i(k_2 z - \omega t)} & z>0 \end{cases}$$

⇓

$$\tilde{A}_R = \frac{k_1 - k_2}{k_1 + k_2} \tilde{A}_I$$

$$\tilde{A}_T = \frac{2k_1}{k_1 + k_2} \tilde{A}_I$$

$$\Rightarrow \quad A_R e^{i\delta_R} = \frac{v_2 - v_1}{v_2 + v_1} A_I e^{i\delta_I}$$

$$A_T e^{i\delta_T} = \frac{2v_2}{v_2 + v_1} A_I e^{i\delta_I}$$

1. $\mu_2 < \mu_1 \Rightarrow v_2 > v_1 \Rightarrow \delta_R = \delta_T = \delta_I$ same phase ~~for~~ all 3 waves

$$A_R = \frac{v_2 - v_1}{v_2 + v_1} \qquad A_T = \frac{2v_2}{v_2+v_1} A_I$$

2. $\mu_2 > \mu_1 \Rightarrow v_1 > v_2$

$$A_R e^{i\delta_R} = -\frac{v_1 - v_2}{v_1+v_2} A_I e^{i\delta_I} \Rightarrow \left.\begin{array}{l} \delta_R = \delta_I - \pi \\ \delta_T = \delta_I \end{array}\right\}$$

transmitted wave have the same phase as incident

$$A_R = \frac{v_1 - v_2}{v_1 + v_2} A_I$$

$$A_T = \frac{2v_2}{v_1+v_2} A_I$$

reflected wave is out of phas by π (180°)

If the string is nailed down at $z=0$ (limit $\mu_2 \to \infty$, $v_2 = 0$) all the wave is reflected back

$$A_R = A_I \quad (\text{and } A_T = 0)$$

Problem 9.5

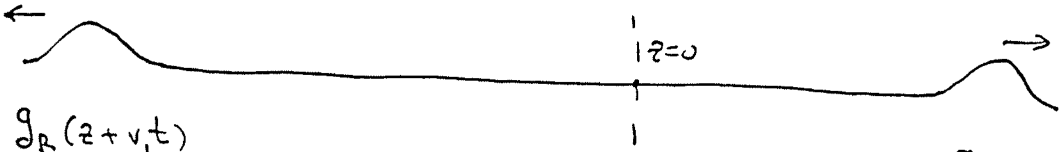
("Particle scattering" in 1 dimension)

$t = -\infty$

$g_I(z - v_1 t)$

initial pulse ("phonon")

$t = +\infty$



$g_R(z + v_1 t)$

reflected pulse

$g_T(z - v_2 t)$

transmitted pulse

Formally, we need to solve the differential eqs

$$\frac{\partial^2 g}{\partial z^2} = \frac{1}{v_1^2}\frac{\partial^2 g}{\partial t^2} \quad z<0, \qquad \frac{\partial^2 g}{\partial z^2} = \frac{1}{v_2^2}\frac{\partial^2 g}{\partial t^2} \quad z>0$$

with the boundary conditions (*)

$$g(0^-, t) = g(0^+, t), \quad \frac{\partial g}{\partial z}(0^-, t) = \frac{\partial g}{\partial z}(0^+, t)$$

and initial condition

$$g(z,t) \to g_I(z-v_1 t) \quad \text{at} \quad t=-\infty$$

the scattering problem is solved by (formal) decomposition of initial pulse into plane waves. (In mathematics it is called Fourier transformation)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\, \hat{f}(k)$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx}\, f(x)$$

$\hat{f}(k) \equiv$ "Fourier transform" of $f(x)$

Proof: $\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\int_{-\infty}^{\infty} dx'\, e^{-ikx'} f(x') = \frac{1}{2\pi}\int dx'\, f(x') \underbrace{\int dk\, e^{ik(x-x')}}_{2\pi\delta(x-x')}$

$= \int dx'\, f(x')\, \delta(x-x') = f(x)$

Now

$$g_I(z-v_1 t) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\, \tilde{g}_I(k)\, e^{ik(z-v_1 t)} \qquad (**)$$

Each $\tilde{g}_I(k)\, e^{ik(z-v_1 t)}$ is a plane wave which is transmitted (and reflected) independently of other constituents with different k's (superposition principle ⇐ linear differential eqs (*))

Scattering of $\tilde{g}_I(k)\, e^{ik(z-v_1 t)}$: transmitted wave + reflected wave

$$\tilde{g}_R^{(k)}(z+v_1 t) = \tilde{g}_R(k)\, e^{i(-kz-\underbrace{kv_1}_{\omega} t)}$$

$$\tilde{g}_T^{(k)}(z-v_2 t) = \hat{g}_T(k)\, e^{i(\underbrace{\frac{v_1}{v_2}k}_{k_2=k_1\frac{v_1}{v_2}} z-\overbrace{kv_1} t)}$$

$$\tilde{g}_R(k) = \frac{v_2-v_1}{v_2+v_1}\, \hat{g}_I(k)$$

$$\tilde{g}_T(k) = \frac{2v_2}{v_2+v_1}\, \hat{g}_I(k)$$

Assembling the superposition (**) back, we get

$$g_R = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \tilde{g}_R(k) e^{-ik(z+v_1 t)} = \frac{v_2-v_1}{v_2+v_1}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\, \tilde{g}_I(k) e^{-ik(z+v_1 t)} =$$

$$= \frac{v_2-v_1}{v_2+v_1}\, g_I(-z-v_1 t)$$

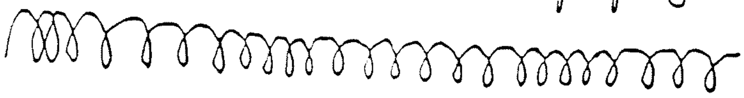
$$g_T = \frac{1}{2\pi}\int dk\, \tilde{g}_T(k)\, e^{ik(z\frac{v_1}{v_2}-v_1 t)} = \frac{2v_2}{v_2+v_1}\, g_I(z\frac{v_1}{v_2}-v_1 t)$$

The solution is

$$g(z<0,t) = g_I(z-v_1 t) + \frac{v_2-v_1}{v_2+v_1}\, g_I(-z-vt), \quad g(z>0,t) = \frac{2v_2}{v_2+v_1}\, g(z\frac{v_1}{v_2}-v_1 t)$$

Transverse waves: displacement is $\perp$ to the direction of the propagation

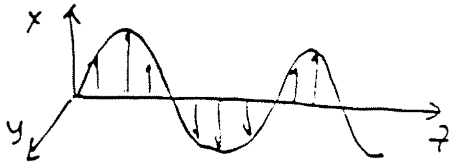
Longitudinal waves: displacement is $\uparrow\uparrow$ to the direction of propagation



← "compression waves"

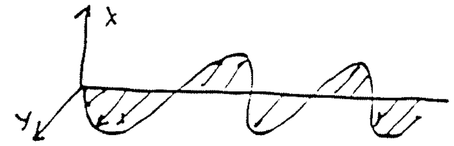
Polarization of the transverse waves

$\vec{\tilde{f}}(z,t) = \hat{A}\, e^{i(kz-\omega t)}\, \hat{e}_1$ „vertical polarization"



$\vec{\tilde{f}}(z,t) = \hat{A}\, e^{i(kz-\omega t)}\, \hat{e}_2$

"horizontal polarization"



Linear polarization

$\vec{\tilde{f}}(z,t) = \hat{A}\, e^{i(kz-\omega t)}\, \hat{n}$ $\qquad \hat{n}\cdot\hat{e}_3 = 0$

$\hat{n} = \cos\theta\, \hat{e}_1 + \sin\theta\, \hat{e}_2$ $\qquad \theta \equiv$ "polarization angle"

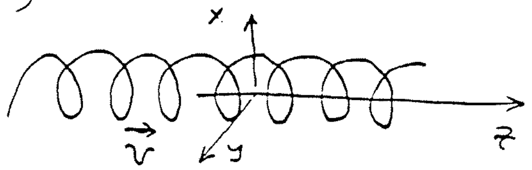
Circular polarization

$$\vec{\tilde{f}}(z,t) = A\, e^{i(kz-\omega t)}\, \hat{e}_1 + A\, e^{i(kz-\omega t-\frac{\pi}{2})}\, \hat{e}_2 = A\, e^{i(kz-\omega t)}(\hat{e}_1 - i\hat{e}_2)$$

$f_1(z,t) = A\cos(kz-\omega t)$

$f_2(z,t) = A\cos(kz-\omega t-\frac{\pi}{2}) = A\sin(kz-\omega t)$

$f_1^2 + f_2^2 = A^2$ - circular motion



Unlike linearly polarized wave, the wave with circular polarization carries angular momentum

Electromagnetic waves in vacuum

$$\left.\begin{array}{ll}\vec\nabla\cdot\vec E = 0 & \vec\nabla\times\vec E = -\frac{\partial\vec B}{\partial t} \\ \vec\nabla\cdot\vec B = 0 & \vec\nabla\times\vec B = \mu_0\epsilon_0\frac{\partial\vec E}{\partial t}\end{array}\right\}$$ Maxwell's eqs in a free space (without charges or currents)

$$\left.\begin{array}{l}\vec\nabla\times(\vec\nabla\times\vec E) = -\frac{\partial}{\partial t}(\vec\nabla\times\vec B) = -\mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\vec E \\ \parallel \\ \vec\nabla(\cancel{\vec\nabla\cdot\vec E}) - \nabla^2\vec E\end{array}\right\} \Rightarrow \nabla^2\vec E = \mu_0\epsilon_0\frac{\partial^2\vec E}{\partial t^2}$$

or each component of $\vec E$, this is a 3-dimensional wave quation

$$\nabla^2 f(\vec r,t) = \frac{1}{v^2}\frac{\partial^2 f(\vec r,t)}{\partial t^2}$$ 3d wave equation
v – speed of the wave

$$\nabla^2 E_1 = \mu_0\epsilon_0\frac{\partial^2 E_1}{\partial t^2}, \quad \nabla^2 E_2 = \mu_0\epsilon_0\frac{\partial^2 E_2}{\partial t^2}, \quad \nabla^2 E_3 = \mu_0\epsilon_0\frac{\partial^2 E_3}{\partial t^2}$$

$\mu_0\epsilon_0 = \frac{1}{v^2}$ $\quad v = \frac{1}{\sqrt{\mu_0\epsilon_0}}$ – speed of the electromagnetic waves (= speed of light c)

Similarly,

$$\vec\nabla\times(\vec\nabla\times\vec B) = \vec\nabla(\underset{=0}{\cancel{\vec\nabla\cdot\vec B}}) - \nabla^2\vec B = -\mu_0\epsilon_0\frac{\partial^2\vec B}{\partial t^2} \Rightarrow \nabla^2\vec B = \mu_0\epsilon_0\frac{\partial^2\vec B}{\partial t^2}$$

same wave equation

Monochromatic plane waves

$$\tilde{\vec E}(z,t) = \tilde{\vec E}_0 e^{i(kz-\omega t)}$$
$$\tilde{\vec B}(z,t) = \tilde{\vec B}_0 e^{i(kz-\omega t)}$$

(for simplicity, we consider the wave propagating in $\hat z$ direction at first)

$$\left.\begin{array}{l}\nabla^2\vec E = -k^2\vec E \\ \frac{\partial^2}{\partial t^2}\vec E = -\omega^2\vec E\end{array}\right.$$ ⇒ wave eqn. is satisfied if $\omega = kv = kc$

$$\left.\begin{array}{l}\vec\nabla\cdot\vec E = 0 \Rightarrow k\cdot E_3 = 0 \Rightarrow E_3 = 0 \\ \vec\nabla\cdot\vec B = 0 \Rightarrow k B_3 = 0 \Rightarrow B_3 = 0\end{array}\right\}$$ electromagnetic waves are transverse

$$\vec\nabla\times\vec E = -\frac{\partial\vec B}{\partial t} \Rightarrow i\vec k\times\tilde{\vec E}_0 = -i\omega\tilde{\vec B}_0 \Rightarrow \tilde B_0 = -\frac{1}{\omega}\vec k\times\tilde{\vec E}_0 \qquad (\vec k \equiv k\hat e_3)$$

$\Rightarrow$ $\vec E$ and $\vec B$ are in phase and mutually perpendicular

$B_0 = \frac{k}{\omega}E_0 = \frac{1}{c}E_0$

Monochromatic plane waves traveling in an arbitrary direction

$$\tilde{\vec E}(\vec r,t) = \tilde{\vec E}_0\, e^{i(\vec k\cdot\vec r-\omega t)}$$

$$\vec B(\vec r,t) = \frac{\hat k}{c}\times\tilde{\vec E}_0\, e^{i(\vec k\cdot\vec r-\omega t)}$$

Check:

$$\left.\begin{aligned}\nabla^2\tilde{\vec E}(\vec r,t) &= -k^2\tilde{\vec E}(\vec r,t)\\ \frac{\partial^2}{\partial t^2}\tilde{\vec E}(\vec r,t) &= -\omega^2\tilde{\vec E}(\vec r,t)\end{aligned}\right\}\text{ wave eqn is satisfied}$$

$\vec\nabla\cdot\tilde E = 0 \Rightarrow \hat k\cdot\tilde{\vec E}_0 = 0$

$$\left.\begin{aligned}\vec\nabla\times\tilde{\vec E} &= i(\vec k\times\tilde{\vec E}_0)\, e^{i(\vec k\cdot\vec r-\omega t)}\\ \frac{\partial\tilde{\vec B}}{\partial t} &= -\frac{i\omega}{c}(k\times\tilde{\vec E}_0)\, e^{i(k\cdot r-\omega t)}\end{aligned}\right\}\Rightarrow \vec\nabla\times\tilde{\vec E} = -\frac{\partial\tilde{\vec B}}{\partial t}\qquad \left(k=\frac{\omega}{c}\right)$$

For the linearly polarized wave $\tilde{\vec E}_0 = \underset{\uparrow}{\hat n}\,\tilde E_0$ $(n^2=1)$

$\tilde{\vec E} = \tilde E_0\hat n\, e^{i(\vec k\cdot\vec r-\omega t)}$, $\tilde{\vec B} = \frac{\hat k}{c}\times\hat n E_0 e^{i(\dots)}$ polarization vector

The actual (real) electric and magnetic fields are

$$\vec E(\vec r,t) = E_0\hat n\cos(\vec k\cdot\vec r-\omega t+\delta),\quad \vec B = \frac{\hat k\times\hat n}{c}E_0\cos(\vec k\cdot\vec r-\omega t+\delta)$$

Energy and momentum in electromagnetic waves

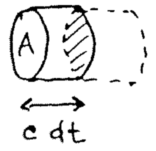
$u = \frac{1}{2}\left(\epsilon_0E^2+\frac{1}{\mu_0}B^2\right)$ – energy density

In the wave $B^2 = \frac{E^2}{c^2} = \mu_0\epsilon_0E^2 \Rightarrow u = \frac{1}{2}(\underset{\uparrow}{\epsilon_0E^2}+\underset{\uparrow}{\epsilon_0E^2}) = \epsilon_0E^2$

electric and magnetic contributions are equal

$\Rightarrow u = \epsilon_0E_0^2\cos^2(kz-\omega t+\delta)$

Flux of energy



Energy transported by the wave thru ▨ per time dt is $uAc\,dt$ $\Rightarrow$ flux of energy $= uc$

General formula is $\vec S = \frac{1}{\mu_0}\vec E\times\vec B \Rightarrow$ for the wave

$S = c\epsilon_0E_0^2\cos^2(kz-\omega t+\delta) = cu$

Momentum density

$\vec{\mathcal{P}} = \mu_0 \epsilon_0 \vec{S} = \frac{1}{c^2}\vec{S}$ $\Rightarrow$ for the monochromatic plane wave

$$\vec{\mathcal{P}} = \frac{1}{c}\epsilon_0 E_0^2 \cos^2(kz-\omega t+\delta)\hat{e}_3 = \frac{1}{c}u\hat{e}_3$$

For the light, $\cos^2(kz-\omega t+\delta)$ oscillates very quickly ($T \sim 10^{-15}s$) $\Rightarrow$ macroscopic measurements are sensitive to the average values only.

$\langle\ \rangle \equiv$ time average over a complete cycle ($\Leftrightarrow$ over many cycles)

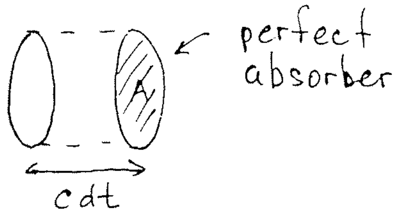
$\langle\cos^2\rangle = \langle\sin^2\rangle = \frac{1}{2}$ (Proof : $\frac{1}{T}\int_0^T dt\, \cos^2(kz - 2\pi\frac{t}{T}+\delta) = \frac{1}{T}\int_0^T dt\, \sin^2(kz-2\pi\frac{t}{T}+\delta) = \frac{1}{2}$

$\Rightarrow \langle u\rangle = \frac{\epsilon_0}{2}E_0^2$ $\langle\vec{S}\rangle = \frac{c\epsilon_0}{2}E_0^2\hat{e}_3$ $\langle\vec{\mathcal{P}}\rangle = \frac{\epsilon_0}{2c}E_0^2\hat{e}_3$

Intensity

$I \equiv \langle S\rangle = \frac{\epsilon_0 c}{2}E_0^2$ – average power per unit area transported by an electromagnetic wave

Radiation pressure

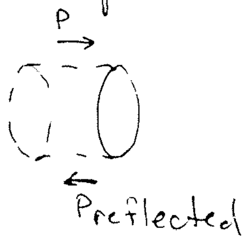


The momentum transferred to the area A per time dt is

$d\vec{p} = \langle\vec{\mathcal{P}}\rangle A\, c\, dt \Rightarrow$

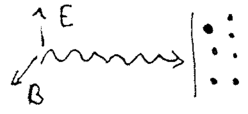
$\Rightarrow P =$ force per unit area $= \frac{1}{A}\frac{d\vec{p}}{dt} = \frac{\epsilon_0 E_0^2}{2} = \frac{I}{c}$

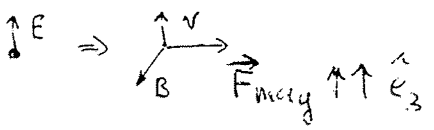
For the perfect reflector, the pressure is twice as great



$\Rightarrow d\vec{p} = 2\langle\vec{\mathcal{P}}\rangle A c dt \Rightarrow P = \epsilon_0 E_0^2 = 2\frac{I}{c}$

Microscopically, the light pressure is due to the Lorentz forces acting on the charges on the surface





Electromagnetic waves in matter.

Propagation in linear media:

Without free charges or currents

$$\left.\begin{array}{ll} \vec\nabla\cdot\vec D = 0 & \vec\nabla\times\vec E = -\frac{\partial\vec B}{\partial t} \\ \vec\nabla\cdot\vec B = 0 & \vec\nabla\times\vec H = \frac{\partial\vec D}{\partial t} \end{array}\right\} \text{Maxwell's eqns}$$

In linear medium $\vec D = \epsilon\vec E$, $\vec H = \frac{1}{\mu}\vec B \Rightarrow$

$$\left.\begin{array}{ll} \vec\nabla\cdot\vec E = 0 & \vec\nabla\times\vec E = -\frac{\partial\vec B}{\partial t} \\ \vec\nabla\cdot\vec B = 0 & \vec\nabla\times\vec B = \mu\epsilon\frac{\partial\vec E}{\partial t} \end{array}\right\}$$

same as Maxwell's eqns in a free space but with the replacement $\mu_0\epsilon_0 \to \mu\epsilon$

$\Rightarrow$ plane waves propagate thru a linear homogeneous medium at a speed $v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}$, $n \equiv \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$ "index of refraction"

Usually $\mu \simeq \mu_0 \Rightarrow n \simeq \sqrt{\epsilon_r}$

$$\tilde{\vec E}(\vec r,t) = \tilde E_0\, e^{i(\vec k\cdot\vec r-\omega t)} \qquad \omega = kv$$

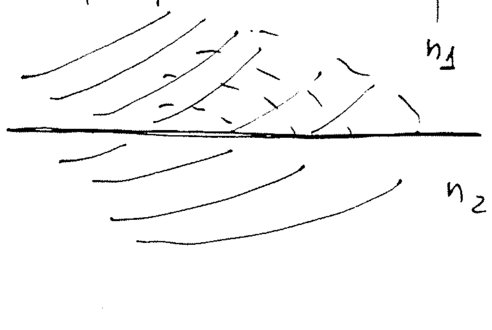
$$\tilde{\vec B}(\vec r,t) = \frac{1}{v}\hat k\times\tilde{\vec E}_0\, e^{i(\vec k\cdot\vec r-\omega t)} = \frac{1}{v}\hat k\times\tilde{\vec E} \qquad B_0 = \frac{1}{v}E_0$$

The energy density is $u = \frac{1}{2}(\epsilon E^2 + \frac{1}{\mu}B^2)$

Poynting vector $\vec S = \frac{1}{\mu}\vec E\times\vec B$

Intensity $I = \frac{1}{2}\epsilon v E_0^2$

A very practical question: How does the wave propagate thru the boundary between two linear media



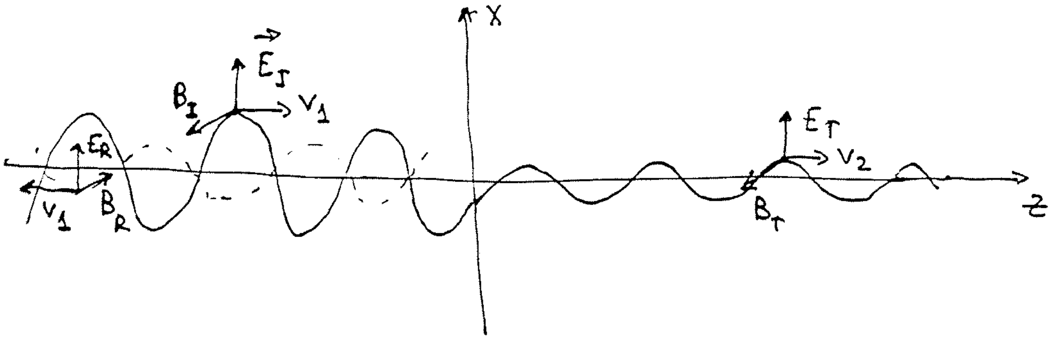
From the analogy with mechanical waves, we shoud expect both reflection and transmission

Boundary conditions

$$\left.\begin{array}{ll} \epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp & \vec E_1^\parallel = \vec E_2^\parallel \\ B_1^\perp = B_2^\perp & \frac{1}{\mu_1}\vec B_1^\parallel = \frac{1}{\mu_2}\vec B_2^\parallel \end{array}\right\}$$

analog of the rule $\left.\frac{\partial f}{\partial z}\right|_{0^+} = \left.\frac{\partial f}{\partial z}\right|_{0^-}$ in 1-dim mech. problem

Reflection and transmission at normal incidence



Incident wave (vertically polarized)

$$\tilde{\vec{E}}_I(z,t) = \tilde{E}_{0_I}\, e^{i(k_1 z - \omega t)}\, \hat{e}_1$$

$$\tilde{\vec{B}}_I(z,t) = \frac{1}{v_1} \tilde{E}_{0_I}\, e^{i(k_1 z - \omega t)}\, \hat{e}_2$$

Both reflected wave and transmitted wave will also be vertically polarized

Proof: suppose reflected wave has θ_R and transmitted wave has θ_T

$$\tilde{\vec{E}}_T(z,t) = \tilde{E}_{0_T}(\cos\theta_T\, \hat{e}_1 + \sin\theta_T\, \hat{e}_2)\, e^{i(k_2 z - \omega t)}$$

$$\tilde{\vec{E}}_R(z,t) = \tilde{E}_{0R}(\cos\theta_R\, \hat{e}_1 + \sin\theta_R\, \hat{e}_2)\, e^{i(-k_1 z - \omega t)}$$

Boundary conditions

$E_1^{\perp} = E_2^{\perp} = 0$ trivial

$$\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}\Big|_{z=0} \Rightarrow \{\hat{e}_1 \tilde{E}_{0_I} + \tilde{E}_{0_R}(\hat{e}_1 \cos\theta_R + \hat{e}_2 \sin\theta_R)\}\, e^{-i\omega t} = \tilde{E}_{0_T}(\cos\theta_T\, \hat{e}_1 + \sin\theta_T\, \hat{e}_2)\, e^{-i\omega}$$

$$\Rightarrow \tilde{E}_{0_I} + \tilde{E}_{0_R}\cos\theta_R = \tilde{E}_{0_T}\cos\theta_T \qquad \tilde{E}_{0_R}\sin\theta_R = \tilde{E}_{0_T}\sin\theta_T$$

Magnetic fields

$$\tilde{\vec{B}}_T(z,t) = \frac{1}{v_2}\hat{k}\times\tilde{\vec{E}}_T = \frac{1}{v_2}\tilde{E}_{0_T}\, \hat{e}_3 \times (\hat{e}_1\cos\theta_T + \hat{e}_2 \sin\theta_T)\, e^{i(k_2 z - \omega t)} =$$

$$= \frac{1}{v_2}\tilde{E}_{0_T}(-\hat{e}_1 \sin\theta_T + \hat{e}_2\cos\theta_T)\, e^{i(k_2 z - \omega t)}$$

$$\tilde{\vec{B}}_R(z,t) = \frac{1}{v_1}\hat{k}\times\tilde{\vec{E}}_R = -\frac{1}{v_1}\tilde{E}_{0_R}\, \hat{e}_3 \times (\hat{e}_1\cos\theta_R + \hat{e}_2\sin\theta_R)\, e^{i(-k_1 z - \omega t)}$$

$$= \frac{1}{v_1}\tilde{E}_{0R}(\hat{e}_1\sin\theta_R - \hat{e}_2\cos\theta_R)\, e^{i(-k_2 z - \omega t)}$$

Boundary conditions $B_1^{\perp} = B_2^{\perp} = 0$ trivial

$$\frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}\Big|_{z=0} \Rightarrow \frac{1}{\mu_1}\left(\tilde{\vec{B}}_I(0_-,t) + \tilde{\vec{B}}_R(0_-,t)\right) = \frac{1}{\mu_2}\tilde{\vec{B}}_T(0_+,t)$$

$$\Rightarrow \frac{1}{\mu_1}\left[\hat{e}_2 \frac{1}{v_1}\tilde{E}_{0_I} + \frac{1}{v_1}\tilde{E}_{0_R}(\underline{\hat{e}_1 \sin\theta_R} - \hat{e}_2\cos\theta_R)\right]e^{-i\omega t} = \frac{1}{\mu_2}\frac{\tilde{E}_{0_T}}{v_2}(\underline{-\hat{e}_1\sin\theta_T} + \hat{e}_2\cos\theta_T)e^{-i\omega}$$

$$\Rightarrow \frac{1}{\mu_1 v_1}\tilde{E}_{0_R}\sin\theta_R = -\frac{1}{\mu_2 v_2}\tilde{E}_{0_T}\sin\theta_T$$

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0_I} - \tilde{E}_{0_R}\cos\theta_R) = \frac{1}{\mu_2 v_2}\tilde{E}_{0_T}\cos\theta_T$$

We have 4 eqns

$$\tilde{E}_{0_I} + \tilde{E}_{0_R}\cos\theta_R = \tilde{E}_{0_T}\cos\theta_T \qquad \tilde{E}_{0_I} - E_{0_R}\cos\theta_R = \frac{\mu_1 v_1}{\mu_2 v_2}\tilde{E}_{0_T}\cos\theta_T$$

$$E_{0_R}\sin\theta_R = \underbrace{\tilde{E}_{0_T}\sin\theta_T} \qquad \tilde{E}_{0_R}\sin\theta_R = \underbrace{-\frac{\mu_1 v_1}{\mu_2 v_2}E_{0_T}\sin\theta_T} \qquad (*)$$

$$\tilde{E}_{0_T}\sin\theta_T = -\beta\tilde{E}_{0_T}\sin\theta_T \quad\Rightarrow\quad \theta_T = 0 \Rightarrow \theta_R = 0 \qquad \frac{\mu_1 v_1}{\mu_2 v_2} \equiv \beta$$

⇒ both transmitted and reflected waves have the same polarization as the incident wave

$$\left.\begin{aligned}\vec{\tilde{E}}_I &= \tilde{E}_{0_I}\hat{e}_1 e^{i(k_1 z-\omega t)}\\ \vec{\tilde{B}}_I &= \frac{1}{v_1}\tilde{E}_{0_I}\hat{e}_2 e^{i(k_1 z-\omega t)}\end{aligned}\right\} \Rightarrow \quad \left.\begin{aligned}\vec{\tilde{E}}_T &= \tilde{E}_{0_T}\hat{e}_1 e^{i(k_2 z-\omega t)}\\ \vec{\tilde{B}}_T &= \frac{1}{v_2}\tilde{E}_{0_T}\hat{e}_2 e^{i(k_2 z-\omega t)}\end{aligned}\right\}\text{transmitted wave}$$

$$\left.\begin{aligned}\vec{\tilde{E}}_R &= E_{0_R}\hat{e}_1 e^{i(-k_1 z-\omega t)}\\ \vec{\tilde{B}}_R &= -\frac{\hat{e}_2}{v_1}E_{0_R}e^{i(-k_1 z-\omega t)}\end{aligned}\right\}\text{reflected wave}$$

From the eqns (*) we get

$$\left.\begin{aligned}\tilde{E}_{0_I} + \tilde{E}_{0_R} &= \tilde{E}_{0_T}\\ \tilde{E}_{0_I} - \tilde{E}_{0_R} &= \beta\tilde{E}_{0_T}\end{aligned}\right\} \Rightarrow \quad \begin{aligned}\tilde{E}_{0_T} &= \frac{2}{1+\beta}\tilde{E}_{0_I}\\ \tilde{E}_{0_R} &= \frac{1-\beta}{1+\beta}\tilde{E}_{0_I}\end{aligned}$$

Usually $\mu_2 \simeq \mu_1 \simeq \mu_0 \Rightarrow \beta = \frac{v_1}{v_2} \Rightarrow$

$$\Rightarrow \tilde{E}_{0_R} = \frac{v_2 - v_1}{v_2 + v_1}\tilde{E}_{0_I} \qquad \tilde{E}_{0_T} = \frac{2v_2}{v_2+v_1}\tilde{E}_{0_I}$$

⇒ same results as for the mechanical wave in the string with $\mu_1 \neq \mu_2$:

$$E_{0_R} = \left|\frac{v_2 - v_1}{v_2 + v_1}\right| E_{0_I} \qquad E_{0_T} = \frac{2v_2}{v_2 + v_1}E_{0_I}$$

transmitted wave is in phase, the reflected wave is in phase at $v_2 > v_1$ and out of phase ($\delta = 180°$) if $v_1 > v_2$

In terms of n

$$E_{0_R} = \left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{0_I}, \qquad E_{0_T} = \frac{2n_1}{n_1 + n_2}E_{0_I} \qquad \left(n = \frac{c}{v}\right)$$

Reflected and transmitted energy

$I = \frac{1}{2} \epsilon v E_0^2$

If $\mu_1 = \mu_2 = \mu_0 \Rightarrow$ $R = \frac{I_R}{I_I} = \frac{E_{0R}^2}{E_{0I}^2} = \frac{(n_1-n_2)^2}{(n_1+n_2)^2}$ ← reflection coefficient

$R+T = 1$ conservation of energy $\Leftarrow \Bigg\{$ $T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2 E_{0T}^2}{\epsilon_1 v_1 E_{0I}^2} = \frac{4 n_1 n_2}{(n_1+n_2)^2}$ ← transmission coefficient

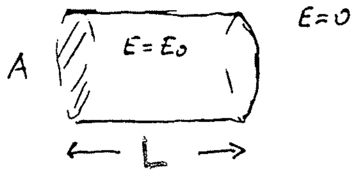
To avoid confusion due to interference between incident and reflected wave, consider an electromagnetic pulse instead of a plane wave. (For simpicity, we take square pulse

Incident pulse: $\vec{E}_I(z-v_1t) = E_0\, s\,(z-v_1t)\,\hat{e}_1$ $\quad s(x) = \begin{cases} 1 & 0<x<L \\ 0 & \text{otherwise} \end{cases}$

$\vec{B}_I(z-v_1t) = \frac{1}{v_1}\hat{e}_2 E_0\, s\,(z-v_1t)$

"square pulse"

Energy stored in the pulse



$W_I = \epsilon_1 \int E_I^2 d^3x = \epsilon_1 E_0^2 \underbrace{AL}_{\text{volume of the pulse}}$

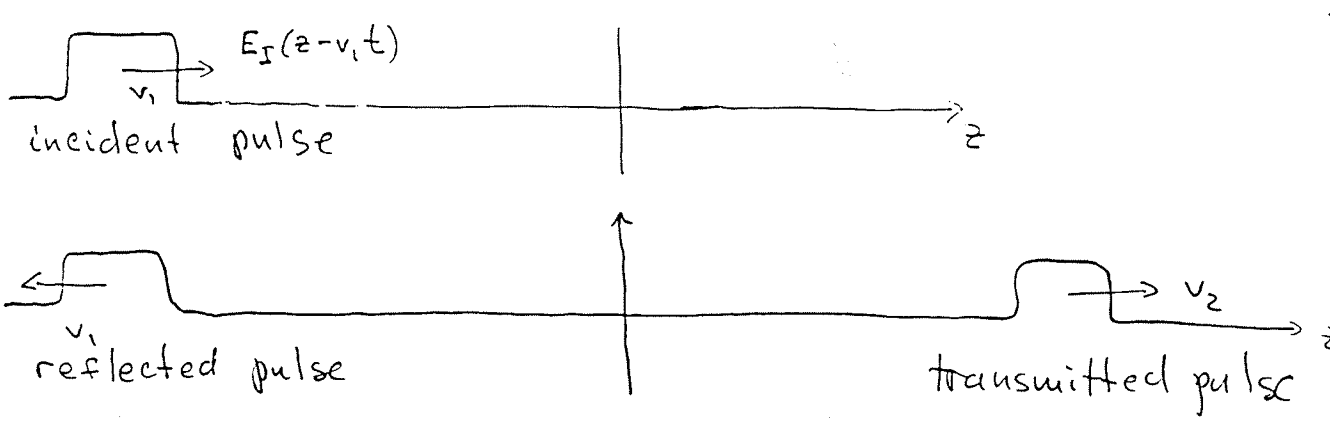
We know the solution of the problem of scattering of the pulse

Propagation of plane waves is the same as in the mechanical string problem ⇒ propagation of pulse is the same

$$\vec{E}(z<0,t) = \underset{\text{incident pulse}}{\vec{E}_I(z-v_1t)} + \underset{\text{reflected pulse}}{\frac{v_2-v_1}{v_2+v_1}\vec{E}_I(-z-v_1t)}$$

$$\vec{E}(z>0,t) = \underset{\text{transmitted pulse}}{\frac{2v_2}{v_2+v_1}\vec{E}_I\left(z\frac{v_1}{v_2}-v_1t\right)}$$

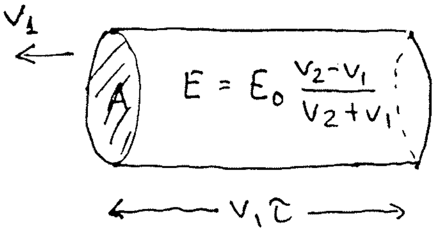
Mathematically, the solutions of electromagnetic and string problems are similar because the differential eqns and boundary conditions (at $z=0$) are the same. (Boundary condition $f(0_-,t) = f(0_+,t)$ is similar to $E_x(0_-,t) = E_x(0_+,t)$ while $\frac{\partial f}{\partial z}(0_-,t) = \frac{\partial f}{\partial z}(0_+,t)$ is equivalent to $\frac{\partial E_x}{\partial z}(0_+,t) = \frac{\partial E_x}{\partial z}(0_-,t) \Leftrightarrow$ $\Leftrightarrow (\vec{\nabla}\times\vec{E})_y(0_-,t) = (\vec{\nabla}\times\vec{E})_y(0_+,t) \Leftrightarrow \frac{\partial B_y}{\partial t}(0_-,t) = \frac{\partial B_y}{\partial t}(0_+,t) \Rightarrow B_y(0_-,t) = B_y(0_+,t)$



$$E_R = \frac{v_2-v_1}{v_2+v_1} E_0 S_L(z-v_1t)\hat{e}_1 \qquad E_T = \frac{2v_2}{v_2+v_1} E_0 \hat{e}_1 S_L\left(z\frac{v_2}{v_1}-v_1t\right)$$

$$\left(\vec{B}_R = -\frac{\hat{e}_2}{v_1} E_0 \frac{v_2-v_1}{v_2+v_1} S_L(z-v_1t)\right) \qquad \left(\vec{B}_T = \frac{\hat{e}_2}{v_2}\frac{2v_2}{v_2+v_1} E_0 S_L\left(z\frac{v_2}{v_1}-v_1t\right)\right)$$

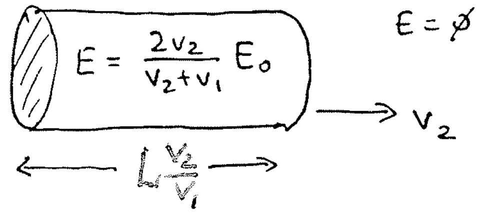
Energy stored in the reflected pulse



$E = 0$
outside

$$W_R = \epsilon_1 \int E_R^2(\vec{r})\, d^3x = \epsilon_1 \frac{(v_2-v_1)^2}{(v_2+v_1)^2} E_0^2 \underbrace{A v_1 \tau}_{\text{volume}}$$

Energy stored in the transmitted pulse



$E = 0$

$$W_T = \epsilon_2 \int E_T^2(\vec{r})\, d^3x = \epsilon_2 \frac{4v_2^2}{(v_2+v_1)^2} E_0^2 A L \frac{v_2}{v_1}$$

$\epsilon_2 v_2^2 = \epsilon_1 v_1^2$ (we assume $\mu_1 \simeq \mu_2 \simeq \mu_0$

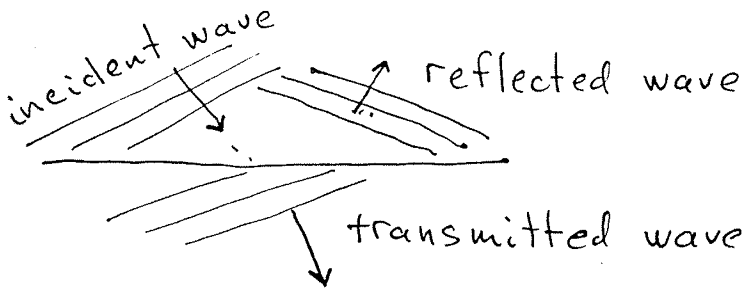
$$W_R + W_T = \epsilon_1 L \frac{(v_2-v_1)^2}{(v_2+v_1)^2} E_0^2 A + \frac{4\epsilon_1 v_1}{(v_2+v_1)^2} v_2 E_0^2 A L = \epsilon_1 L E_0^2 A = W_I$$

$\Rightarrow$ $\boxed{W_I = W_R + W_T}$ $\Rightarrow$ conservation of energy

$$R = \frac{W_R}{W_I} = \frac{(v_2-v_1)^2}{(v_2+v_1)^2} = \frac{(n_1-n_2)^2}{(n_1+n_2)^2} \qquad T = \frac{W_T}{W_I} = \frac{4v_1v_2}{(v_1+v_2)^2} = \frac{4n_1n_2}{(n_1+n_2)^2}$$

$R + T = 1$

Reflection and transmission at oblique incidence



$$\left.\begin{aligned}\tilde{\vec{E}}_I(\vec{r},t) &= \tilde{\vec{E}}_{0_I}\, e^{i(\vec{k}_I\cdot\vec{r}-\omega t)}\\ \tilde{\vec{B}}_I(\vec{r},t) &= \frac{1}{v_1}(\hat{k}_I\times\tilde{\vec{E}}_I)\end{aligned}\right\}\ \text{incident wave}$$

ω is the same for all 3 waves

$$\underbrace{\begin{aligned}\tilde{\vec{E}}_R(\vec{r},t) &= \tilde{\vec{E}}_{0_R}\, e^{i(\vec{k}_R\cdot\vec{r}-\omega t)}\\ \vec{B}_R(\vec{r},t) &= \frac{1}{v_1}(\hat{k}_R\times\tilde{\vec{E}}_R)\end{aligned}}_{\text{reflected wave}} \qquad \underbrace{\begin{aligned}\tilde{\vec{E}}_T(\vec{r},t) &= \tilde{\vec{E}}_{0_T}\, e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}\\ \vec{B}_T(\vec{r},t) &= \frac{1}{v_2}(\hat{k}_T\times\tilde{\vec{E}}_T)\end{aligned}}_{\text{transmitted wave}}$$

$$\omega = kv \Rightarrow k_R = k_I \qquad k_T = \frac{v_1}{v_2}k_I = \frac{n_2}{n_1}k_I$$

Boundary conditions

$$\vec{E}(\vec{r},t)\big|_{z<0} = \tilde{\vec{E}}_{0_I}\, e^{i(\vec{k}_I\vec{r}-\omega t)} + \tilde{\vec{E}}_{0_R}\, e^{i(\vec{k}_R\cdot\vec{r}-\omega t)}$$

$$\vec{B}(\vec{r},t)\big|_{z<0} = \frac{1}{v_1}(\hat{k}_I\times\tilde{\vec{E}}_{0_I})\, e^{i(\vec{k}_I\vec{r}-\omega t)} + \frac{1}{v_1}(\hat{k}_R\times\tilde{\vec{E}}_{0_R})\, e^{i(\vec{k}_R\vec{r}-\omega t)}$$

$$\vec{E}(\vec{r},t)\big|_{z>0} = \tilde{\vec{E}}_{0_T}\, e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}$$

$$\vec{B}(\vec{r},t)\big|_{z>0} = \frac{1}{v_2}(\hat{k}_T\times\tilde{\vec{E}}_{0_T})\, e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}$$

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}\big|_{z=0} \Rightarrow \epsilon_1 \tilde{E}^z_{0_I}\, e^{i(\vec{k}_I\cdot\vec{s}-\omega t)} + \epsilon_1 \tilde{E}^z_{0_R}\, e^{i(\vec{k}_R\cdot\vec{s}-\omega t)} = \epsilon_2 E^z_{0_T}\, e^{i(\vec{k}_T\vec{s}-\omega t)}$$

$$\vec{s} \equiv \hat{e}_x x + \hat{e}_y y$$

$$B_1^{\perp} = B_2^{\perp}\big|_{z=0} \Rightarrow \frac{1}{v_2}(\hat{k}_T\times\tilde{\vec{E}}_{0_T})\, e^{i(\vec{k}_T\vec{s}-\omega t)} =$$

$$= \frac{1}{v_1}(\hat{k}_I\times\tilde{\vec{E}}_{0_I})_z\, e^{i(\vec{k}_I\vec{s}-\omega t)} + \frac{1}{v_1}(\hat{k}_R\times\tilde{\vec{E}}_{0_R})_z\, e^{i(\vec{k}_R\cdot\vec{r}-\omega t)}$$

$$\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}\big|_{z=0} \Rightarrow \tilde{\vec{E}}^{\parallel}_{0_I}\, e^{i(\vec{k}_I\vec{s}-\omega t)} + \tilde{\vec{E}}^{\parallel}_{0_R}\, e^{i(\vec{k}_R\cdot\vec{s}-\omega t)} = \tilde{\vec{E}}^{\parallel}_{0_T}\, e^{i(\vec{k}_T\cdot\vec{s}-\omega t)}$$

$$\frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}\Big|_{z=0} \Rightarrow \hat{k}_I\times\tilde{\vec{E}}^{\parallel}_{0_I}\, e^{i(\vec{k}_I\cdot\vec{s}-\omega t)} + \hat{k}_R\times\tilde{\vec{E}}^{\parallel}_{0_R}\, e^{i(\vec{k}_R\vec{s}-\omega t)} =$$

$$= \frac{v_2\epsilon_2}{v_1\epsilon_1}\,\hat{k}_T\times\tilde{\vec{E}}^{\parallel}_{0_-}\, e^{i(k_T\cdot s-\omega t)}$$

1. For any s

$$\vec{k}_I \cdot \vec{s} = \vec{k}_R \cdot \vec{s} = \vec{k}_T \cdot \vec{s}$$

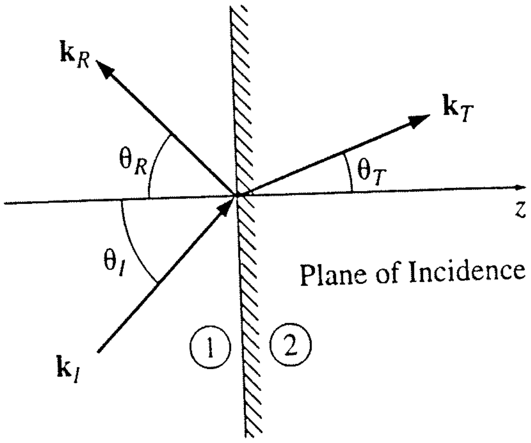
$$x k_{I_x} + y k_{I_y} = x k_{R_x} + y k_{R_y} = x k_{T_x} + y k_{T_y}$$

$y=0 \Rightarrow x k_{I_x} = x k_{R_x} = x k_{T_x} \Rightarrow k_{I_x} = k_{R_x} = k_{T_x}$

Similarly, $k_{I_y} = k_{R_y} = k_{T_y}$

For simplicity, assume $k_{I_y} = 0$ ($\vec{k}_I$ lies in the xz plane)
$\Rightarrow k_{R_y} = k_{T_y} = 0 \Rightarrow \vec{k}_R$ and $\vec{k}_T$ lie in the same plane (xz) as $\vec{k}_I$

First law : $\vec{k}_I, \vec{k}_R, \vec{k}_T$ and the normal (to the boundary plane) lie in the same "plane of incidence"



$$(k_I)_x = (k_R)_x = (k_T)_x \Leftrightarrow \underset{\substack{\uparrow \\ \text{angle of} \\ \text{incidence}}}{k_I \sin\theta_I} = \underset{\substack{\downarrow \\ \text{angle of} \\ \text{reflection}}}{k_R \sin\theta_R} = \underset{\substack{\searrow \\ \text{angle of} \\ \text{refraction}}}{k_T \sin\theta_T}$$

Second law :

$k_I = k_R \Rightarrow$

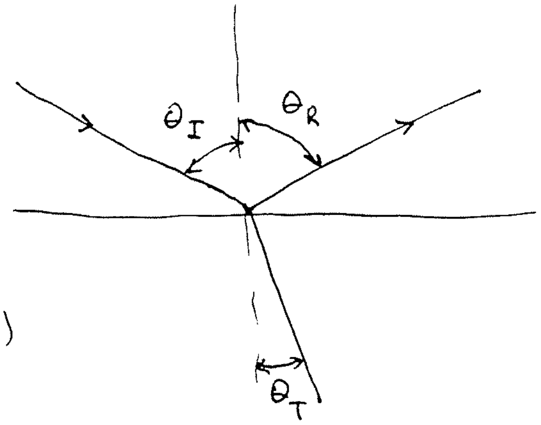
$\Rightarrow \underline{\theta_I = \theta_R}$

Third law $\quad \dfrac{\sin\theta_T}{\sin\theta_I} = \dfrac{k_I}{k_T} = \dfrac{v_2}{v_1} = \dfrac{n_1}{n_2}$

$\underline{\dfrac{\sin\theta_T}{\sin\theta_I} = \dfrac{n_1}{n_2}}$ "law of refraction" or Snell's law.

These are the general laws for many different types of waves (e. m., sound waves, etc).

Oblique incidence



Incident wave

$$\tilde{\vec E}_I(\vec r,t) = \tilde{\vec E}_{0_I} e^{i(\vec k_I\cdot\vec r-\omega t)}$$
$$\tilde{\vec B}_I(\vec r,t) = \frac{1}{v_1}(\hat k_I \times \tilde{\vec E}_I)$$

Reflected wave

$$\tilde{\vec E}_R(\vec r,t) = \tilde{\vec E}_{0_R} e^{i(\vec k_R\cdot\vec r-\omega t)}$$
$$\tilde{\vec B}_R(\vec r,t) = \frac{1}{v_1}(\hat k_R \times \tilde{\vec E}_R)$$

Transmitted wave

$$\tilde{\vec E}_T(\vec r,t) = \tilde{\vec E}_{0_T} e^{i(\vec k_T\cdot\vec r-\omega t)}$$
$$\tilde{\vec B}_T(\vec r,t) = \frac{1}{v_2}(\hat k_T \times \tilde{\vec E}_T)$$

Boundary conditions

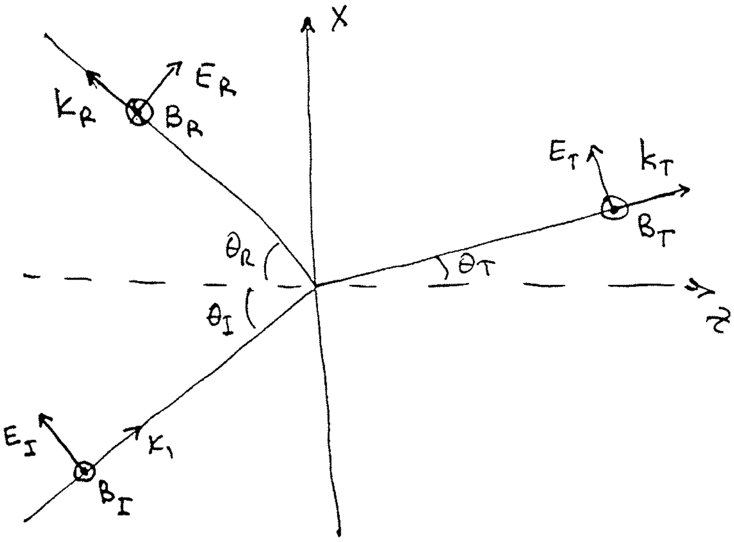
$$\left.\begin{array}{ll} \epsilon_1 E_\perp^{(1)} = \epsilon_2 E_\perp^{(2)}\Big|_{z=0} & B_\perp^{(1)} = B_\perp^{(2)}\Big|_{z=0} \\ \vec E_\parallel^{(1)} = \vec E_\parallel^{(2)}\Big|_{z=0} & \frac{1}{\mu_1}\vec B^{\parallel}_{(1)} = \frac{1}{\mu_2}\vec B^{\parallel}_{(2)}\Big|_{z=0} \end{array}\right\}$$

$$\Rightarrow \quad \epsilon_1 \tilde E^z_{0_I} e^{i\vec k_I\cdot\vec s} + \epsilon_1 \tilde E^z_{0_R} e^{i\vec k_R\cdot\vec s} = \epsilon_2 E^z_{0_T} e^{i\vec k_T\vec s} \qquad \vec s \equiv x\hat e_x + y\hat e_y$$
$$\tilde{\vec E}^{\parallel}_{0_I} e^{i\vec k_I\cdot\vec s} + \tilde{\vec E}^{\parallel}_{0_R} e^{i\vec k_R\cdot\vec s} = \tilde{\vec E}^{\parallel}_{0_T} e^{i\vec k_T\cdot\vec s}$$
$$(\hat k_I\times\tilde{\vec E}_{0_I})_z e^{i\vec k_I\cdot\vec s} + (\hat k_R\times\tilde{\vec E}_{0R})_z e^{i\vec k_R\cdot\vec s} = \frac{v_1}{v_2}(\hat k_T\times\tilde{\vec E}_{0_T})_z e^{i\vec k_T\cdot\vec s}$$
$$(\hat k_I\times\tilde{\vec E}_{0_I})^{\parallel} e^{i\vec k_I\cdot\vec s} + (\hat k_R\times\tilde{\vec E}_{0_R})^{\parallel} e^{i\vec k_R\cdot\vec s} = \frac{v_2\epsilon_2}{v_1\epsilon_1}(\hat k_T\times\tilde{\vec E}_{0_T})^{\parallel} e^{ik_T\cdot\vec s}$$

$\vec k_I\cdot\vec s = \vec k_R\cdot\vec s = \vec k_T\cdot\vec s \Rightarrow$ 3 laws of refraction ($\hat k_I, \hat k_R, k_T$ lie in the same plane, $\theta_R=\theta_I$, and $\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$)

$$\Rightarrow \quad \tilde E^z_{0_I} + \tilde E^z_{0_R} = \frac{\epsilon_2}{\epsilon_1}\tilde E^z_{0_T}$$
$$E^{x,y}_{0_I} + E^{x,y}_{0_R} = \tilde E^{x,y}_{0_T}$$
$$(\hat k_I\times\tilde{\vec E}_{0_I})_z + (\hat k_R\times\tilde{\vec E}_{0_R})_z = \frac{v_1}{v_2}(\hat k_T\times\tilde{\vec E}_{0_T})_z$$
$$(\hat k_I\times\tilde{\vec E}_{0_I})_{x,y} + (\hat k_R\times\tilde{\vec E}_{0_R})_{x,y} = \frac{v_2\epsilon_2}{v_1\epsilon_1}(k_T\times E_{0_T})_{x,y}$$

For the incident wave polarized ↑↑ to the plane of incidene



$$-\tilde{E}_{0_I}\sin\theta_I + \hat{E}_{0_R}\sin\theta_R = -\frac{\epsilon_2}{\epsilon_1}\hat{E}_{0_T}\sin\theta_T \qquad \frac{k_T}{k_R} = \frac{v_1}{v_2}$$

$$\tilde{E}_{I_0}\cos\theta_I + \tilde{E}_{0_R}\cos\theta_R = \hat{E}_{0_T}\cos\theta_T$$

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \frac{v_2\epsilon_2}{v_1\epsilon_1}E_{0_T}$$

$$\Rightarrow \begin{cases} \tilde{E}_{0_I} - \tilde{E}_{0_R} = \beta\hat{E}_{0_T} \\ \tilde{E}_{0_I} + \tilde{E}_{0_R} = \alpha\hat{E}_{0_T} \end{cases} \qquad \beta = \frac{v_2\epsilon_2}{v_1\epsilon_1} = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I}$$

$$\Rightarrow \quad \hat{E}_{0_R} = \frac{\alpha-\beta}{\alpha+\beta}\hat{E}_{0_I}, \quad \tilde{E}_{0_T} = \frac{2}{\alpha+\beta}\tilde{E}_{0_I}$$

Fresnel's equs

$$\alpha = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{1}{\cos\theta_I}\sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta_I}$$

For the normal incidence ($\theta_I=0$) $\alpha = 1$ and we recover

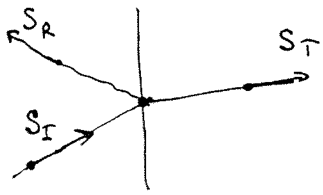
$$\tilde{E}_{0_R} = \frac{1-\beta}{1+\beta}E_{0_I}, \quad \tilde{E}_{0_T} = \frac{2}{1+\beta}\tilde{E}_{0_I}$$

Brewster's angle : no reflected wave at $\alpha=\beta$

$$\sin\theta_B = \sqrt{\frac{1-\beta^2}{\frac{n_1^2}{n_2^2}-\beta^2}}$$

Typically, $\mu_2 \simeq \mu_1 \Rightarrow \tan\theta_B = \frac{n_2}{n_1}$

Power per unit area of interface



Power $= S_z = \frac{1}{2}\epsilon_1 v_1 E_{0_I}^2\cos\theta_I$
for the incident wave

$$I_R = \frac{1}{2}\epsilon_1 v_1 E_{0_R}^2 \cos\theta_R$$
$$I_T = \frac{1}{2}\epsilon_2 v_2 E_{0_T}^2 \cos\theta_T$$
} power per unit of xy plane for the reflected and transmitted waves

$$\Rightarrow R = \frac{I_R}{I_I} = \left(\frac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$$ – reflection coefficient

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2 = \frac{4\alpha\beta}{(\alpha+\beta)^2}$$ – transmission coefficient

$R+T = 1$ conservation of energy

Electromagnetic waves in conductors

Ohm's law: $\vec{J}_f = \sigma\vec{E}$ ⇒ Maxwell's equs take the form

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon}\rho_f \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}$$

Continuity equ for free charge

$$\frac{\partial\rho_f}{\partial t} = -\vec{\nabla}\cdot\vec{J}_f = -\sigma(\vec{\nabla}\cdot\vec{E}) = -\frac{\sigma}{\epsilon}\rho_f \Rightarrow \rho_f(t) = e^{-\frac{\sigma}{\epsilon}t}\rho_f(0)$$

$$\rho_f(t) = e^{-\frac{t}{\tau}}\rho_f \qquad \tau = \frac{\epsilon}{\sigma}$$ – Characteristic time

perfect conductor $\sigma = \infty \Rightarrow \tau = 0$

"good" conductor $\tau \ll \frac{1}{\omega}$

"poor" conductor $\tau \gg \frac{1}{\omega}$

At $t \gg \tau$ accumulated free charge disappears $\Leftrightarrow \rho_f = 0$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\nabla^2 E = -\frac{\partial}{\partial t}\left(\mu\sigma\vec{E} + \mu\epsilon\frac{\partial E}{\partial t}\right) \Leftrightarrow \nabla^2\vec{E} = \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma\frac{\partial\vec{E}}{\partial t}$$
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = -\nabla^2 B = -\mu\sigma\frac{\partial\vec{B}}{\partial t} - \mu\epsilon\frac{\partial^2 B}{\partial t^2} \Rightarrow \nabla^2\vec{B} = \mu\epsilon\frac{\partial^2\vec{B}}{\partial t^2} + \mu\sigma\frac{\partial\vec{B}}{\partial t}$$

Monochromatic waves

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{i(\tilde{k}z-\omega t)} \qquad \tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_0 e^{i(\tilde{k}z-\omega t)} \Rightarrow$$

$$\Rightarrow \tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$$ complex wave number

$\tilde{k} = k + i\varkappa$ $$\left.\begin{array}{l} k^2 - \varkappa^2 = \mu\epsilon\omega^2 \\ k\varkappa = \frac{1}{2}\mu\sigma\omega \end{array}\right\} \Rightarrow k^2 = \frac{\mu\epsilon\omega^2}{2}\left(1+\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}\right)$$

$$\varkappa^2 = \frac{\mu\epsilon\omega^2}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}-1\right)$$

Imaginary part of $\tilde{k}$ ⇒ attenuation of the wave

$$\vec{\tilde{E}}(z,t) = \vec{\tilde{E}}_0 e^{-\varkappa z} e^{i(kz-\omega t)} \qquad \vec{\tilde{B}}(z,t) = \tilde{B}_0 e^{-\omega z} e^{i(kz-\omega t)}$$

"Skin depth" $d = \frac{1}{\varkappa}$ - how far the wave penetrates into the conductor

The fields $\vec{E}$ and $\vec{B}$ are transverse, as before. In general,

$$\vec{\tilde{E}}(\vec{r},t) = \vec{\tilde{E}}_0 e^{i(\vec{\tilde{k}}\cdot\vec{r}-\omega t)} = \tilde{E}_0 \hat{n} e^{i(\vec{\tilde{k}}\cdot\vec{r}-\omega t)} = \tilde{E}_0 \hat{n} e^{-\vec{\varkappa}\cdot\vec{r}} e^{i(\vec{k}\cdot\vec{r}-$$

$$\vec{\tilde{B}}(\vec{r},t) = \frac{1}{\omega}(\vec{\tilde{k}} \times \vec{\tilde{E}}) = \frac{1}{\omega}(\vec{\tilde{k}} \times \hat{n})\tilde{E}_0 e^{i(\vec{\tilde{k}}\cdot\vec{r}-\omega t)} = \frac{\vec{\tilde{k}}\times\hat{n}}{\omega}\tilde{E}_0 e^{-\vec{\varkappa}\cdot\vec{r}} e^{i(\vec{k}\vec{r}-$$

Modulus and phase of $\tilde{k}$ $\vec{\varkappa} \uparrow\uparrow \vec{k}$

$$\tilde{k} = K e^{i\varphi} \qquad K = \omega\sqrt{\mu\epsilon}\left(1+\frac{\sigma^2}{\omega^2\epsilon^2}\right)^{1/4} \qquad \varphi = \tan^{-1}\frac{\varkappa}{k}$$

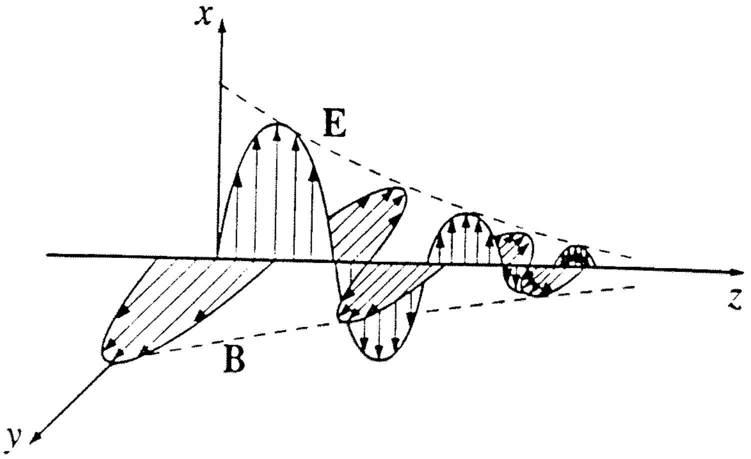
$$\vec{\tilde{B}}_0 = \frac{\vec{\tilde{k}} \times \vec{\tilde{E}}_0}{\omega} \Rightarrow \tilde{B}_0 = \frac{\tilde{k}}{\omega}\tilde{E}_0 \Leftrightarrow B_0 e^{i\delta_B} = \frac{K}{\omega} e^{i\varphi} E_0 e^{i\delta_E}$$

$\Rightarrow \delta_B - \delta_E = \varphi$ - magnetic field lags behind the electric fiel

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\mu\epsilon}\left(1+\frac{\sigma^2}{\omega^2\epsilon^2}\right)^{1/4}$$

For the vertically polarized wave propagating in z direction

$$\vec{E}(z,t) = \hat{e}_1 E_0 e^{-\varkappa z}\cos(kz-\omega t+\delta_E)$$
$$\vec{B}(z,t) = \hat{e}_2 B_0 e^{-\varkappa z}\cos(kz-\omega t+\delta_E+\varphi)$$



Reflection at a conducting surface

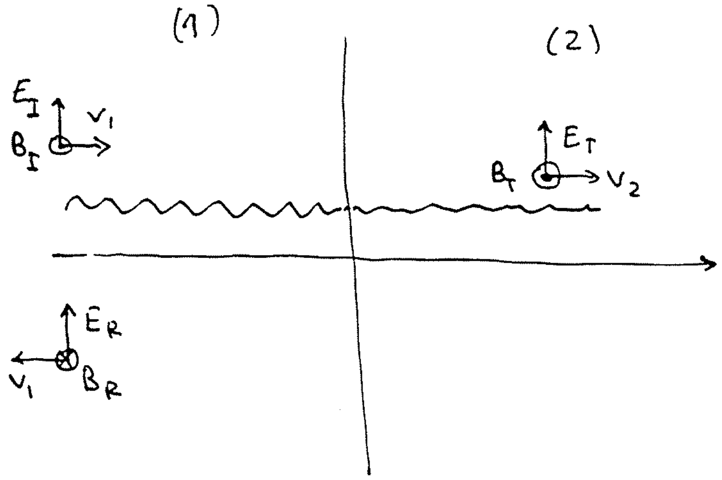
Boundary conditions

(i) $\epsilon_1 E^{\perp}_{(1)} - \epsilon_2 E^{\perp}_{(2)} = \sigma_f$ $\qquad$ $\vec{E}^{\parallel}_{(1)} = \vec{E}^{\parallel}_{(2)}$ (iii)

(ii) $B^{\perp}_{(1)} = B^{\perp}_{(2)}$ $\qquad$ $\frac{1}{\mu_1}\vec{B}^{\parallel}_{(1)} - \frac{1}{\mu_2}\vec{B}^{\parallel}_{(2)} = \vec{K}_f \times \hat{n}$ (iv)

$\downarrow$ unit vector ⊥ to the surface and pointing from (2) to (1)

Normal incidence



Incident wave

$$E_I(z,t) = \tilde{E}_{0_I}\hat{e}_1 e^{i(k_1 z - \omega t)}$$

$$B_I(z,t) = \frac{1}{v_1}\hat{e}_2 \tilde{E}_{0_I} e^{i(k_1 z - \omega t)}$$

Reflected wave

$$E_R(z,t) = \tilde{E}_{0_R}\hat{e}_1 e^{i(-k_1 z - \omega t)}$$

$$B_R(z,t) = -\frac{1}{v_1}\hat{e}_2 \tilde{E}_{0_R} e^{i(-k_1 z - \omega t)}$$

Transmitted wave

$$E_T(z,t) = \hat{e}_1 \tilde{E}_{0_T} e^{i(\tilde{k}_2 z - \omega t)}$$

$$B_T(z,t) = \frac{\tilde{k}_2}{\omega}\hat{e}_2 E_{0_T} e^{i(\tilde{k}_2 z - \omega t)}$$

$$\text{(iii)} \Rightarrow \tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T}$$

$$\text{(iv)} \Rightarrow \frac{1}{\mu_1 v_1}(\tilde{E}_{0_I} - \tilde{E}_{0_R}) = \frac{\tilde{k}_2}{\mu_2 \omega}\tilde{E}_{0_T} \Leftrightarrow \tilde{E}_{0_I} - \tilde{E}_{0_R} = \tilde{\beta}\tilde{E}_{0_T} \qquad \tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$$

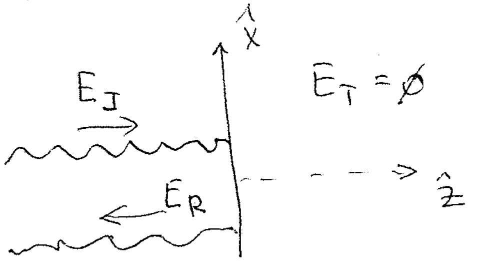
$$\Rightarrow \tilde{E}_{0_R} = \frac{1-\tilde{\beta}}{1+\tilde{\beta}}\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \frac{2}{1+\beta}\tilde{E}_{0_I}$$

For the perfect conductor $(\sigma = \infty)$ $\quad k_2 = \omega\sqrt{\frac{\mu\epsilon}{2}}\left(\sqrt{1+\frac{\sigma^2}{\epsilon^2\omega^2}}+1\right)^{1/2} = \infty$

$$\Rightarrow \tilde{\beta} = \infty \Rightarrow \tilde{E}_{0_R} = -\tilde{E}_{0_I}, \quad E_{0_T} = 0$$

no transmission, total reflection with 180° phase shift

For a perfect conductor $\sigma = \infty \Leftrightarrow E = B = 0$ inside



$$\tilde{\vec{E}}_I = \tilde{E}_{0I}\,\hat{e}_1\, e^{i(k_1 z-\omega t)} \qquad \tilde{\vec{B}}_I = \frac{1}{v_1}\hat{e}_2\, \tilde{E}^0_I e^{i(k_1 z-\omega t)}$$

$$\tilde{\vec{E}}_R = \tilde{E}_{0R}\,\hat{e}_1\, e^{i(k_1 z-\omega t)} \qquad \tilde{\vec{B}}_R = -\frac{1}{v_1}\hat{e}_2\, \tilde{E}_{0R}\, e^{i(-k_1 z-\omega t)}$$

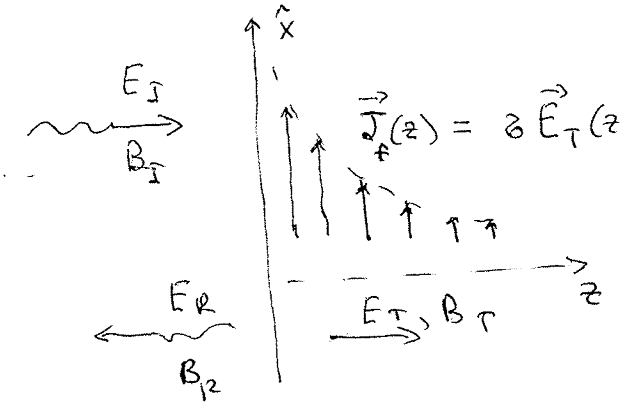
Boundary conditions : $\tilde{\vec{E}}^{\parallel}_{(1)} = \tilde{\vec{E}}^{\parallel}_{(2)} = \emptyset \qquad \tilde{E}^{\perp}, \tilde{B}^{\perp} = \emptyset$

$$\left.\begin{array}{l}\frac{1}{\mu_1}\tilde{\vec{B}}^{(\parallel)}_1 = \tilde{\vec{K}}_f \times \hat{n} \Leftrightarrow \tilde{\vec{K}}_f = \frac{1}{\mu_1}\hat{n}\times\vec{B}_{(1)}\end{array}\right\} \Rightarrow$$

$\Rightarrow \tilde{E}^R_0 = -\tilde{E}^I_0$ phase $= \pi$

$$\tilde{\vec{K}}_f = \frac{1}{\mu_1}(-\hat{e}_3)\times\hat{e}_2\,\frac{2}{v_1}\tilde{E}_{0I}\cos kz\, e^{-i\omega t}\Big|_{z=0} = \frac{2}{\mu_1 v_1}\hat{e}_1\tilde{E}_{0I}e^{-i\omega t}$$

Check : take $\sigma \to \infty$



$$\vec{J}_f(z) = \sigma\tilde{\vec{E}}_T(z) = \sigma\tilde{E}_{0T}\hat{e}_1\, e^{i(\tilde{k}_2 z-\omega t)}$$

$\tilde{k}_2 = k + i\varkappa \qquad \tilde{k}_2^2 = \mu_2\epsilon_2\omega^2 + i\mu_2\omega\sigma$

For $\sigma\to\infty \quad k_2 = \varkappa_2 = \sqrt{\frac{\mu_2\omega\sigma}{2}} \to \infty$

$$\tilde{E}_{0T} = \frac{2}{1+\tilde{\beta}}\tilde{E}_{0I} \qquad \tilde{\beta} = \frac{\mu_1 v_1}{\mu_2\omega}\tilde{k}_2 \to \infty$$

In the limit $\sigma\to\infty \qquad \vec{J}_f(z) \simeq \frac{2\sigma}{\tilde{\beta}}\hat{e}_1\, e^{i(\tilde{k}_2 z-\omega t)}$

$$\vec{K}_f = \int_0^\infty dz\,\vec{J}_f(z) = \int_0^\infty dz\,\frac{2\sigma}{\tilde{\beta}}\hat{e}_1\, e^{i(\tilde{k}_2 z-\omega t)}\tilde{E}_{0I} = \frac{2\sigma i}{\tilde{k}_2\tilde{\beta}}\hat{e}_1 e^{-i\omega t}$$

$$= \hat{e}_1\tilde{E}_{0I}\frac{2\sigma i\mu_2\omega}{\mu_1 v_1\tilde{k}_2^2} \simeq \frac{2\sigma i\mu_2\omega}{\mu_1 v_1 i\mu_2\omega\sigma}\hat{e}_1\tilde{E}_{0I} = \frac{2}{\mu_1 v_1}\tilde{E}_{0I}\hat{e}_1\,(\cdot\, e^{-i\omega t})$$

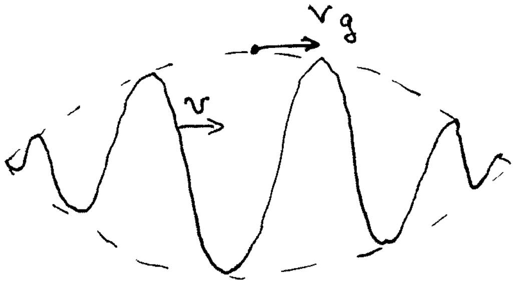
$\Rightarrow K_f = \frac{2}{\mu_1 v_1}\hat{e}_1\tilde{E}_{0I}$ is smeared over $\Delta z \sim \frac{1}{\varkappa_2} \simeq \frac{1}{\sqrt{\mu_2\omega\sigma}}$

The frequency dependence of permittivity

In real life $\epsilon = \epsilon(\omega)$, $\mu = \mu(\omega)$, $\sigma = \sigma(\omega)$

$\Rightarrow n = n(\omega)$ "dispersion": speed of wave depends on its frequency

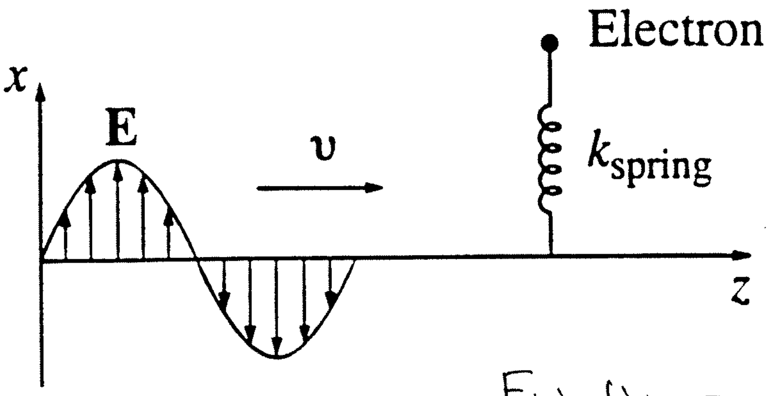
Wave packet



$v = \frac{\omega}{k}$ wave (phase) velocity

$v_g = \frac{\partial \omega}{\partial k}$ group velocity

A model for frequency dependence of permittivity



$F_{binding} = -k_{spring}\, x = -m\omega_0^2 x$

Motivation:

in general, $U(x) = U(0) + x\,\underset{0}{U'(0)} + \frac{1}{2}x^2 U''(0) + \frac{1}{3}\cancel{x^3 U'''}(0)$

irrelevant overall constant

0 if equilibrium

neglect if small oscillations

Damping force

$F_{damping} = -m\gamma\frac{dx}{dt}$ (again, $F_{damping} = f(\text{velocity}) \simeq$ $\simeq$ coeff × velocity at small v)

Driving force

$F_{driving} = qE = qE_0 \cos\omega t$

2nd law: $m\frac{d^2x}{dt^2} = F_{binding} + F_{damping} + F_{driving}$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = qE_0 \cos\omega t$$

differential eqn. for damped harmonic oscillator

Solution:

$$m\frac{d^2\tilde{x}}{dt^2} + m\gamma\frac{d\tilde{x}}{dt} + m\omega_0^2\tilde{x} = qE_0 e^{-i\omega t}$$

$\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t}$ – in a steady state the system should oscillate with driving frequency

$$(-m\omega^2\tilde{x}_0 - mi\gamma\omega\tilde{x}_0 + m\omega_0^2\tilde{x}_0)\, e^{-i\omega t} = qE_0 e^{-i\omega t} \Rightarrow$$

$$\Rightarrow \tilde{x}_0 = \frac{qE_0}{m}\frac{1}{\omega_0^2-\omega^2-i\gamma\omega} = \frac{qE_0}{m}\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}\, e^{i\delta_\omega}$$

$$\tan\delta_\omega = \frac{\gamma\omega}{\omega_0^2-\omega^2}$$

$$\Rightarrow \quad \tilde{x}(t) = \frac{qE_0}{m}\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}\, e^{-i(\omega t-\delta_\omega)}$$

$$x_0 = \frac{qE}{m\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}$$

$$\Rightarrow \quad x(t) = \frac{qE_0}{m\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}\cos(\omega t-\delta_\omega)$$

Dipole moment $\quad p = q\,x(t) \qquad p = \mathrm{Re}\,\tilde{p}$

$$\tilde{p}(t) = q\tilde{x}(t) = \frac{q^2/m}{\omega_0^2-\omega^2-i\gamma\omega}\, E_0 e^{-i\omega t}$$

If there are N molecules per unit volume

$$\tilde{\vec{P}} = \frac{Nq^2}{m}\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\gamma_j\omega}$$

f_j – # of electrons in the molecule

ω_j, γ_j

$\tilde{\vec{P}} = \epsilon_0\tilde{\chi}_e\tilde{\vec{E}}$ "complex susceptibility" $\tilde{\chi}_e$

(for the real parts, $\vec{P} \not\upuparrows \vec{E}$ due to the phase shift)

$$\Rightarrow \tilde{\epsilon}_r = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\gamma_j\omega}$$

"complex dielectric constant"

In a dispersive medium

$$\nabla^2\tilde{\vec{E}} = \tilde{\epsilon}\mu_0\frac{\partial^2\tilde{\vec{E}}}{\partial t^2}$$

Similarly to the case of conductors,

$$\vec{\tilde{E}}(z,t) = \tilde{\vec{E}}_0\, e^{i(\tilde{k}z-\omega t)}$$

$$\tilde{k} = \sqrt{\tilde{\epsilon}\mu_0\omega} = k + i\varkappa \Rightarrow \quad \vec{\tilde{E}}(z,t) = \tilde{\vec{E}}_0\, e^{-\varkappa z} e^{i(kz-\omega t)}$$

attenuated wave

$\alpha \equiv 2\varkappa$ absorption coefficient (factor 2 because the intensity $\sim E^2$ falls off as $e^{-2\varkappa z}$)

Formulas are similar to the case of conductors, but the underlying physics is quite different.

For gases, $\hat{\epsilon}_r = 1 + (\text{small } \# r)$ so $\sqrt{1+\#} \simeq 1 + \frac{\#}{2}$

$$\tilde{k} = \frac{\omega}{c}\sqrt{\tilde{\epsilon}_r} \simeq \frac{\omega}{c}\left(1 + \frac{Nq^2}{2m\epsilon_0}\sum \frac{f_j}{\omega_j^2-\omega^2-i\gamma_j\omega}\right) \quad \tilde{k} = k + i\varkappa \Rightarrow$$

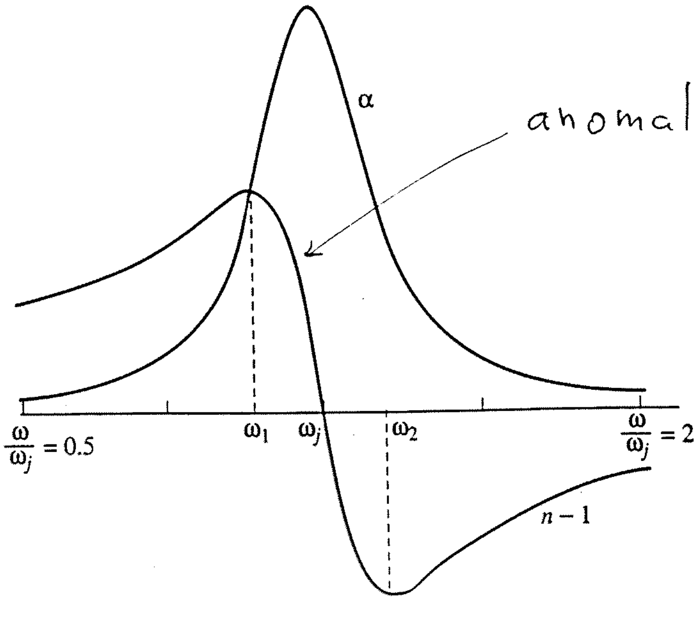
$$\Rightarrow k = \frac{\omega}{c}\left(1 + \sum \frac{Nq^2}{2m\epsilon_0}\frac{f_j(\omega_j^2-\omega^2)}{(\omega_j^2-\omega^2)^2+\gamma_j^2\omega^2}\right), \quad \varkappa = \frac{\omega^2}{c}\frac{q^2}{2m\epsilon_0}\sum\frac{f_j\gamma_j}{(\omega_j^2-\omega^2)^2+\gamma_j^2\omega}$$

The index of refraction is defined as $\frac{c}{\omega/k} \Rightarrow$

$$\Rightarrow n = 1 + \frac{Nq^2}{2m\epsilon_0}\sum\frac{f_j(\omega_j^2-\omega^2)}{(\omega_j^2-\omega^2)^2+\gamma_j^2\omega^2}$$

the absorption coefficient is

$$\alpha = 2\varkappa = \frac{Nq^2\omega^2}{m\epsilon_0 c}\sum_i \frac{f_j\gamma_i}{(\omega_j^2-\omega^2)^2+\gamma_i^2\omega^2}$$



anomalous dispersion

Away from the resonance frequencies the damping ($\sim\gamma_i$) can be ignored

$$n = 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2-\omega^2}$$

For the transparent materials typically $\omega < \omega_j$

$$\Rightarrow \frac{1}{\omega_j^2-\omega^2} \approx \frac{1}{\omega_j^2}\left(1+\frac{\omega^2}{\omega_j^2}\right) \Rightarrow$$

$$\Rightarrow n = 1 + \left(\frac{Nq^2}{2m\epsilon_0}\sum \frac{f_j}{\omega_j^2}\right) + \omega^2\left(\frac{Nq^2}{2m\epsilon_0}\sum \frac{f_j}{\omega_j^4}\right)$$

$$\underset{\substack{\uparrow \\ \text{wavelength}}}{\lambda} = \frac{2\pi c}{\omega}$$

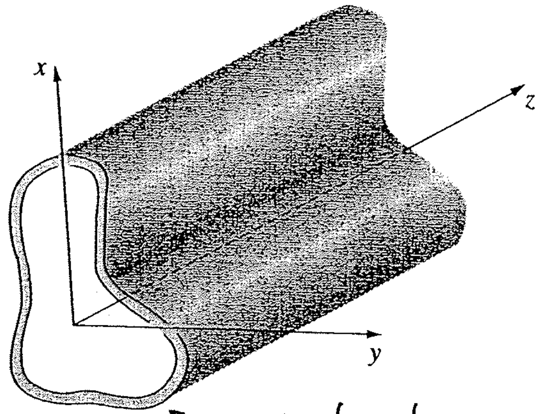
$$\boxed{n = 1 + A\left(1+\frac{B}{\lambda^2}\right)}$$ Cauchy's formula

A - coeff. of refraction

B - coeff. of dispersion

Guided waves

$$\left.\begin{array}{ll}\text{(i)} & \mathbf{E}^{\|} = 0, \\ \text{(ii)} & B^{\perp} = 0.\end{array}\right\}$$ on the surface of the conductor



← conductor $\Rightarrow E = 0,\ B = 0$ (in a perfect conductor)

Monochromatic waves propagating in z direction

$$\vec{\tilde{E}}(\vec{r},t) = \vec{\tilde{E}}_0(x,y)\, e^{i(kz-\omega t)}$$
$$\vec{\tilde{B}}(\vec{r},t) = \vec{\tilde{B}}_0(x,y)\, e^{i(kz-\omega t)}$$

Maxwell's eqns

$$\vec{\nabla}\cdot\vec{\tilde{E}} = 0 \qquad \vec{\nabla}\times\vec{\tilde{E}} = -\frac{\partial\vec{\tilde{B}}}{\partial t} = i\omega\vec{\tilde{B}}$$
$$\vec{\nabla}\cdot\vec{\tilde{B}} = 0 \qquad \vec{\nabla}\times\vec{\tilde{B}} = \frac{1}{c^2}\frac{\partial\vec{\tilde{E}}}{\partial t} = -\frac{i\omega}{c^2}\vec{\tilde{E}}$$

In components

$$\frac{\partial}{\partial z}e^{ikz} = ike^{ikz}$$

$$\partial_x\tilde{E}_1 + \partial_y\tilde{E}_2 + \partial_z\tilde{E}_3 = 0 \Rightarrow \partial_x\tilde{E}_1 + \partial_y\tilde{E}_2 + ik\tilde{E}_3 = 0$$
$$\partial_x\tilde{B}_1 + \partial_y\tilde{B}_2 + \partial_z\tilde{B}_3 = 0 \Rightarrow \partial_x\tilde{B}_1 + \partial_y\tilde{B}_2 + ik\tilde{B}_3 = 0$$
$$\partial_x\hat{E}_2 - \partial_y\hat{E}_1 = i\omega\hat{B}_3$$
$$\partial_y\hat{E}_3 - \partial_z\hat{E}_2 = i\omega\tilde{B}_1 \Rightarrow \partial_y\tilde{E}_3 - ik\hat{E}_2 = i\omega\hat{B}_1$$
$$\left.\begin{array}{l}\partial_x\hat{E}_3 - \partial_z\hat{E}_1 = -i\omega\tilde{B}_2 \Rightarrow \partial_x\tilde{E}_3 - ik\tilde{E}_1 = -i\omega\hat{B}_2 \\ \partial_x\hat{B}_2 - \partial_y\tilde{B}_1 = -\frac{i\omega}{c^2}\tilde{E}_3 \\ \partial_y\hat{B}_3 - \partial_z\hat{B}_2 = -\frac{i\omega}{c^2}\tilde{E}_1 \Rightarrow \partial_y\tilde{B}_3 - ik\hat{B}_2 = -\frac{i\omega}{c^2}\tilde{E}_1\end{array}\right\} \Rightarrow \begin{array}{l}\tilde{B}_2 = \frac{\partial_x\tilde{E}_3 - ik\tilde{E}_1}{-i\omega} = \\ = \frac{\partial_y\tilde{B}_3 + \frac{i\omega}{c^2}\tilde{E}_1}{ik} \Rightarrow \\ \Rightarrow \tilde{E}_1 = \frac{i(k\partial_x\tilde{E}_3 + \omega\partial_y\tilde{B}_3)}{\frac{\omega^2}{c^2} - k^2}\end{array}$$
$$\partial_x\hat{B}_3 - \partial_z\hat{B}_1 = \frac{i\omega}{c^2}\tilde{E}_2 \Rightarrow \partial_x\hat{B}_3 - ik\hat{B}_1 = \frac{i\omega}{c^2}\tilde{E}_2$$

Similarly,

$$\tilde{E}_2 = \frac{i}{\frac{\omega^2}{c^2} - k^2} \left(k \partial_y \tilde{E}_3 - \omega \partial_x \hat{B}_3\right)$$

$$\hat{B}_1 = \frac{i}{\frac{\omega^2}{c^2} - k^2} \left(k \partial_x \tilde{B}_3 - \frac{\omega}{c^2} \partial_y E_3\right)$$

$$\hat{B}_2 = \frac{i}{\frac{\omega^2}{c^2} - k^2} \left(k \partial_y \tilde{B}_3 + \frac{\omega}{c^2} \partial_x \tilde{E}_3\right)$$

For the E_3, B_3 components we obtain

$$\partial_x \tilde{E}_2 - \partial_y \hat{E}_1 = i\omega \tilde{B}_3 \Rightarrow \frac{i}{\frac{\omega^2}{c^2} - k^2} \left(k \partial_x \partial_y \hat{E}_3 - \omega \partial_x^2 B - k \partial_y \partial_x \tilde{E}_3 - \omega \partial_y^2 \tilde{B}_3\right) = i\omega \tilde{B}_3$$

$$\Rightarrow (\partial_x^2 + \partial_y^2) B_3 + \left(\frac{\omega^2}{c^2} - k^2\right) B_3 = 0$$

Similarly,

$$\partial_x \hat{B}_2 - \partial_y \hat{B}_1 = -\frac{i\omega}{c^2} \hat{E}_3 \quad \Rightarrow \quad (\partial_x^2 + \partial_y^2) \tilde{E}_3 + \left(\frac{\omega^2}{c^2} - k^2\right) \tilde{E}_3 = 0$$

$B_3 = 0$ – transverse magnetic (TM) waves

$E_3 = 0$ – transverse electric (TE) waves

$E_3 = B_3 = 0$ TEM waves

Property: TEM waves cannot occur in a hollow wave guide

Proof:

$E_3, B_3 = 0 \Rightarrow$ Gauss law $\Rightarrow$ $\partial_x E_x + \partial_y E_y = 0$

Faraday's law $\Rightarrow$ $\partial_x E_y - \partial_y E_x = 0$

$\Rightarrow \vec{E} = -\vec{\nabla}\Phi(x,y)$ where Φ is a scalar potential in (x,y) plane

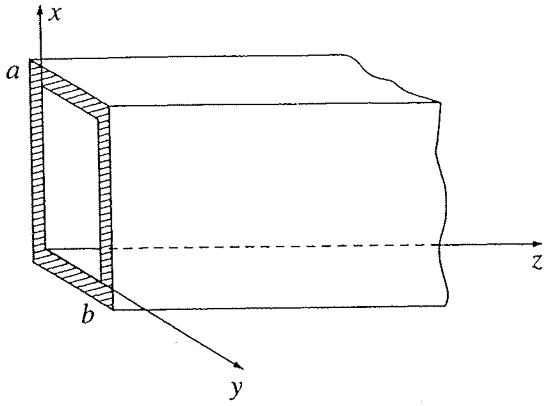
Boundary condition $\vec{E}^{\parallel} = 0 \Rightarrow$ surface of the wave guide is equipotential

Laplace eqn $\nabla^2 \Phi = 0$ admits no local maxima or minima

} $\Phi = \text{const}$ throughout the interior of the wave guide

$\Rightarrow \vec{E} = 0 \Rightarrow$ no wave

TE waves in a rectangular wave guide



$E_3 = 0 \Rightarrow$ we should solve the eqn

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right)\tilde{B}_3(x,y) = 0$$

with the boundary condition $B^{\perp}(x,y) = 0$

Ansatz $\tilde{B}_3(x,y) = \tilde{X}(x)\tilde{Y}(y)$ (separation of variables)

$$\tilde{Y}\partial_x^2\tilde{X} + \tilde{X}\partial_y^2\tilde{Y} + \left(\frac{\omega^2}{c^2} - k^2\right)\tilde{X}\tilde{Y} = 0 \Rightarrow \frac{\partial_x^2\tilde{X}(x)}{X(x)} + \frac{\partial_y^2\tilde{Y}(y)}{Y(y)} + \frac{\omega^2}{c^2} - k^2 = 0$$

depends only on x ⇒ ⇒ must be a #   same   number

$$\Rightarrow \frac{\partial_x^2\tilde{X}}{\tilde{X}} = -k_1^2$$

$$\frac{\partial_y^2\tilde{Y}}{\tilde{Y}} = -k_2^2 \qquad k_1^2 + k_2^2 = \frac{\omega^2}{c^2} - k^2$$

$$\Rightarrow \partial_x^2 X = -k^2 X \Rightarrow \tilde{X}(x) = A\sin k_1 x + B\cos k_1 x$$

$$\partial_y^2 Y = -k^2 Y \Rightarrow \tilde{Y}(y) = C\sin k_2 y + D\cos k_2 y$$

Boundary conditions: $B_1(0,y) = B_1(a,y) = 0$

$$\Rightarrow \left.\frac{\partial\tilde{B}_3}{\partial x}\right|_{x=0} = \left.\frac{\partial\tilde{B}_3}{\partial x}\right|_{x=a} = 0 \Rightarrow \tilde{X}'(0) = \tilde{X}'(a) = 0 \Rightarrow A = 0,\ k_1 a = \pi m$$

m-integer

$$\Rightarrow \tilde{X}(x) = B\cos\frac{\pi m}{a}x$$

Similarly, $\tilde{Y}(y) = D\cos\frac{\pi n}{b}y$ n - integer

$$\Rightarrow \tilde{B}_3(x,y) = \tilde{B}_0\cos\frac{mx\pi}{a}\cos\frac{ny\pi}{b}$$

"TE_{mn} mode"

$k = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2 m^2}{a^2} - \frac{\pi^2 n^2}{b^2}}$ - wave number

If $\frac{\omega}{c} < \pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$ wave #r is imaginary ⇒

⇒ exponentially attenuated fields $\sim e^{-z\sqrt{\frac{\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2} - \frac{\omega^2}{c^2}}}$ instead of traveling waves. ⇒

⇒ $\omega_{mn} = \pi c\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$ is a cutoff frequency

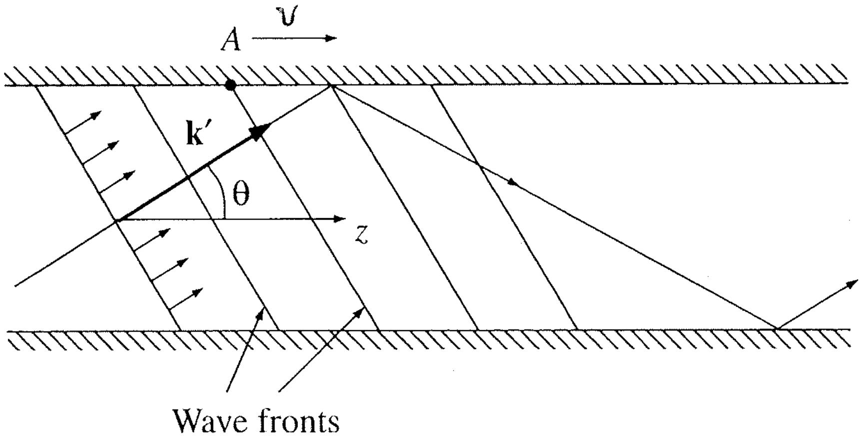
The waves with $\omega < \omega_{10} = \pi\frac{c}{a}$ do not propagate at all

$k = \frac{1}{c}\sqrt{\omega^2-\omega_{mn}^2} \Rightarrow \quad v = \frac{c}{\sqrt{1-\frac{\omega_{mn}^2}{\omega^2}}}$ - wave velocity

Note that $v > c$, but

$$v_g = \frac{d\omega(k)}{dk} = \frac{1}{\frac{dk(\omega)}{d\omega}} = \frac{c\sqrt{\omega^2-\omega_{mn}^2}}{\omega} = c\sqrt{1-\frac{\omega_{mn}^2}{\omega^2}} < c$$

An illustration: ordinary plane wave traveling at angle θ to z axis and reflecting perfectly off each conducting surface



In x, y directions - standing waves

$\lambda_x = \frac{2a}{m} \quad \lambda_y = \frac{2b}{n} \Rightarrow k_x = \frac{2\pi}{\lambda_x} = \pi\frac{m}{a}, \; k_y = \pi\frac{n}{b}$

In the z direction - traveling wave with $k_3 = k$

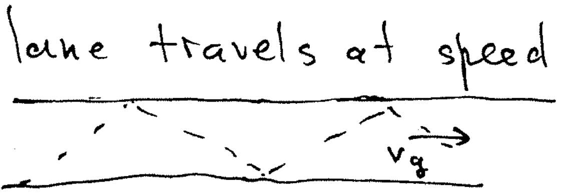
$\Rightarrow \vec{k}' = \frac{\pi m}{a}\hat{e}_1 + \frac{\pi n}{b}\hat{e}_2 + k\hat{e}_3$ - wave vector for the "original" plane wave

$\Rightarrow \omega = c|\vec{k}'| = \sqrt{c^2k^2 + \omega_{mn}^2}$

m, n integer $\Rightarrow \cos\theta = \frac{k}{|\vec{k}'|} = \sqrt{1-\frac{\omega_{mn}^2}{\omega^2}} \Rightarrow \theta$ is "quantized"

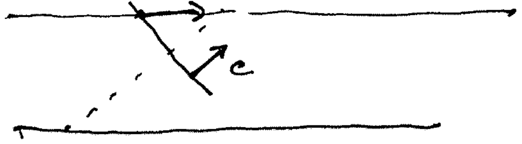
The plane travels at speed c, but at angle $\theta \Rightarrow$

$\Rightarrow$



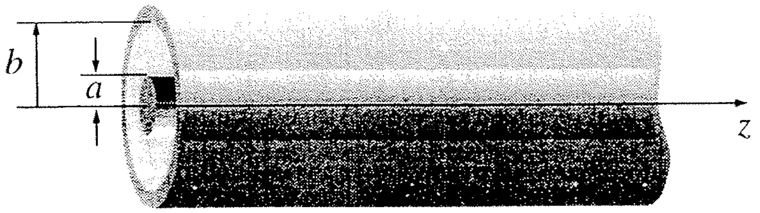
$v_g = c\cos\theta = c\sqrt{1-\frac{\omega_{mn}^2}{\omega^2}}$

Wave velocity is the speed of wave fronts



$$v = \frac{c}{\cos\theta} = \frac{c}{\sqrt{1-\frac{\omega_{mn}^2}{\omega^2}}}$$

The coaxial transmission line



$$\tilde{\vec{E}}(\vec{r},t) = \tilde{\vec{E}}_0(x,y)\,e^{i(kz-\omega t)}$$
$$\hat{\vec{B}}(\vec{r},t) = \tilde{\vec{B}}_0(x,y)\,e^{i(kz-\omega t)}$$

Now it is possible to have $E_3^{(0)} = B_3^{(0)} = 0$

Equations

$$\left.\begin{array}{ll} \partial_x \tilde{E}_2 = \partial_y \tilde{E}_1 & \partial_x \tilde{B}_2 = \partial_y \tilde{B}_1 \\ \tilde{E}_2 = -\frac{\omega}{k}\tilde{B}_1 & \hat{B}_2 = \frac{\omega}{c^2}\hat{E}_1 \\ \tilde{B}_2 = \frac{k}{\omega}\tilde{E}_1 & \tilde{E}_2 = -\frac{c^2 k}{\omega}\tilde{B}_1 \end{array}\right\} \Rightarrow \begin{array}{l} k = \frac{\omega}{c} \\ \tilde{B}_2 = \frac{1}{c}\tilde{E}_1\,,\ \tilde{B}_1 = -\frac{1}{c}\tilde{E}_2 \end{array}$$

Gauss law: $\left.\begin{array}{l}\partial_x \tilde{E}_1 + \partial_y \tilde{E}_2 = 0 \\ \partial_x \tilde{E}_2 - \partial_y \hat{E}_1 = 0\end{array}\right\}$ eqns. for the electrostatics with cylindrical symmetry

$$\begin{array}{l}\partial_x \tilde{E}_1(x,y) + \partial_y \hat{E}_2(x,y) = 0 \\ \partial_x \hat{E}_2(x,y) - \partial_y \tilde{E}_1(x,y) = 0\end{array}$$

same eqns as for the charged line

$$\Rightarrow \tilde{\vec{E}}_0(s,\varphi) = \frac{\hat{A}}{s}\hat{s} \Rightarrow$$
$$\Rightarrow \tilde{\vec{B}}_0(s,\varphi) = \frac{\tilde{A}}{cs}\hat{\varphi}$$

(same as for the infinite wire)

$$\Rightarrow \vec{E}(s,\varphi,z,t) = \frac{A}{s}\hat{s}\cos(kz-\omega t)$$
$$\vec{B}(s,\varphi,z,t) = \frac{A}{cs}\hat{\varphi}\cos(kz-\omega t)$$