Potentials and Fields

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = \phi \qquad \vec{\nabla}\times\vec{B} = \mu_0 J + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

Maxwell's eqns

A problem: given $\rho(\vec{r},t)$ and $\vec{J}(\vec{r},t)$, what are $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$?

It is easier to proceed in terms of potentials (as in the case of electro/magnetostatics)

In electrostatics: $\vec{\nabla}\times\vec{E} = \phi \Rightarrow \vec{E} = \vec{\nabla}(-V)$

Now $\vec{\nabla}\times\vec{E} \neq 0 \Rightarrow$ more complicated

Still,

$$\vec{\nabla}\cdot\vec{B} = \phi \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$$ as in magnetostatics

$$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A} \Rightarrow \vec{\nabla}\times\underbrace{\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)}_{\text{a gradient of a scalar}} = \phi \Rightarrow$$

$$\Rightarrow \vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}V \Rightarrow$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$
$$\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t}$$

fields in terms of potentials $(*)$

With $(*)$, eqns $\vec{\nabla}\cdot\vec{B} = \phi$ and $\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$ are trivially satisfied

Two other eqns give

$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\epsilon_0}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} - \mu_0\epsilon_0\vec{\nabla}\frac{\partial V}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\Leftrightarrow \begin{cases} \left(\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J} \\ \nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\epsilon_0} \end{cases}$$

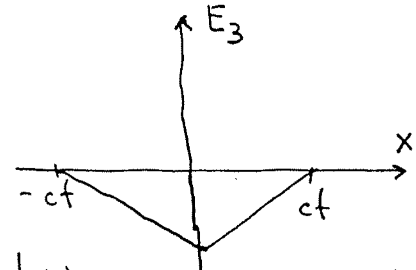
Maxwell's eqns in terms of potentials

Example 10.1

Find $\vec{E}, \vec{B}$ and ρ, J for the potentials

$$V = 0 \qquad \vec{A} = \frac{\mu_0 k}{4c}(ct - |x|)^2 \hat{e}_3 \, \theta(ct - |x|)$$

1. $$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 k}{2}(ct - |x|)\hat{e}_3 \, \theta(ct - |x|)$$



It is easy to see that

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial}{\partial z} E_3 = \emptyset \quad \Rightarrow \rho = \emptyset$$

$$(\vec{\nabla} \times \vec{E})_2 = \cancel{\frac{\partial}{\partial z} E_1} - \frac{\partial}{\partial x} E_3 = \frac{\mu_0 k}{2} \frac{\partial}{\partial x}(ct - |x|)\, \theta(ct - |x|)$$

$$= \begin{cases} -\frac{\mu_0 k}{2} & x > 0 \\ \frac{\mu_0 k}{2} & x < 0 \end{cases} = -\frac{\mu_0 k}{2} \epsilon(x)\, \theta(ct - |x|)$$

$$\epsilon(x) = \theta(x) - \theta(-x) = \begin{cases} 1 \text{ if } x > 0 \\ -1 \text{ if } x < 0 \end{cases}$$

(sign x)

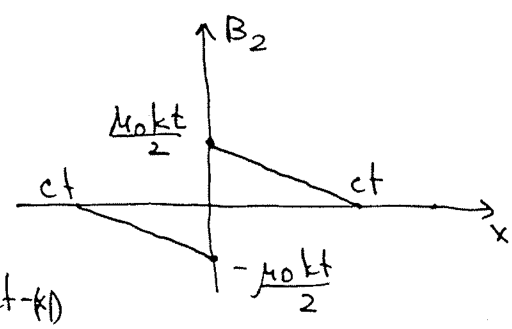
$$\Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\mu_0 k}{2} \epsilon(x)\, \theta(ct - |x|)\, \hat{e}_2$$

2. $$\vec{B} = \vec{\nabla} \times \vec{A} \qquad B_1 = B_3 = \emptyset$$

$$B_2 = (\vec{\nabla} \times \vec{A})_2 = \frac{\partial}{\partial z} A_1 - \frac{\partial}{\partial x} A_3 = -\frac{\mu_0 k}{4c} \frac{\partial}{\partial x}(ct - |x|)^2 \theta(ct - |x|)$$

$$= \begin{cases} \frac{\mu_0 k}{2c}(ct - |x|) & x > 0 \\ -\frac{\mu_0 k}{2c}(ct - |x|) & x < 0 \end{cases} = \frac{\mu_0 k}{2c}(ct - |x|)\, \epsilon(x)\, \theta(ct - |x|)$$

$$\Rightarrow \vec{B} = \frac{\mu_0 k}{2c}(ct - |x|)\, \epsilon(x)\, \theta(ct - |x|)\, \hat{e}_2$$



$$\vec{\nabla} \cdot \vec{B} = \frac{\partial}{\partial y} B_2 = \emptyset$$

$$(\vec{\nabla} \times \vec{B})_3 = \frac{\partial}{\partial x} B_2 = \frac{\mu_0 k}{2c} \frac{\partial}{\partial x}(ct - |x|)\, \epsilon(x)\, \theta(ct - |x|)$$

$$\begin{matrix} -\frac{\mu_0 k}{2c}\, \theta(ct - |x|) & x > 0 \\ -\frac{\mu_0 k}{2c}\, \theta(ct - |x|) & x < 0 \end{matrix} = -\frac{\mu_0 k}{2c}\, \theta(ct - |x|)$$

$$\frac{\partial \vec{E}}{\partial t} = \frac{\mu_0 k c}{2} \hat{e}_3\, \theta(ct - |x|)$$

$$\Rightarrow \rho = \emptyset \quad \text{and} \quad \vec{J} = \emptyset,$$

But $$\vec{B}_1^{\parallel} - \vec{B}_2^{\parallel} = \mu_0 \vec{K}_f \times \vec{n} \Rightarrow \mu_0 k t \hat{e}_2 = \mu_0 \vec{K} \times \hat{e}_1 \Rightarrow$$

$$\Rightarrow kt\hat{e}_2 = \vec{K} \times \hat{e}_1 \Rightarrow K = kt\hat{e}_3$$

Gauge transformations

Maxwell's equs $\quad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$

$$\vec{\nabla}\cdot\vec{B} = \emptyset \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

Potentials V and A $\quad \left.\begin{array}{l} \vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} \\ B = \vec{\nabla}\times\vec{A} \end{array}\right\} \Rightarrow$

$$\left.\begin{array}{l} \nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\epsilon_0} \\ (\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}) - \vec{\nabla}(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}) = -\mu_0 J \end{array}\right\} \text{Maxwell's equs in terms of potentials.}$$

NB: V and A are <u>not</u> uniquely defined by $\vec{E}$ and $\vec{B}$

Suppose we have V, A and V', A' corresponding to the same $\vec{E}$ and $\vec{B}$. By how much can they differ?

$V' = V + \beta \qquad \vec{A}' = A + \alpha \qquad \vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times(\vec{A}+\vec{\alpha}) \Rightarrow$

$\Rightarrow \vec{\nabla}\times\vec{\alpha} = \emptyset \Rightarrow \vec{\alpha} = \vec{\nabla}\Lambda$

scalar function (of $\vec{r}$ and t)

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}(V+\beta) - \frac{\partial}{\partial t}(\vec{A}+\vec{\alpha}) \Rightarrow$$

$$\Rightarrow \vec{\nabla}(\beta + \frac{\partial\Lambda}{\partial t}) = \emptyset \Rightarrow \beta + \frac{\partial\Lambda}{\partial t} = k(t) \Rightarrow$$

$$\Rightarrow \begin{array}{l} \vec{\alpha} = \vec{\nabla}\Lambda(\vec{r},t) \\ \beta = -\frac{\partial\Lambda}{\partial t} + k(t) \end{array}$$

We can redefine Λ: $\lambda = \Lambda + \int_0^t dt' k(t') \Rightarrow$

$\Rightarrow \begin{array}{l} \vec{\alpha} = \vec{\nabla}\lambda \\ \beta = -\frac{\partial\lambda}{\partial t} \end{array}$ where $\lambda(\vec{r},t)$ is an arbitrary (scalar) function

$$\Rightarrow \left.\begin{array}{l} \vec{A}' = \vec{A} + \vec{\nabla}\lambda(\vec{r},t) \\ V' = V - \frac{\partial\lambda(\vec{r},t)}{\partial t} \end{array}\right\} \text{does not change } \vec{E} \text{ and } \vec{B}$$

Such transformations of the potentials are called gauge transformations.

Coulomb gauge and Lorentz gauge

Coulomb gauge

In magnetostatics, we took $\vec{\nabla}\cdot\vec{A} = \emptyset \Rightarrow$
$\Rightarrow \nabla^2 V = -\frac{\rho}{\epsilon_0}$ and the solution was

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|} \quad \text{– Coulomb potential}$$

We can make the same choice of gauge $\vec{\nabla}\cdot\vec{A} = \emptyset$ in electrodynamics, then

$$\nabla^2 V(\vec{r},t) = -\frac{\rho(\vec{r},t)}{\epsilon_0}$$

$$\nabla^2\vec{A}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2} = -\mu_0\vec{J}(r,t) + \frac{1}{c^2}\vec{\nabla}\frac{\partial V(r,t)}{\partial t}$$

Solution of the first equ. is the Coulomb potential

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|},$$

But the solution of the second equ (for $\vec{A}$) is very complicated.

This is no surprize because:

$V(\vec{r},t)$ is determined by $\rho(\vec{r}';t)$ at the same $t \Rightarrow$
$\Rightarrow V(\vec{r},t)$ instantaneously reflect changes in $\rho(\vec{r}';t)$ at some remote $\vec{r}'$ $\Rightarrow$ contradicts to the theory of special relativity.

$\vec{A}(\vec{r},t)$ is organized in such a way that $\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$ depends only on the distribution of charges $\rho(\vec{r}',t')$ at $|\vec{r}'-\vec{r}| < c(t'-t) \Rightarrow \vec{A}$ is complicated

Lorentz gauge

$$\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \Rightarrow$$

$$(*)\quad \begin{aligned} \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} &= -\frac{\rho}{\epsilon_0} \\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= -\mu_0 J \end{aligned} \iff \left.\begin{aligned} \Box^2 \vec{V} &= -\frac{\rho}{\epsilon_0} \\ \Box^2\vec{A} &= -\mu_0 J \end{aligned}\right\} \text{Maxwell's eqns in the Lorentz gauge}$$

$$\Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \quad \text{"d'Alembertian"}$$

How to solve these eqns?
In the static case, we had

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \Rightarrow V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d^3x'$$

$$\nabla^2 A = -\mu_0 J \qquad \vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d^3x'$$

A guess for the solution of the eqns (*)

$$(**)\quad \begin{aligned} V(\vec{r},t) &= \frac{1}{4\pi\epsilon_0}\int d^3x' \,\frac{\rho(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|} \\ \vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\int d^3x' \,\frac{\vec{J}(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|} \end{aligned} \qquad t_r \equiv t - \frac{|\vec{r}-\vec{r}'|}{c} \quad \text{"retarded time"}$$

The potentials are determined by $\rho(\vec{r}')$ at the time t_r "when the signal was sent"

We will prove that (**) is actually the solution of (*)
NB: for the fields, the corresponding guess does <u>not</u> give the right result:

$$\vec{E}(\vec{r},t) \neq \frac{1}{4\pi\epsilon_0}\int d^3x' \,\frac{\rho(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|}, \quad \vec{B}(r,t) \neq \frac{\mu_0}{4\pi}\int d^3x' \,\frac{\vec{J}(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|}$$

Thus, we are lucky that for the potentials our simple guess (**) is right.

Proof

1. Let $V(\vec r,t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec r', t_r)}{|\vec r-\vec r'|}$

$$\vec\nabla V = \frac{1}{4\pi\epsilon_0}\int d^3x' \left\{\rho(\vec r',t)\,\vec\nabla_r \frac{1}{|\vec r-\vec r'|} + \frac{1}{|\vec r-\vec r'|}\vec\nabla_r\, \rho(\vec r',t_r)\right\}$$

where $\rho(\vec r', t_r)$ depends on $\vec r$ via the retarded time t_r.

$$\frac{\partial}{\partial x_i}\rho(\vec r',t_r) = \frac{\partial t_r}{\partial x_i}\frac{\partial\rho}{\partial t}\Big|_{t=t_r} = \dot\rho(\vec r',t_r)\left[\frac{\partial}{\partial x_i}\frac{-\sqrt{(\vec r-\vec r')^2}}{c} = -\frac{2(x-x')_i}{2c\sqrt{(\vec r-\vec r')^2}}\right]$$

$$= -\dot\rho(\vec r',t_r)\frac{(x-x')_i}{c|\vec r-\vec r'|} \Rightarrow \left.\begin{array}{l}\vec\nabla_r\rho(\vec r',t_r) = -\dot\rho(\vec r',t_r)\frac{\vec r-\vec r'}{c|\vec r-\vec r'|} \\ \vec\nabla t_r = -\frac{\vec r-\vec r'}{c|\vec r-\vec r'|} \qquad \nabla_r \frac{1}{|\vec r-\vec r'|} = -\frac{|\vec r-\vec r'|}{|\vec r-\vec r'|^3}\end{array}\right\}\Rightarrow$$

$$\Rightarrow \vec\nabla V = \frac{1}{4\pi\epsilon_0}\int d^3x'\left(-\frac{\dot\rho(\vec r,t_r)}{c}\frac{\vec r-\vec r'}{|\vec r-\vec r'|^2} - \frac{\rho(\vec r,t_r)\,(\vec r-\vec r')}{|\vec r-\vec r'|^3}\right)$$

Now, $$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int d^3x'\Bigg(-\frac{\dot\rho}{c}\vec\nabla_r\cdot\frac{\vec r-\vec r'}{|\vec r-\vec r'|^2} - \frac{(\vec r-\vec r')}{c(\vec r-\vec r')^2}\cdot\overbrace{\vec\nabla_r\dot\rho(\vec r',t_r)}^{-\ddot\rho\frac{\vec r-\vec r'}{|\vec r-\vec r'|c}} -$$

$$- \rho(\vec r',t_r)\,\vec\nabla_r\cdot\frac{(\vec r-\vec r')}{|\vec r-\vec r'|^3} - \frac{(\vec r-\vec r')}{|\vec r-\vec r'|^3}\underbrace{\nabla_r\rho(\vec r,t_r)}_{-\dot\rho\frac{(\vec r-\vec r')}{|\vec r-\vec r'|c}}\Bigg)$$

$$\nabla_r\frac{\vec r-\vec r'}{(\vec r-\vec r')^2} = \frac{1}{(\vec r-\vec r')^2}$$

$$\nabla_r\frac{(\vec r-\vec r')}{|\vec r-\vec r'|^3} = 4\pi\delta^3(\vec r-\vec r')$$

$$\Rightarrow \nabla^2 V = \frac{1}{4\pi\epsilon_0}\int d^3x'\left(-\cancel{\frac{\dot\rho}{c}\frac{1}{(\vec r-\vec r')^2}} + \frac{\ddot\rho}{c^2}\frac{1}{|\vec r-\vec r'|} - \rho(\vec r',t_r)\,4\pi\delta^3(\vec r-\vec r') +\right.$$

$$\left. + \cancel{\frac{\dot\rho}{c}\frac{1}{(\vec r-\vec r')^2}}\right) = \frac{1}{4\pi\epsilon_0}\int d^3x'\left(\frac{1}{c^2}\frac{\frac{\partial^2}{\partial t^2}\rho(\vec r',t-\frac{|\vec r-\vec r'|}{c})}{|\vec r-\vec r'|}\right) - \frac{\rho(\vec r,t)}{\epsilon_0} =$$

$$= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underbrace{\left(\frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\rho(\vec r',t_r)}{|\vec r-\vec r'|}\right)}_{V(r,t)} - \frac{\rho(\vec r,t)}{\epsilon_0} = \frac{\partial^2}{c^2\partial t^2}V(r,t) - \frac{\rho(\vec r,t)}{\epsilon_0}\Rightarrow$$

$$\Rightarrow \nabla^2 V(\vec r,t) - \frac{1}{c^2}\frac{\partial^2 V(\vec r,t)}{\partial t^2} = -\frac{\rho(\vec r,t)}{\epsilon_0}$$ q.e.d.

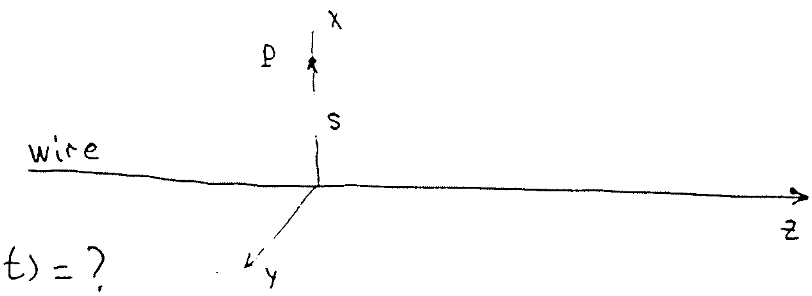
2. For $\vec A(\vec r,t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\vec J(\vec r',t_r)}{|\vec r-\vec r'|}$ – same story

E.g. $A_1(\vec r,t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{J_1(\vec r',t_r)}{|\vec r-\vec r'|}$ Replacing $\frac{1}{\epsilon_0}\rho(\vec r',t_r)$ by $\mu_0\vec J_1(t,r)$, we get $\nabla^2A_1 - \frac{1}{c^2}\frac{\partial^2A_1}{\partial t^2} = -\mu_0 J_1(\vec r,t)$

Similarly, $\nabla^2 A_2(\vec r,t) - \frac{1}{c^2}\frac{\partial^2 A_2}{\partial t^2} = -\mu_0 J_2$ and $\nabla^2 A_3(\vec r,t) - \frac{1}{c^2}\frac{\partial^2 A_3}{\partial t^2} = -\mu_0 J_3$

$$\Rightarrow \nabla^2\vec A(\vec r,t) - \frac{1}{c^2}\frac{\partial^2\vec A(\vec r,t)}{\partial t^2} = -\mu_0\vec J(\vec r,t)$$

Example 10.2



$I(t) = I_0\,\theta(t)$
(someone turns on the current I_0 at time $t = \emptyset$)

$\vec{E}(s,t) = ?$
$\vec{B}(s,t) = ?$

Since $\rho(\vec{r},t) = \emptyset \Rightarrow V(\vec{r},t) = 0$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\hat{e}_3 \int dz \frac{I(t_r)}{|\vec{r}-z\hat{e}_3|} = \frac{\mu_0}{4\pi}\hat{e}_3 \int dz \frac{I_0\,\theta(t_r)}{|\vec{r}-z\hat{e}_3|}$$

For $\vec{r} = s\hat{e}_1$ we get $|\vec{r}-z\hat{e}_3| = \sqrt{s^2+z^2}$ $\quad t_r = t - \frac{\sqrt{s^2+z^2}}{c}$

$$\Rightarrow \vec{A}(s,t) = \frac{\mu_0}{4\pi}\hat{e}_3 I_0 \int_{-\infty}^{\infty} dz \frac{1}{\sqrt{s^2+z^2}}\,\theta\left(t - \frac{\sqrt{s^2+z^2}}{c}\right) = \frac{\mu_0}{4\pi}\hat{e}_3 I_0 \int_{-\sqrt{c^2t^2-s^2}}^{\sqrt{c^2t^2-s^2}} dz \frac{\theta(ct-s)}{\sqrt{s^2+z^2}}$$

$$= \frac{\mu_0 I_0}{2\pi}\hat{e}_3 \int_0^{\sqrt{c^2t^2-s^2}} dz \frac{\theta(ct-s)}{\sqrt{s^2+z^2}} = \frac{\mu_0 I_0}{2\pi} e_3 \ln\left(z+\sqrt{s^2+z^2}\right)\Big|_{\emptyset}^{\sqrt{c^2t^2-s^2}} \theta(ct-s)$$

$$= \frac{\mu_0 I_0}{2\pi}\hat{e}_3 \ln\frac{ct+\sqrt{c^2t^2-s^2}}{s}\,\theta(ct-s)$$

The electric field is

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} = -\frac{\mu_0 I_0 c}{2\pi}\frac{\hat{e}_3\,\theta(t-s/c)}{\sqrt{c^2t^2-s^2}}$$

The magnetic field

$\vec{B} = \vec{\nabla}\times\vec{A}$ $\qquad$ In cylindrical coordinates $A_s = A_\varphi = \emptyset$

$$A_3 = \frac{\mu_0 I_0}{2\pi}\theta(ct-s)\ln\frac{ct+\sqrt{c^2t^2-s^2}}{s}$$

$$\Rightarrow \vec{\nabla}\times\vec{A} = \cancel{\hat{s}\frac{1}{s}\frac{\partial A_3}{\partial\varphi}} - \hat{\varphi}\frac{\partial A_3}{\partial s} = -\hat{\varphi}\frac{\mu_0 I_0}{2\pi}\theta(ct-s)\left(-\frac{1}{s}\frac{ct}{\sqrt{c^2t^2-s^2}}\right) \Rightarrow$$

$$\vec{B}(s,t) = \frac{\mu_0 I_0}{2\pi s}\hat{\varphi}\frac{ct}{\sqrt{c^2t^2-s^2}}\,\theta\left(t-\frac{s}{c}\right)$$

Note that at $t\to\infty$ $B(s) \to \frac{\mu_0 I}{2\pi s}\hat{\varphi}$ as in magnetostatics

Pr. 10.8.

Show that the retarded potentials

$$V(\vec r,t)=\frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\rho(\vec r',t_r)}{|\vec r-\vec r'|}\qquad \vec A(\vec r,t)=\frac{\mu_0}{4\pi}\int d^3x'\,\frac{\vec J(\vec r',t_r)}{|\vec r-\vec r'|}$$

satisfy Lorentz gauge condition

$$\vec\nabla\cdot\vec A=-\mu_0\epsilon_0\frac{\partial V}{\partial t}$$

$$\vec\nabla_r\frac{1}{|\vec r-\vec r'|}=-\vec\nabla_{r'}\frac{1}{|\vec r-\vec r'|}$$

Proof:

$$\vec\nabla\cdot\vec A(\vec r,t)=\frac{\mu_0}{4\pi}\int d^3x'\left\{\vec J(\vec r',t_r)\cdot\vec\nabla\frac{1}{|\vec r-\vec r'|}+\frac{1}{|\vec r-\vec r'|}\vec\nabla\cdot\vec J(\vec r',t_r)\right\}=\frac{\mu_0}{4\pi}\int d^3x'\cdot$$

$$\cdot\left\{-\vec J(\vec r',t_r)\cdot\vec\nabla_{r'}\frac{1}{|\vec r-\vec r'|}+\frac{1}{|\vec r-\vec r'|}\vec\nabla\cdot\vec J(\vec r',t_r)\right\}=\text{by parts}=\frac{\mu_0}{4\pi}\int d^3x'\left\{\frac{1}{|\vec r-\vec r'|}\times\right.$$

$$\left.\times\vec\nabla_{r'}\cdot\vec J(\vec r',t_r)+\frac{1}{|\vec r-\vec r'|}\vec\nabla\cdot\vec J(\vec r',t_r)\right\}=\frac{\mu_0}{4\pi}\int d^3x'\,\frac{1}{|\vec r-\vec r'|}\left\{\vec\nabla_{r'}\cdot\vec J(\vec r',\tilde t)\Big|_{\tilde t=t-\frac{|\vec r-\vec r'|}{c}}\right.$$

$$\left.-\dot{\vec J}(\vec r',t_r)\cdot\vec\nabla_{r'}\frac{|\vec r-\vec r'|}{c}-\dot{\vec J}(\vec r',t_r)\cdot\vec\nabla_r\frac{|\vec r-\vec r'|}{c}\right\}=\frac{\mu_0}{4\pi}\int d^3x'\,\frac{-1}{|\vec r-\vec r'|}\,\frac{\partial\rho(\vec r',\tilde t)}{\partial\tilde t}\Big|_{\tilde t=t_r}$$

$$=\frac{\mu_0}{4\pi}\frac{d}{dt}\int d^3x'\,\frac{-1}{|\vec r-\vec r'|}\rho(\vec r',t_r)=-\mu_0\epsilon_0\frac{d}{dt}\frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\rho(\vec r',t_r)}{|\vec r-\vec r'|}=-\mu_0\epsilon_0\frac{\partial V(\vec r,t)}{\partial t}\quad \text{q.e.d.}$$

Jefimenko's eqns.

$$\vec\nabla t_r=-\frac{\vec r-\vec r'}{c|\vec r-\vec r'|}$$

1. $\vec E=-\vec\nabla V-\frac{\partial\vec A}{\partial t}$

$$\left.\begin{aligned}\vec\nabla V&=-\frac{1}{4\pi\epsilon_0}\int d^3x'\left(\rho(\vec r',t_r)\frac{(\vec r-\vec r')}{|\vec r-\vec r'|^3}+\frac{\dot\rho(\vec r',t_r)}{c}\frac{(\vec r-\vec r')}{|\vec r-\vec r'|^2}\right)\\ \frac{\partial\vec A}{\partial t}&=\frac{\mu_0}{4\pi}\int d^3x'\,\frac{\dot{\vec J}(\vec r',t_r)}{|\vec r-\vec r'|}\end{aligned}\right\}=$$

$$\Rightarrow\vec E(\vec r,t)=\frac{1}{4\pi\epsilon_0}\int d^3x'\left\{\rho(\vec r',t_r)\frac{(\vec r-\vec r')}{|\vec r-\vec r'|^3}+\frac{\dot\rho(\vec r',t_r)}{c}\frac{(\vec r-\vec r')}{|\vec r-\vec r'|^2}-\frac{1}{c^2}\frac{\dot{\vec J}(\vec r',t_r)}{|\vec r-\vec r'|}\right\}$$

2. $\vec B=\vec\nabla\times\vec A=\frac{\mu_0}{4\pi}\int d^3x'\,\vec\nabla\times\frac{\vec J(\vec r',t_r)}{|\vec r-\vec r'|}=$ $\Big[\ \vec\nabla\times(f\vec A)=f(\vec\nabla\times\vec A)-\vec A\times\vec\nabla f$

$$=\frac{\mu_0}{4\pi}\int d^3x'\left\{\frac{1}{|\vec r-\vec r'|}\vec\nabla\times\vec J(\vec r',t_r)-\vec J(\vec r',t_r)\times\vec\nabla\frac{1}{|\vec r-\vec r'|}\right\}=\frac{\mu_0}{4\pi}\int d^3x'\left\{\frac{\vec\nabla\times\vec J}{|\vec r-\vec r'|}+\frac{\vec J\times(\vec r-\vec r')}{|\vec r-\vec r'|^3}\right.$$

$$\left(\vec\nabla\times\vec J(\vec r',t_r)\right)_1=\frac{\partial}{\partial x_2}J_3\!\left(\vec r',t-\frac{|\vec r-\vec r'|}{c}\right)-\frac{\partial}{\partial x_3}J_2\!\left(\vec r',t-\frac{|\vec r-\vec r'|}{c}\right)=$$

$$-\dot J_3(\vec r',t_r)\frac{1}{c}\frac{\partial}{\partial x_2}|\vec r-\vec r'|+\dot J_2(\vec r',t_r)\frac{1}{c}\frac{\partial}{\partial x_3}|\vec r-\vec r'|=-\dot J_3(\vec r',t_r)\frac{(x_2-x_2')}{c|\vec r-\vec r'|}+\dot J_2(\vec r',t_r)\frac{x_3-x_3'}{c|\vec r-\vec r'|}$$

$$=\left(\dot{\vec J}(\vec r',t_r)\times\frac{(\vec r-\vec r')}{c|\vec r-\vec r'|^3}\right)_1\Rightarrow\ \vec\nabla\times\vec J(\vec r',t_r)=\frac{1}{c}\dot{\vec J}(\vec r',t_r)\times\frac{(\vec r-\vec r')}{|\vec r-\vec r'|^3}\Rightarrow$$

$$\Rightarrow\vec B(\vec r,t)=\frac{\mu_0}{4\pi}\int d^3x'\left(\frac{\vec J(\vec r',t_r)}{|\vec r-\vec r'|^3}+\frac{\dot{\vec J}(\vec r',t_r)}{c|\vec r-\vec r'|^2}\right)\times(\vec r-\vec r')$$

Pr. 10.12. The accuracy of the quasistatic approximation.

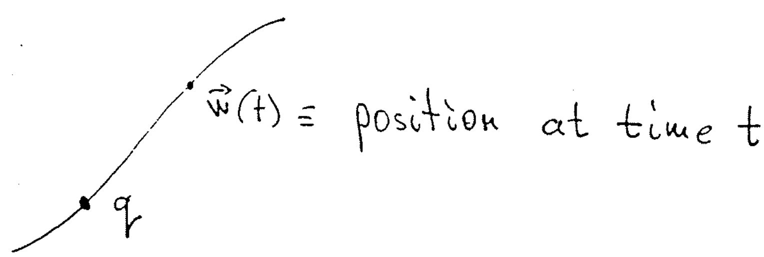
For the slowly varying current density

$$\vec{J}(r', t_r) = \vec{J}\left(r', t - \frac{|r-r'|}{c}\right) \simeq J(\vec{r}', t) - \frac{|\vec{r}-\vec{r}'|}{c}\dot{\vec{J}}(\vec{r}', t) + O\left(\frac{1}{c^2}\right)$$

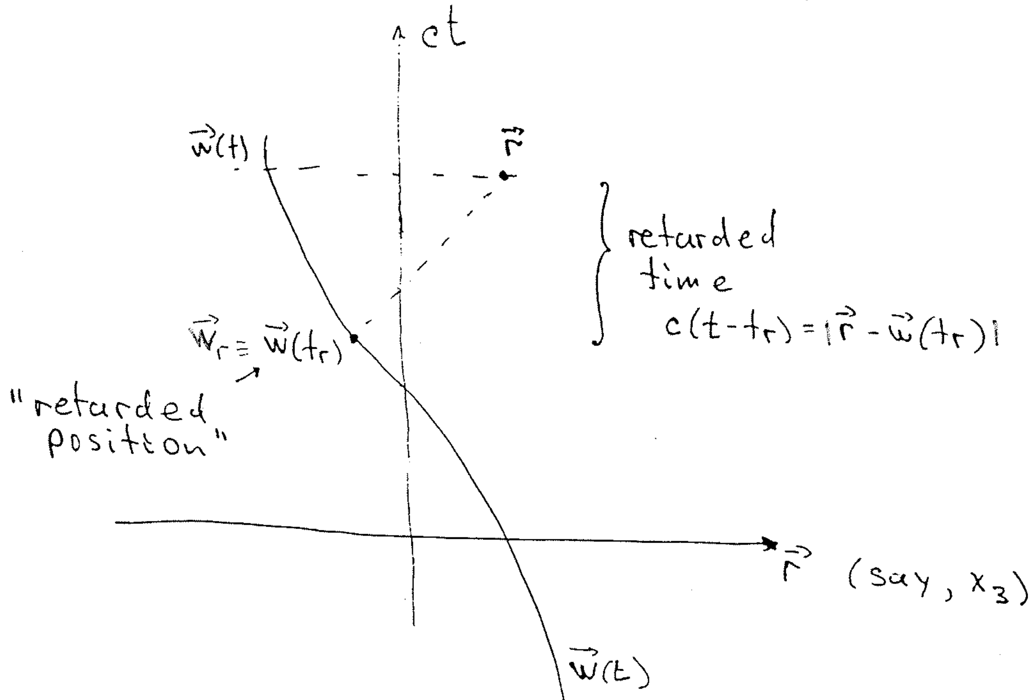
$$\vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi}\int d^3x' \left\{ \frac{\vec{J}(\vec{r}', t_r)}{|\vec{r}-\vec{r}'|^3} + \frac{\dot{\vec{J}}(\vec{r}', t_r)}{c|\vec{r}-\vec{r}'|^2} \right\} \times (\vec{r}-\vec{r}') \simeq \frac{\mu_0}{4\pi}\int d^3x' \left\{ \frac{\vec{J}(\vec{r}', t)}{|\vec{r}-\vec{r}'|^3} - \frac{|\vec{r}-\vec{r}'|}{c}\frac{\dot{\vec{J}}(\vec{r}', t)}{|\vec{r}-\vec{r}'|^3} + \frac{1}{c}\frac{\dot{\vec{J}}(\vec{r}', t)}{|\vec{r}-\vec{r}'|^2} + O\left(\frac{1}{c^2}\right) \right\} \times (\vec{r}-\vec{r}') = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{r}', t) \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$\Rightarrow$ accuracy of the quasistatic approximation is $\frac{v^2}{c^2}$ rather than $\frac{v}{c}$.

Lienard-Wiechert potentials



Retarded time and "retarded position"



Speed of charge < speed of light $\Rightarrow$ only one retarded point contributes to the potentials at any given moment.

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{1}{|\vec{r}-\vec{r}'|}\rho(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c})$$

Formally,

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\, dt' \frac{1}{|\vec{r}-\vec{r}'|}\rho(\vec{r}',t')\,\delta(t'-t+\frac{|\vec{r}-\vec{r}'|}{c})$$

For a point charge, $\rho(\vec{r},t) = q\,\delta^{(3)}(\vec{r}-\vec{w}(t))$

$$\Rightarrow V(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\int d^3x'\, dt'\, \delta^{(3)}(\vec{r}'-\vec{w}(t'))\,\delta(t'-t-\frac{|\vec{r}-\vec{r}'|}{c})\frac{1}{|\vec{r}-\vec{r}'|} =$$

$$= \left[\text{perform } \int dx' \text{ using } \delta^3(\vec{r}'-\vec{w}(t'))\right] = \frac{q}{4\pi\epsilon_0}\int dt' \frac{\delta(t'-t-\frac{|\vec{r}-\vec{w}(t')|}{c})}{|\vec{r}-\vec{w}(t')|}$$

$$\delta(F(t')) = \frac{\delta(t'-t'_*)}{F'(t_*)} \quad \text{where} \quad F(t_*) = 0 \Rightarrow$$

$$\Rightarrow \int dt' \frac{\delta(t'-t-\frac{|\vec{r}-\vec{w}(t')|}{c})}{|\vec{r}-w(t')|} = \int dt' \frac{\delta(t'-t'_*)}{\frac{\partial}{\partial t'}(t'-t+\frac{|\vec{r}-\vec{w}(t')|}{c})\Big|_{t'=t_*}|\vec{r}-\vec{w}(t')|}$$

$$= \int dt' \frac{\delta(t'-t'_*)}{\left|1 - \frac{\dot{\vec{w}}(t')\cdot(\vec{r}-\vec{w}(t'))}{c|\vec{r}-\vec{w}(t')|}\right| |\vec{r}-\vec{w}(t')|} = c\int dt' \frac{\delta(t'-t_*)}{c|\vec{r}-\vec{w}(t')| - \dot{\vec{w}}(t')\cdot(\vec{r}-\vec{w}(t'))}$$

$$= \frac{c}{c|\vec{r}-\vec{w}(t_*)| - \vec{v}(t_*)\cdot(\vec{r}-\vec{w}(t_*))} \quad \text{where} \quad t_* = t - \frac{|\vec{r}-\vec{w}(t_*)|}{c} \Rightarrow t_* \equiv t_r$$

$\vec{v}(\vec{r},t) \equiv \dot{\vec{w}}(\vec{r},t)$ velocity

$$\Rightarrow V(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{c}{c|\vec{r}-\vec{w}(t_r)| - \vec{v}(t_r)\cdot(\vec{r}-\vec{w}(t_r))}$$

Similarly

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{1}{|\vec{r}-\vec{r}'|}\vec{J}(\vec{r}',t-\frac{|\vec{r}-\vec{r}'|}{c}) = \frac{\mu_0}{4\pi}\int d^3x'\,dt' \frac{\vec{J}(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\delta(t'-t+\frac{|\vec{r}-\vec{r}'|}{c})$$

For a rigid object

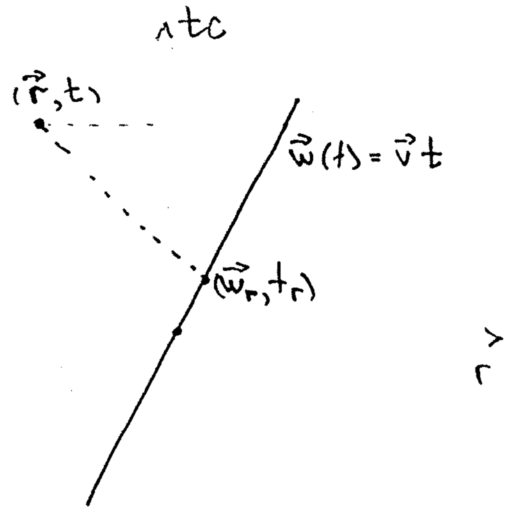
$\vec{J}(\vec{r},t) = \rho(r,t)\vec{v}(\vec{r},t) \Rightarrow$ for a point charge $\vec{J}(\vec{r},t) = q\,\vec{v}(t)\,\delta^{(3)}(\vec{r}-\vec{w}(t))$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3x'\,dt'\, q\,\vec{v}(t')\frac{\delta^3(\vec{r}'-\vec{w}(t'))}{|\vec{r}-\vec{w}(t')|}\delta(t'-t+\frac{|\vec{r}-\vec{r}'|}{c}) =$$

$$= \frac{\mu_0 q}{4\pi}\int dt'\,\vec{v}(t')\frac{\delta(t'-t+\frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{w}(t')|} = \frac{\mu_0 q\,\vec{v}(t_r)}{4\pi}\frac{c}{c|\vec{r}-\vec{w}(t_r)| - \vec{v}(t_r)\cdot(\vec{r}-w(t_r))}$$

$$\left.\begin{aligned} V(\vec{r},t) &= \frac{qc}{4\pi\epsilon_0(c|\vec{r}-\vec{w}_r| - \vec{v}_r\cdot(\vec{r}-\vec{w}_r))} \\ A(\vec{r},t) &= \frac{\vec{v}}{c^2}V(\vec{r},t) \\ \vec{w}_r \equiv \vec{w}(t_r)&, \quad \vec{v}_r \equiv \vec{v}(t_r) \quad c(t-t_r) = |\vec{r}-\vec{w}_r| \end{aligned}\right\} \text{Lienard-Wiechert potentials}$$

Example: potentials of a point charge moving with constant velocity



$$c(t-t_r) = |\vec r - \vec w_r| = |\vec r - \vec v t_r| \Rightarrow$$

$$\Rightarrow t_r = \frac{t - \vec r\cdot\vec v \pm \sqrt{(c^2t - \vec r\cdot\vec v)^2 - (c^2-v^2)(c^2t^2-r^2)}}{c^2-v^2}$$

For the retarded time, we need (-) sign (+ sign leads to an "advanced time")

Check: at $v \to 0$ $\quad t_r = t - \frac{r}{c}$

$$\Rightarrow \quad t_r = \frac{c^2t - \vec r\cdot\vec v - \sqrt{(c^2t-\vec r\cdot\vec v)^2 - (c^2-v^2)(c^2t^2-r^2)}}{c^2-v^2}$$

Now,

$$V(\vec r, t) = \frac{qc}{4\pi\epsilon_0}\,\frac{1}{c|\vec r - \vec w_r| - \vec v\cdot(\vec r - \vec w_r)} = \frac{qc}{4\pi\epsilon_0}\,\frac{1}{c^2(t-t_r) - \vec v\cdot(\vec r - \vec v t_r)}$$

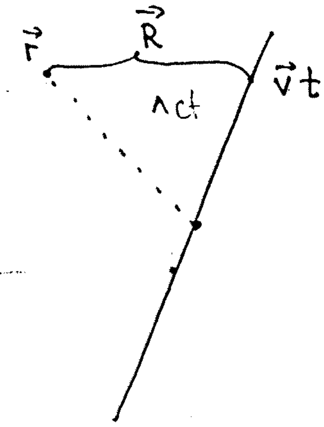
$$= \frac{qc}{4\pi\epsilon_0}\left(c^2t - (c^2-v^2)t_r - \vec v\cdot\vec r\right)^{-1} = \frac{1}{4\pi\epsilon_0}\,\frac{qc}{\sqrt{(c^2t-\vec v\cdot\vec r)^2 - (c^2-v^2)(c^2t^2-r^2)}}$$

Consequently,

$$\vec A(\vec r, t) = \frac{\vec v}{c^2}V(r,t) = \frac{\mu_0}{4\pi}\,\frac{qc\vec v}{\sqrt{(c^2t-\vec v\cdot\vec r)^2 - (c^2-v^2)(c^2t^2-r^2)}}$$

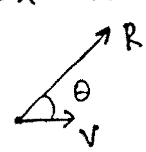
Problem 10.14: show that

$$V(\vec r, t) = \frac{q}{4\pi\epsilon_0 R}\,\frac{1}{\sqrt{1 - \frac{v^2}{c^2}\sin^2\theta}}$$



$\vec R = \vec r - \vec v t$

θ - angle between $\vec R$ and $\vec v$ (in 3d space)



Proof

$$(c^2t-\vec{v}\cdot\vec{r})^2-(c^2-v^2)(c^2t^2-r^2) = (c^2t-\vec{v}\cdot(\vec{R}+\vec{v}t))^2-(c^2-v^2)(c^2t^2-(\vec{R}+\vec{v}t)^2$$

$$=((c^2-v^2)t-\vec{v}\cdot\vec{R})^2-(c^2-v^2)((c^2-v^2)t^2-2\vec{v}\cdot\vec{R}t-\vec{R}^2) = (c^2-v^2)R^2+(\vec{v}\cdot\vec{R})^2$$

$$\Rightarrow \sqrt{(c^2t-\vec{v}\cdot\vec{r})^2-(c^2-v^2)(c^2t^2-r^2)} = \sqrt{c^2R^2-v^2R^2-(\vec{v}\cdot\vec{R})^2} = \sqrt{c^2R^2-v^2R^2\sin^2\theta}$$

$$\Rightarrow V(\vec{r},t) = \frac{qc}{4\pi\epsilon_0}\frac{1}{\sqrt{c^2R^2-v^2R^2\sin^2\theta}} = \frac{qc}{4\pi\epsilon_0 R}\frac{1}{\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}} \rightarrow$$

- Lorentz-contracted Coulomb potential

The fields of a moving point charge

$$V(\vec{r},t) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r}-\vec{w}_r| - \frac{1}{c}\vec{v}_r\cdot(\vec{r}-\vec{w}_r)} \qquad \vec{A}(r,t) = \frac{\vec{v}_r}{c^2} V(\vec{r},t)$$

$\vec{w}_r \equiv w(t_r)$ $\quad \vec{v}_r \equiv \vec{v}(t_r)$ $\quad |\vec{r}-\vec{w}_r| = c(t-t_r)$ $\quad t_r \equiv$ retarded time

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$

1. $\nabla V = ?$ $\qquad V = \frac{q}{4\pi\epsilon_0}\frac{1}{\Phi(\vec{r},t)} \Rightarrow \vec{\nabla}V = -\frac{q}{4\pi\epsilon_0}\frac{\vec{\nabla}\Phi(\vec{r},t)}{(\Phi(\vec{r},t))^2}$

$\Phi(\vec{r},t) = |\vec{r}-\vec{w}_r| - \frac{1}{c}\vec{v}_r\cdot(\vec{r}-\vec{w}_r)$ = function of $\vec{r}$ and t_r.

$$\frac{\partial}{\partial x_i}\Phi(\vec{r},t) = \underbrace{\frac{\partial}{\partial x_i}|\vec{r}-\vec{w}(t_r)|}_{\frac{(\hat{e}_i - \frac{\partial \vec{w}_r}{\partial x_i})\cdot(\vec{r}-\vec{w}_r)}{|r-w_r|}} - \frac{1}{c}\underbrace{\frac{\partial \vec{v}(t_r)}{\partial x_i}}_{\vec{a}(t_r)\frac{\partial t_r}{\partial x_i}}\cdot(\vec{r}-\vec{w}(t_r)) - \frac{\vec{v}(t_r)}{c}\cdot(\hat{e}_i - \underbrace{\frac{\partial \vec{w}(t_r)}{\partial x_i}}_{\vec{v}(t_r)\frac{\partial t_r}{\partial x_i}}) =$$

$$\frac{\partial}{\partial x_i}|\vec{f}| = \frac{\vec{f}\cdot\frac{\partial \vec{f}}{\partial x_i}}{|\vec{f}|}$$

$$= (\hat{e}_i - \vec{v}_r\cdot\frac{\partial t_r}{\partial x_i})\cdot\frac{\vec{r}-\vec{w}_r}{|\vec{r}-\vec{w}_r|} - \frac{\vec{a}_r}{c}\cdot(\vec{r}-\vec{w}_r)\frac{\partial t_r}{\partial x_i} - \frac{v_i(t_r)}{c} + \frac{v_r^2}{c}\frac{\partial t_r}{\partial x_i}$$

$$\frac{\partial t_r}{\partial x_i} = ?$$

$$\frac{\partial}{\partial x_i}|\vec{r}-\vec{w}(t_r)| = \frac{(\vec{r}-\vec{w}_r)\cdot(\hat{e}_i - \frac{\partial \vec{w}(t_r)}{\partial x_i})}{|\vec{r}-\vec{w}_r|} = \frac{\vec{r}-\vec{w}_r}{|r-w_r|}(\hat{e}_i - \vec{v}_r\frac{\partial t_r}{\partial x_i}) = \frac{(r-w_r)_i}{|\vec{r}-\vec{w}_r|} - \frac{(\vec{r}-\vec{w}_r)\cdot\vec{v}_r}{|\vec{r}-\vec{w}_r|}$$

On the other hand

$$\frac{\partial}{\partial x_i}|\vec{r}-\vec{w}(t_r)| = \frac{\partial}{\partial x_i}c(t-t_r) = -c\frac{\partial t_r}{\partial x_i} \Rightarrow \frac{(r-w_r)_i}{|\vec{r}-\vec{w}_r|} - \frac{\vec{v}_r\cdot(\vec{r}-\vec{w}_r)}{|r-w_r|}\frac{\partial t_r}{\partial x_i} = -c\frac{\partial t_r}{\partial x_i}$$

$$\Rightarrow \frac{\partial t_r}{\partial x_i} = -\frac{(r-w_r)_i}{|\vec{r}-\vec{w}_r|}\frac{1}{c - \frac{\vec{v}_r\cdot(\vec{r}-w_r)}{|\vec{r}-\vec{w}_r|}} = -\frac{\mathscr{r}_i}{c-\vec{v}\cdot\mathscr{r}} \qquad \vec{\mathscr{r}} \equiv \vec{r}-\vec{w}_r \qquad \hat{\mathscr{r}} \equiv \frac{\vec{\mathscr{r}}}{|\vec{\mathscr{r}}|}$$

$$\Rightarrow \frac{\partial}{\partial x_i}\Phi(\vec{r},t) = \hat{\mathscr{r}}_i - \frac{v_{r_i}}{c} + \frac{\partial t_r}{\partial x_i}\left(-\vec{v}_r\cdot\hat{\mathscr{r}} - \frac{\vec{a}_r\cdot\vec{\mathscr{r}}}{c} + \frac{v_r^2}{c}\right) \qquad -\frac{v_{r_i}}{c} - \frac{\mathscr{r}_i}{c-\vec{v}\cdot\hat{\mathscr{r}}}\left(\frac{v}{c}\right. \left. -\vec{v}_r\cdot\hat{\mathscr{r}} - \frac{1}{c}\vec{a}_r\cdot\vec{\mathscr{r}}\right)$$

$$\Rightarrow \vec{\nabla}\Phi(\vec{r},t) = \hat{\mathscr{r}} - \frac{\vec{v}_r}{c} + \frac{\hat{\mathscr{r}}}{c-\vec{v}_r\cdot\mathscr{r}}\left(\vec{v}_r\cdot\hat{\mathscr{r}} + \frac{1}{c}\vec{a}_r\cdot\vec{\mathscr{r}} - \frac{v_r^2}{c}\right) =$$

$$= \frac{\hat{\mathscr{r}}(c - \cancel{\vec{v}_r\cdot\hat{\mathscr{r}}}) + \cancel{(\vec{v}_r\cdot\hat{\mathscr{r}})}\hat{\mathscr{r}} + \frac{1}{c}(\vec{a}_r\cdot\vec{\mathscr{r}})\hat{\mathscr{r}} - \frac{v_r^2}{c}\hat{\mathscr{r}}}{c-\vec{v}_r\cdot\hat{\mathscr{r}}} - \frac{\vec{v}_r}{c} = \frac{1}{c}\left(\frac{\hat{\mathscr{r}}(c^2-v_r^2+\vec{a}_r\cdot\vec{\mathscr{r}})}{c-\vec{v}_r\cdot\hat{\mathscr{r}}} - \vec{v}_r\right)$$

$$\Rightarrow \vec{\nabla}V(\vec{r},t) = -\frac{q}{4\pi\epsilon_0}\frac{1}{(\mathscr{r} - \frac{\vec{v}_r}{c}\cdot\vec{\mathscr{r}})^2}\frac{1}{c}\left(\frac{\vec{\mathscr{r}}(c^2-v_r^2+\vec{a}_r\cdot\vec{\mathscr{r}})}{c\mathscr{r}-\vec{v}_r\cdot} - \vec{v}_r\right) =$$

$$= \frac{qc}{4\pi\epsilon_0}\frac{1}{(c\mathscr{r}-\vec{v}_r\cdot\vec{\mathscr{r}})^3}\left(\vec{v}_r(c\mathscr{r}-\vec{v}_r\cdot\vec{\mathscr{r}}) - \vec{\mathscr{r}}(c^2-v_r^2+\vec{a}_r\cdot\vec{\mathscr{r}})\right)$$

$$\frac{\partial}{\partial t}\vec{A}(r,t) = \frac{q}{4\pi\epsilon_0 c^2}\frac{\partial}{\partial t}\vec{v}(t_r)\frac{1}{|\vec{r}-\vec{w}(t_r)| - \frac{1}{c}\vec{v}(t_r)\cdot(\vec{r}-\vec{w}(t_r))} =$$

$$= \frac{q}{4\pi\epsilon_0 c^2}\frac{\partial t_r}{\partial t}\frac{\partial}{\partial t_r}\vec{v}(t_r)\frac{1}{|\vec{r}-\vec{w}(t_r)| - \frac{\vec{v}(t_r)}{c}\cdot(\vec{r}-\vec{w}(t_r))} = \frac{q}{4\pi\epsilon_0 c^2}\frac{\partial t_r}{\partial t}\left\{\frac{\vec{a}_r}{\mathscr{r} - \frac{1}{c}\vec{v}_r\cdot\vec{\mathscr{r}}} -\right.$$

$$\left. - \vec{v}_r\frac{1}{(\mathscr{r} - \frac{1}{c}\vec{v}_r\cdot\vec{\mathscr{r}})^2}\frac{\partial}{\partial t_r}\left(|\vec{r}-\vec{w}(t_r)| - \frac{\vec{v}(t_r)}{c}\cdot(\vec{r}-\vec{w}(t_r))\right)\right\} =$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial t_r}|\vec{r}-\vec{w}(t_r)| = \frac{-\vec{v}_r\cdot(\vec{r}-\vec{w}_r)}{|r-w_r|} = -\vec{v}_r\cdot\hat{\mathscr{r}}\\
&\frac{\partial}{\partial t_r}\vec{v}(t_r)\cdot(\vec{r}-\vec{w}(t_r)) = \vec{a}_r\cdot(\vec{r}-\vec{w}(t_r)) - v^2(t_r) = \vec{a}_r\cdot\vec{\mathscr{r}} - v_r^2
\end{aligned}\right\} \begin{aligned}&\frac{\partial}{\partial t_r}\left\{|r-w_r| - \frac{\vec{v}_r}{c}\cdot(\vec{r}-\vec{w}_r)\right\} =\\ &= -\vec{v}_r\cdot\hat{\mathscr{r}} - \frac{\vec{a}_r\cdot\vec{\mathscr{r}}}{c} + \frac{v_r^2}{c}\end{aligned}$$

$$= \frac{q}{4\pi\epsilon_0 c}\frac{\partial t_r}{\partial t}\left\{\frac{\vec{a}_r}{c\mathscr{r} - \vec{v}_r\cdot\vec{\mathscr{r}}} + \frac{\vec{v}_r(\vec{v}_r\cdot\hat{\mathscr{r}}c + \vec{a}_r\cdot\vec{\mathscr{r}} - v_r^2)}{(c\mathscr{r} - \vec{v}_r\cdot\vec{\mathscr{r}})^2}\right\}$$

$$\frac{\partial t_r}{\partial t} = ? \qquad \left.\begin{aligned}&\frac{\partial}{\partial t}|\vec{r}-\vec{w}(t_r)| = \frac{\partial}{\partial t}c(t-t_r) = c\left(1 - \frac{\partial t}{\partial t_r}\right)\\ &\quad \text{''}\ -\frac{\left(\frac{\partial}{\partial t}\vec{w}(t_r)\right)\cdot(\vec{r}-\vec{w}(t_r))}{|\vec{r}-\vec{w}(t_r)|} = -\vec{v}_r\cdot\hat{\mathscr{r}}\frac{\partial t}{\partial t_r}\end{aligned}\right\} \begin{aligned}&c = (c - v_r\cdot\hat{\mathscr{r}})\frac{\partial t}{\partial t_r} \Rightarrow\\ &\Rightarrow \frac{\partial t}{\partial t_r} = \frac{c}{c - \vec{v}_r\cdot\hat{\mathscr{r}}}\end{aligned}$$

$$\Rightarrow \frac{\partial\vec{A}}{\partial t} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathscr{r}\vec{a}_r}{(c\mathscr{r} - \vec{v}_r\cdot\vec{\mathscr{r}})^2} + \frac{\mathscr{r}\vec{v}_r(c\vec{v}_r\cdot\hat{\mathscr{r}} + a_r\cdot\vec{\mathscr{r}} - v_r^2)}{(c\mathscr{r} - \vec{v}_r\cdot\vec{\mathscr{r}})^3}\right\} = \frac{qc}{4\pi\epsilon_0}\frac{1}{(c\mathscr{r} - \vec{v}_r\cdot\vec{\mathscr{r}})^3}\times$$

$$\times\left\{\left(\frac{\vec{a}_r}{c}\mathscr{r} - \vec{v}_r\right)(c\mathscr{r} - \vec{v}_r\cdot\vec{\mathscr{r}}) + \vec{v}_r\frac{\mathscr{r}}{c}(c^2 - v_r^2 + \vec{\mathscr{r}}\cdot\vec{a}_r)\right\}$$

Once more

$$\vec{\nabla}V = \frac{qc}{4\pi\epsilon_0}\frac{1}{(c\mathscr{r} - \vec{v}_r\cdot\vec{\mathscr{r}})^3}\left\{\vec{v}_r(c\mathscr{r} - \vec{v}_r\cdot\vec{\mathscr{r}}) - \vec{\mathscr{r}}(c^2 - v_r^2 + \vec{a}_r\cdot\vec{\mathscr{r}})\right\}$$

$$\Rightarrow \frac{\partial\vec{A}}{\partial t} + \vec{\nabla}V = \frac{qc}{4\pi\epsilon_0}\frac{1}{(c\mathscr{r} - \vec{v}_r\cdot\vec{\mathscr{r}})^3}\left\{\vec{a}_r\frac{\mathscr{r}}{c}(c\mathscr{r} - \vec{v}_r\cdot\vec{\mathscr{r}}) + \left(\vec{v}_r\frac{\mathscr{r}}{c} - \vec{\mathscr{r}}\right)(c^2 - v_r^2 + \vec{a}_r\cdot\vec{\mathscr{r}})\right\}$$

$$\vec{u} \equiv c\hat{\mathscr{r}} - \vec{v}_r$$

Thus,

$$\vec{E}(\vec{r},t) = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}V = \frac{qc}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left\{-\frac{\vec{a}_r}{c}(\vec{\mathscr{r}}\cdot\vec{u}) + \frac{\vec{u}}{c}(c^2 - v_r^2 + \vec{a}_r\cdot\vec{\mathscr{r}})\right\} =$$

$$= \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left\{\vec{u}(c^2 - v_r^2) + \underbrace{\vec{u}(\vec{a}_r\cdot\vec{\mathscr{r}}) - \vec{a}_r(\vec{\mathscr{r}}\cdot\vec{u})}_{\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})}\right\} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left\{\vec{u}(c^2 - v^2) + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right\}$$

Similarly (see textbook)

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A} = -\frac{q}{4\pi\epsilon_0 c}\frac{\vec{\mathscr{r}}}{(\vec{u}\cdot\vec{\mathscr{r}})^3}\times\left\{(c^2 - v_r^2)\underset{-\vec{u}}{\vec{v}_r} + (\vec{\mathscr{r}}\cdot\vec{a})\underset{-u}{\vec{v}_r} + (\vec{\mathscr{r}}\cdot\vec{u})\vec{a}\right\} =$$

$$= \frac{q}{4\pi\epsilon_0 c}\frac{\vec{\mathscr{r}}\times}{(\vec{u}\cdot\vec{\mathscr{r}})^3}\left\{\vec{u}(c^2 - v_r^2) + (\vec{\mathscr{r}}\cdot\vec{a})\vec{u} - (\vec{\mathscr{r}}\cdot\vec{u})\vec{a}\right\} = \frac{\hat{\mathscr{r}}\times\vec{E}}{c}$$

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left\{\vec{u}(c^2 - v^2) + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right\}$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}\hat{\mathscr{r}}\times\vec{E}(\vec{r},t)$$

$v \equiv v(t_r) \qquad a \equiv a(t_r)$

$\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$

$\vec{u} \equiv c\hat{\mathscr{r}} - v(t_r)$

Electric & magnetic fields of a point charge moving with constant velocity

General formula:

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}\hat{\mathscr{r}}\times\vec{E}(\vec{r},t)$$

$|\vec{r}-\vec{w}(t_r)| = c(t-t_r)$

$\vec{\mathscr{r}} = \vec{r}-\vec{w}(t_r)$

$\vec{v} \equiv \vec{v}(t_r)$

$u \equiv c\hat{\mathscr{r}} - \vec{v}$

$\vec{a} \equiv a(t_r)$

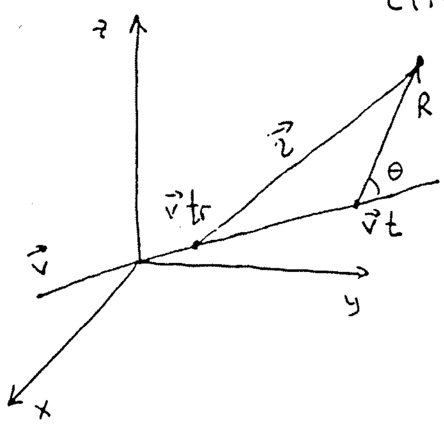
In our case, $\vec{a}=\emptyset$. Let us also take $\vec{w}=\vec{v}t$ for simplicity.

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}\vec{u}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}(c^2-v^2) = \frac{q(c^2-v^2)}{4\pi\epsilon_0}\frac{\vec{\mathscr{r}}c-\mathscr{r}\vec{v}}{(c\mathscr{r}-\vec{v}\cdot\vec{\mathscr{r}})^3}$$

$$\vec{\mathscr{r}}c-\mathscr{r}\vec{v} = (\vec{r}-\vec{v}t_r)c - \underbrace{|\vec{r}-\vec{v}t_r|}_{c(t-t_r)}\vec{v} = c\vec{r} - \cancel{ct_r\vec{v}} - ct\vec{v} + \cancel{ct_r\vec{v}} = (\vec{r}-t\vec{v})c = c\vec{R}$$

$\vec{R} \equiv \vec{r}-\vec{v}t$

distance from the present position of



$$c\mathscr{r}-\vec{v}\cdot\vec{\mathscr{r}} = \overbrace{c^2(t-t_r) - \vec{v}\cdot(\vec{R}+\vec{v}(t-t_r))}^{=(c^2-v^2)(t-t_r)-\vec{v}\cdot\vec{R}} =$$

$$c(t-t_r) = |\vec{R}+\vec{v}(t-t_r)| \Rightarrow c^2(t-t_r)^2 = (\vec{R}+\vec{v}(t-t_r))^2$$

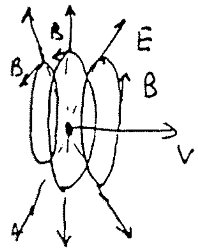
$$\Rightarrow (c^2-v^2)(t-t_r)^2 = 2\vec{v}\cdot\vec{R}(t-t_r) + R^2$$

$$(c\mathscr{r}-\vec{v}\cdot\vec{\mathscr{r}})^2 = (c^2-v^2)^2(t-t_r)^2 - 2\vec{v}\cdot\vec{R}(c^2-v^2)(t-t_r) + (\vec{v}\cdot\vec{R})^2 = (c^2-v^2)(\cancel{2\vec{v}\cdot\vec{R}(t-t_r)}+R^2) -$$
$$-2\vec{v}\cdot\vec{R}\cancel{(c^2-v^2)(t-t_r)} + (\vec{v}\cdot\vec{R})^2 = R^2(c^2-v^2) + \vec{v}\cdot\vec{R}^2 = R^2c^2 - R^2v^2\sin^2\theta$$

$$\Rightarrow E(\vec{r},t) = \frac{q(c^2-v^2)}{4\pi\epsilon_0}\frac{c\vec{R}}{(R^2c^2-R^2v^2\sin^2\theta)^{3/2}} = \frac{q}{4\pi\epsilon_0}\frac{1-v^2/c^2}{\sqrt{1-v^2/c^2\sin^2\theta}}\frac{\vec{R}}{R^3}$$

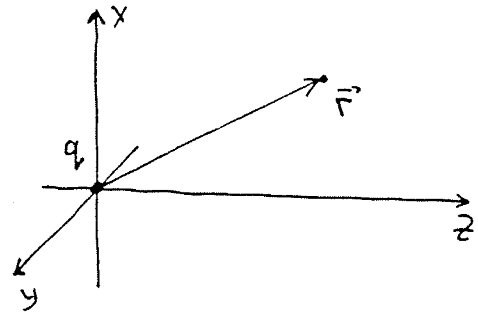
$$\vec{B}(\vec{r},t) = \frac{1}{c\mathscr{r}}\vec{\mathscr{r}}\times\vec{E}(\vec{r},t) = \frac{1}{c\mathscr{r}}(\underset{\vec{R}\times\vec{E}=\emptyset}{\cancel{\vec{R}}}+\vec{v}(t-t_r))\times\vec{E}(\vec{r},t) = \frac{1}{c^2}\vec{v}\times\vec{E}(\vec{r},t)$$

Cartoon



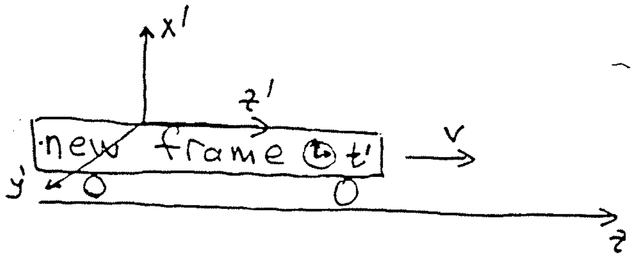
Note that $\vec{E}\uparrow\uparrow R$ which is surprising since the "signal" comes from $\vec{v}t_r$ rather than $\vec{v}t$.

Electric field of a moving charge = Lorentz contraction of the Coulomb field



$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} \quad \text{- Coulomb field}$$

Let us get into a new frame moving with speed v to the right ($\vec{v} = v e_3$)



In the new frame $\quad (\gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}})$

$$\left.\begin{aligned} x' &= x \\ y' &= y \\ z' &= \gamma(z - vt) \\ t' &= \gamma(t - \tfrac{v}{c}z) \end{aligned}\right\} \quad \begin{aligned} z &= \gamma(z' + vt') \\ t &= \gamma(t' + \tfrac{v}{c}z') \end{aligned} \qquad (*) \text{ Lorentz transformations}$$

In addition (to be demonstrated later)

$E'_{\parallel} = E_{\parallel}$ $\quad \vec{E}' \equiv$ electric field in the new frame

$E'_{\perp} = \gamma E_{\perp}$ $\quad$ (component of the electric field orthogonal to $\vec{v}$ is enhanced by $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$)

In our case

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3}{r^3} = \underbrace{\frac{q}{4\pi\epsilon_0}\frac{x\hat{e}_1 + y\hat{e}_2}{r^3}}_{E_\perp} + \underbrace{\frac{q}{4\pi\epsilon_0}\frac{z\hat{e}_3}{r^3}}_{E_\parallel} \Rightarrow$$

$$\Rightarrow \vec{E}' = \gamma\frac{q}{4\pi\epsilon_0}\frac{x\hat{e}_1 + y\hat{e}_2}{r^3} + \frac{q}{4\pi\epsilon_0}\frac{z\hat{e}_3}{r^3}$$

This is $\vec{E}'(\vec{r}, t)$ and we need $\vec{E}'(\vec{r}', t')$ $\Rightarrow$ use (*) to express r in terms of $\vec{r}'$

$$\vec{E}' = \frac{q}{4\pi\epsilon_0}\gamma\,\frac{x'\hat{e}_1 + y'\hat{e}_2}{r^3} + \frac{q}{4\pi\epsilon_0}\,\frac{\gamma(z'+vt')\hat{e}_3}{r^3} = \frac{q\gamma}{4\pi\epsilon_0 r^3}\big(\underbrace{x'\hat{e}_1+y'\hat{e}_2+z'\hat{e}_3}_{\vec{r}'} + vt'\hat{e}_3\big)$$

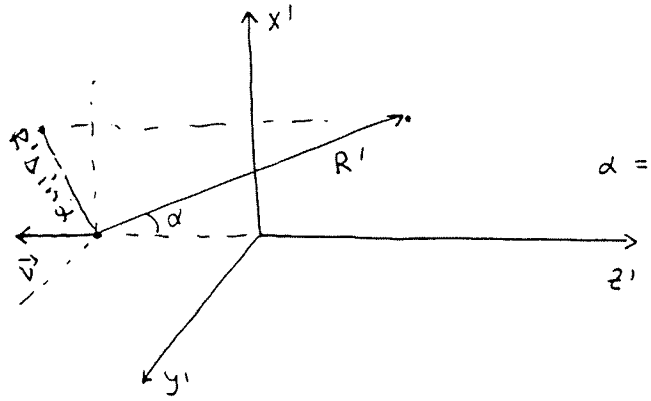
$$= \frac{q\gamma}{4\pi\epsilon_0}\,\frac{\vec{r}' + vt'\hat{e}_3}{r^3}$$

In the new frame, the charge q is moving with speed v to the left $\Rightarrow$ $\vec{R}' = \vec{r}' + vt'\hat{e}_3 = \vec{r}' - (-v\hat{e}_3)t'$ is the distance from the observation point to the position of the particle

$$\vec{E}(\vec{r}', t) = \frac{q\gamma}{4\pi\epsilon_0}\cdot\frac{\vec{R}'}{r^3}$$

Now, we should express $r^2 = x^2+y^2+z^2$ in terms of x', y', z', t'

$$r^2 = x'^2 + y'^2 + \gamma^2(z'+vt')^2 = \gamma^2\left((x'^2+y'^2)(1-\tfrac{v^2}{c^2}) + (z'+vt')^2\right) = \gamma^2\big(x'^2+y'^2+(z'+vt')^2 - \tfrac{v^2}{c^2}(x'^2+y'^2)\big) = \gamma^2\left(R'^2 - \tfrac{v^2}{c^2}R'^2\sin^2\theta\right)$$

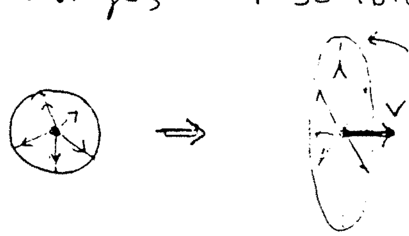


$\alpha = \pi - \theta \Rightarrow x'^2 + y'^2 = R'^2\sin^2\alpha =$
$= R'^2\sin^2\theta$

$$\Rightarrow \vec{E}'(\vec{r}', t') = \frac{q\gamma}{4\pi\epsilon_0}\,\frac{\vec{R}'}{\gamma^3 R'^3}\,\frac{1}{\left(\sqrt{1-v^2/c^2\sin^2\theta}\right)^3} = \frac{q}{4\pi\epsilon_0}\,\frac{\vec{R}'}{R'^3}\,\frac{1-v^2/c^2}{(1-v^2/c^2\sin^2\theta)^{3/2}}$$

$\vec{E}(\vec{r})$ was $\uparrow\uparrow \vec{r}$, $\vec{E}'(\vec{R}', t')$ is also $\uparrow\uparrow R'$: $E_\parallel$ (E_z) gets an extra factor γ from the transformation of coordinates whereas $E_\perp$ (E and E_y) pick up their factors γ from the transformation of the field.

The electric (and magnetic) field of a rapidly moving charges resembles a pancake



$\theta = \frac{\pi}{2}$ $\quad E' = \dfrac{E}{\sqrt{1-v^2/c^2}}$ - enhanced

$\theta = 0$ or π $\quad E' = E(1-\frac{v^2}{c^2})$ - reduced

Problem 10.26

Force exerted on q_2 by q_1:

$$\vec{F}_{12} = (\vec{E} + \vec{v} \times \vec{B})$$

$$\left.\begin{aligned} \vec{E} &= \frac{q_1}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} = \frac{q_1}{4\pi\epsilon_0}\frac{z\hat{e}_3}{z^3} \\ \vec{B} &= \phi \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \vec{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{z\hat{e}_3}{z^3} = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{\hat{e}_3}{v^2t^2}$$

Force exerted on q_1 by q_2

$$\vec{F}_{21} = q_1(\vec{E} + \underbrace{\cancel{\vec{v}\times\vec{B}}}_{=\phi}) \qquad \vec{E} = \frac{q_2}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}\frac{(\overset{\phi}{\vec{r}} - \overset{v\hat{e}_3 t}{\vec{v}t})}{|\vec{r} - \vec{v}t|^3} \Rightarrow \qquad \theta = \pi$$

$$\Rightarrow \vec{F}_{21} = -\frac{q_1 q_2}{4\pi\epsilon_0}\frac{e_3}{v^2t^2}\left(1 - \frac{v^2}{c^2}\right)$$

$\vec{F}_{21} \neq -\vec{F}_{12}$! What happens?

Let us compute the momentum carried by electromagnetic fields

$$\vec{P}_{e.m.} = \mu_0\epsilon_0 \underset{\text{all space}}{\int} d^3x\, \vec{S}(\vec{r},t) = \epsilon_0 \int d^3x\, \vec{E}\times\vec{B}$$

$$\vec{E}(\vec{r},t) = \vec{E}_{(1)}(\vec{r}) + \vec{E}_{(2)}(\vec{r},t) = \frac{q_1}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} + \frac{q_2}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}\frac{\vec{r} - vt\hat{e}_3}{|\vec{r} - vt\hat{e}_3|^3}$$

$$\vec{B}(\vec{r},t) = \underbrace{\cancel{B_{(1)}(\vec{r})}}_{=\phi} + \vec{B}_{(2)}(\vec{r},t) = \frac{1}{c^2}\vec{v}\times\vec{E}_{(2)}(\vec{r},t) = \frac{v\hat{e}_3}{c^2}\times\vec{E}_{(2)}(\vec{r},t)$$

$$\vec{E}\times\vec{B}(\vec{r},t) = \frac{q_1}{4\pi\epsilon_0}\frac{1}{c^2r^3}\vec{r}\times(\vec{v}\times\vec{E}_2) + \vec{E}_{(2)}\times(\vec{v}\times\vec{E}_{(2)})\frac{1}{c^2}$$

$\int d^3x\; E_2(\vec{r},t)\times(\vec{v}\times E_2(\vec{r},t))$ does not depend on time $\Rightarrow$ disregard

Indeed, $E_2(x,y,z,t) = \frac{q_2}{4\pi\epsilon_0}\frac{(1 - v^2/c^2)(x\hat{e}_1 + y\hat{e}_2 + (z-vt)\hat{e}_3)}{\left((x^2+y^2+(z-vt)^2 - \frac{v^2}{c^2}(x^2+y^2)\right)^{3/2}} = \vec{F}(x,y,z-vt)$

$\int dx\,dy\,dz\; \vec{F}(x,y,z-vt)\times(\vec{v}\times\vec{F}(x,y,z-vt)) =$ shift $z \to z+vt$ $=$

$= \int dx\,dy\,dz\; \vec{F}(x,y,z)\times(\vec{v}\times\vec{F}(x,y,z)) =$ does not depend on t

The time-dependent part is

$$\vec P_{e.m.}(t) = \frac{q_1 \cancel{\epsilon_0}}{4\pi\cancel{\epsilon_0} c^2}\int d^3x \, \frac{1}{r^3}\vec r\times(\vec v\times \vec E_{(2)}) = \frac{q_1 q_2 v\epsilon_0}{(4\pi\epsilon_0 c)^2}\left(1-\frac{v^2}{c^2}\right)\int d^3x \, \frac{\vec r}{r^3}\times\left(\hat e_3 \times \frac{\vec r - \cancel{v\hat e_3 t}}{\left(x^2+y^2+(z-vt)^2-\frac{v^2}{c^2}(x^2+y^2)\right)^{3/2}}\right) = \frac{q_1q_2 v}{(4\pi\epsilon_0 c)^2}\left(1-\frac{v^2}{c^2}\right)\int d^3x \, \frac{\hat e_3 r^2 - \vec r z}{(x^2+y^2+\dots)^{3/2}\, r^3}$$

$$= \frac{q_1q_2 v\epsilon_0}{(4\pi\epsilon_0 c)^2}\left(1-\frac{v^2}{c^2}\right)\int dx\,dy\,dz \, \frac{\hat e_3(x^2+y^2+\cancel{z^2}) - (xz\hat e_1 + yz\hat e_2 + \cancel{z^2}\hat e_3)}{\left(x^2+y^2+(z-vt)^2-(x^2+y^2)\frac{v^2}{c^2}\right)^{3/2}(x^2+y^2+z^2)^{3/2}}$$

$$\int_{-\infty}^{\infty} x\,dx\, f(x^2) = \int_{-\infty}^{\infty} y\,dy\, f(y^2) = \emptyset \Rightarrow$$

$$= \frac{q_1q_2 v}{16\pi^2 c^2\epsilon_0}\left(1-\frac{v^2}{c^2}\right)\hat e_3 \int dx\,dy\,dz \, \frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}\left((x^2+y^2)(1-\frac{v^2}{c^2})+(z-vt)^2\right)^{3/2}} =$$

In cylindrical coordinates

$$\int_0^\infty s\,ds\int_0^{2\pi} d\varphi \int_{-\infty}^\infty dz \, \frac{s^2}{(s^2+z^2)^{3/2}\left[s^2(1-\frac{v^2}{c^2})+(z-vt)^2\right]^{3/2}} = \pi\int_{-\infty}^\infty dz\int_0^\infty d\lambda \, \frac{\lambda}{(\lambda+z^2)^{3/2}}\cdot\frac{1}{\left(\lambda(1-v^2/c^2)+(z-vt)^2\right)^{3/2}} = \frac{4\pi}{vt(1-v^2/c^2)}$$

$$= \frac{q_1q_2}{4\pi c^2\epsilon_0}\frac{\hat e_3}{t} = \frac{\mu_0 q_1 q_2}{4\pi t}\hat e_3$$

It is easy to check that

$$\frac{d\vec p_{e.m}}{dt} = -\frac{\mu_0 q_1q_2}{4\pi t^2}\hat e_3 = -(F_{12}+F_{21})$$

so $\frac{d}{dt}\left(\vec P_{e.m} + \vec P_{(1)} + \vec P_{(2)}\right) = \emptyset$