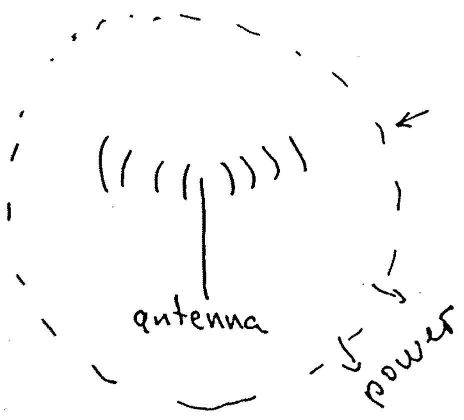
Radiation



large sphere of radius r

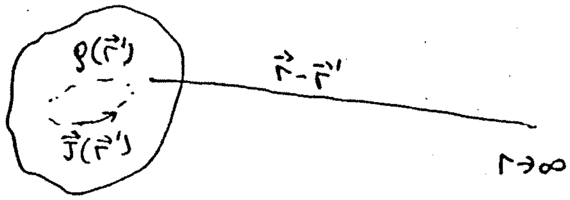
$$P(r) = \frac{1}{\mu_0} \oint (\vec{E} \times \vec{B}) \cdot d\vec{a}$$

$$S = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$P_{rad} \equiv \lim_{r \to \infty} P(r)$$

Signature of radiation : $P_{rad} \neq 0$ – irreversible flow of energy away from the source.

Electro/magnetostatics (static sources and steady currents)



total charge

$$\vec{E}(\vec{r}, \cancel{t}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(r', t)}{|r - r'|^3} |r - r'| \underset{r \to \infty}{\longrightarrow} \frac{Q\hat{r}}{4\pi\epsilon_0 r^2}$$

$$\vec{B}(\vec{r}, \cancel{t}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \to$$

$$\to \frac{\mu_0}{4\pi} \frac{3\vec{m} \cdot r\hat{r} - \vec{m}}{r^3}$$

$$\underbrace{\vec{m} = \frac{1}{2} \int d^3x' \, \vec{r}\,' \times \vec{J}(\vec{r}\,')}_{\text{dipole magnetic moment}}$$

Thus, as $r \to \infty$

$\vec{E}(\vec{r}) \sim \frac{1}{r^2}$, $B(\vec{r}) \sim \frac{1}{r^3} \Rightarrow$

$\Rightarrow S(\vec{r}) \to \frac{1}{r^5} \quad \Rightarrow \quad P_{rad} \approx \lim_{r \to \infty} \int \frac{da}{r^5} \sim \lim_{r \to \infty} \frac{1}{r^3} = \emptyset$

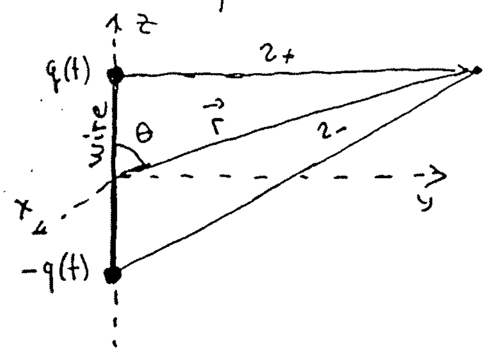
$\Rightarrow$ no radiation in electro/magnetostatics

On the other hand, for moving charges/varying currents

$$\left.\begin{aligned} \vec{E}(\vec{r}, t) &= \frac{1}{4\pi\epsilon_0} \int d^3x' \left\{ \frac{\rho(\vec{r}\,', t_r)(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} + \frac{\dot{\rho}(\vec{r}\,', t_r)(\vec{r} - \vec{r}\,')}{c|\vec{r} - \vec{r}\,'|^2} - \frac{\dot{\vec{J}}(\vec{r}\,', t_r)}{c^2 |\vec{r} - \vec{r}\,'|} \right\} \sim \frac{1}{r} \\ \vec{B}(\vec{r}, t) &= \frac{\mu_0}{4\pi} \int d^3x' \left\{ \frac{\vec{J}(\vec{r}\,', t_r)}{|\vec{r} - \vec{r}\,'|^2} + \frac{\dot{\vec{J}}(\vec{r}\,', t_r)}{c|\vec{r} - \vec{r}\,'|} \right\} \sim \frac{1}{r} \end{aligned}\right\} \Rightarrow$$

$\Rightarrow P_{rad} = \lim_{r \to \infty} \int \frac{da}{r^2} \neq 0 \Rightarrow$ radiation is possible

Electric dipole radiation



$q(t) = q_0 \cos\omega t \Rightarrow$

$\Rightarrow \vec{p}(t) = p_0 \cos\omega t \, \hat{e}_3 \qquad p_0 \equiv q_0 d$

oscillating electric dipole

Retarded potentials are

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|} \quad , \quad A(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|}$$

First,

$$\rho(\vec{r}',t_r) = q(t_r^{(1)})\delta^3(\vec{r}-\hat{e}_3\tfrac{d}{2}) + q(t_r^{(2)})\delta^3(\vec{r}+\hat{e}_3\tfrac{d}{2}) \qquad t_r^{(1)} = t - \frac{z_+}{c}$$

$$\Rightarrow V(\vec{r},t) = \frac{q_0\cos\omega(t-\frac{z_+}{c})}{4\pi\epsilon_0 z_+} - \frac{q_0\cos\omega(t-\frac{z_-}{c})}{4\pi\epsilon_0 z_-} \qquad t_r^{(2)} = t - \frac{z_-}{c}$$

Approximation $\quad d \ll \lambda \ll r$

dipole size $\quad$ wavelength $\lambda = \frac{2\pi c}{\omega}$ $\quad$ separation from the source of radiation

$$z_+ = \sqrt{r^2 - rd\cos\theta + \frac{d^2}{4}} \simeq r\left(1 - \frac{d}{2r}\cos\theta\right) \Rightarrow \frac{1}{z_+} \simeq \frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta\right)$$

$$z_- = \sqrt{r^2 + rd\cos\theta + d^2/4} \simeq r\left(1 + \frac{d}{2r}\cos\theta\right) \qquad \frac{1}{z_-} \simeq \frac{1}{r}\left(1 - \frac{d}{2r}\cos\theta\right)$$

$$\cos\omega(t - \tfrac{z_+}{c}) = \cos(\omega t - \omega\tfrac{r}{c} + \omega\tfrac{d}{2c}\cos\theta) \simeq \cos(\omega t - \omega\tfrac{r}{c})\cos\omega\tfrac{d}{2c}\cos\theta -$$

$$-\sin(\omega t - \tfrac{r\omega}{c})\sin\omega\tfrac{d}{2c}\cos\theta = \cos\omega(t-\tfrac{r}{c})\cos\pi\tfrac{d}{\lambda} - \sin\pi(\omega - \tfrac{r}{c})\sin\pi\tfrac{d}{\lambda}\cos\theta$$

$$\simeq \cos\omega(t-\tfrac{r}{c}) - \frac{\pi d}{\lambda}\cos\theta\sin\omega(t-\tfrac{r}{c}) = \cos\omega(t-\tfrac{r}{c}) - \frac{\omega d}{2c}\sin\omega(t-\tfrac{r}{c})\cos\theta$$

Similarly

$$\cos\omega(t-\tfrac{z_-}{c}) = \cos(\omega t - \omega\tfrac{r}{c} - \omega\tfrac{d}{2c}\cos\theta) \simeq \cos\omega(t-\tfrac{r}{c}) + \frac{\omega d}{2c}\sin\omega(t-\tfrac{r}{c})\cos\theta$$

$$\Rightarrow V(\vec{r},t) = \frac{q_0}{4\pi\epsilon_0}\left[\frac{\cos\omega(t-r/c) - \frac{\omega d}{2c}\sin\omega(t-\frac{r}{c})\cos\theta}{r}\left(1 + \frac{d}{2r}\cos\theta\right)\right.$$

$$\left. - \frac{\cos\omega(t-r/c) + \frac{\omega d}{2c}\sin\omega(t-r/c)\cos\theta}{r}\left(1 - \frac{d}{2r}\cos\theta\right)\right] = \frac{q_0 d\cos\theta}{4\pi\epsilon_0 r} \cdot$$

$$\cdot\left\{-\frac{\omega}{c}\sin\omega(t-\tfrac{r}{c}) + \frac{\cos\omega(t-r/c)}{r}\right\} = \frac{p_0\omega\cos\theta}{4\pi\epsilon_0 c r}\sin\omega(t-\tfrac{r}{c}) + \frac{p_0\cos\theta}{4\pi\epsilon_0 r^2}\cos\omega(t-\tfrac{r}{c})$$

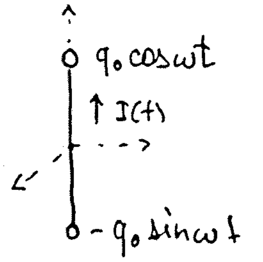
In the static limit $(\omega \to 0)$

$$V(\vec{r}) = \frac{p_0\cos\theta}{4\pi\epsilon_0 r^2} \leftarrow \text{potential of a stationary dipole}$$

We are interested in the $\propto(\frac{1}{r})$ term (recall that $r \gg \frac{c}{\omega}$)

$$V(\vec r,t) \simeq -\frac{p_0\omega\cos\theta}{4\pi\epsilon_0 c r}\sin\omega\left(t-\frac{r}{c}\right)$$

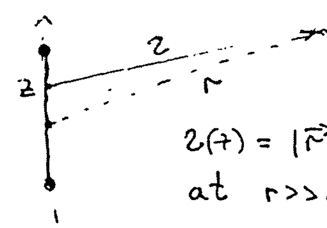
Second,
the magnetic vector potential is determined by the current flowing in the wire



$$\Rightarrow \vec I(t) = \hat e_3 \frac{dq(t)}{dt} = -q_0\omega(\sin\omega t)\hat e_3$$

$$\vec A(\vec r,t) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{J(\vec r\,', t_r)}{|\vec r - \vec r\,'|} \xrightarrow[\text{currents}]{\text{for the line}} \frac{\mu_0}{4\pi}\int dl' \, \frac{\vec I(\vec r\,', t_r)}{|\vec r - \vec r\,'|}$$

$$\vec A(\vec r,t) = -\frac{\mu_0 \hat e_3}{4\pi}\int_{-d/2}^{d/2} dz \, \frac{q_0\omega\sin\omega(t-\frac{\mathcal{R}}{c})}{\mathcal{R}} \simeq$$



$\mathcal{R}(t) = |\vec r - z\hat e_3| \simeq r$
at $r \gg d$

$$\simeq -\frac{\mu_0 q_0 d}{4\pi r}\hat e_3 \sin\omega\left(t-\frac{r}{c}\right)$$

$$\Rightarrow \vec A(r,t) \simeq -\frac{\mu_0 p_0\omega}{4\pi r}\sin\omega\left(t-\frac{r}{c}\right)\hat e_3$$

Fields (at $r\to\infty$):

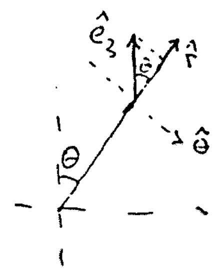
$$\vec E(\vec r,t) = -\nabla\vec V(\vec r,t) - \frac{\partial\vec A(\vec r,t)}{\partial t}, \quad B(\vec r,t) = \vec\nabla\times\vec A(\vec r,t)$$

In spherical coordinates

$$\vec\nabla V(r,\theta,\phi) = \hat r\frac{\partial V(r,\theta)}{\partial r} + \frac{\hat\theta}{r}\frac{\partial V(r,\theta)}{\partial\theta} = \hat r\Big(\underbrace{\frac{p_0\omega\cos\theta}{4\pi\epsilon_0 c r^2}\sin\omega\left(t-\frac{r}{c}\right)}_{\sim O(\frac{1}{r^2})} +$$

$$+ \underbrace{\frac{p_0\omega^2\cos\theta}{4\pi\epsilon_0 c^2 r}\cos\omega\left(t-\frac{r}{c}\right)}_{\sim O(\frac{1}{r})}\Big) + \underbrace{\frac{\hat\theta}{r}\frac{p_0\omega\sin\theta}{4\pi\epsilon_0 c r}\sin\omega\left(t-\frac{r}{c}\right)}_{\sim O(\frac{1}{r^2})} \simeq \frac{\hat r\, p_0\omega^2\cos\theta}{4\pi\epsilon_0 c^2 r}\cos\omega\left(t-\frac{r}{c}\right)$$

$$\frac{\partial\vec A(r,\theta,\phi)}{\partial t} = -\frac{\mu_0 p_0\omega^2}{4\pi r}\hat e_3\cos\omega\left(t-\frac{r}{c}\right)$$



$$\Rightarrow \vec E(r,\theta,\varphi;t) = -\frac{\mu_0\omega^2 p_0}{4\pi r}\cos\omega\left(t-\frac{r}{c}\right)(\hat r\cos\theta - \underbrace{\hat e_3}_{\hat r\cos\theta - \hat\theta\sin\theta})$$

$$\Rightarrow \vec E(\vec r,t) = -\frac{\mu_0 p_0\omega^2}{4\pi}\frac{\sin\theta}{r}\cos\omega\left(t-\frac{r}{c}\right)\hat\theta$$

Similarly,

$\vec{B}(r,\theta,\varphi,t) = \vec{\nabla}\times\vec{A}(r,\theta,\varphi,t)$

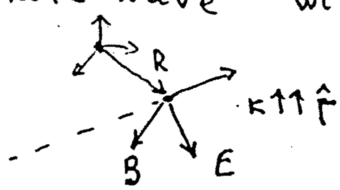
$$\vec{A}(r,\theta;t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin\omega(t-\tfrac{r}{c})(\hat{r}\cos\theta - \hat{\theta}\sin\theta)$$
$$\Downarrow$$
$$A_r = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin\omega(t-\tfrac{r}{c})\cos\theta$$
$$A_\theta = \frac{\mu_0 p_0 \omega}{4\pi r}\sin\omega(t-\tfrac{r}{c})\sin\theta$$
$$A_\varphi = \emptyset$$

$$\vec{\nabla}\times\vec{A} = \frac{\hat{\varphi}}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right) =$$
$$= \frac{\hat{\varphi}}{r}\left\{\frac{\partial}{\partial r}\frac{\mu_0 p_0 \omega}{4\pi}\sin\omega(t-\tfrac{r}{c})\sin\theta + \frac{\partial}{\partial\theta}\frac{\mu_0 p_0\omega}{4\pi r}\cos\theta\sin\omega(t-\tfrac{r}{c})\right\} =$$
$$= \frac{\hat{\varphi}}{r}\left\{-\underbrace{\frac{\mu_0 p_0\omega^2}{4\pi c}\cos\omega(t-\tfrac{r}{c})\sin\theta}_{\sim O(\frac{1}{r})} - \underbrace{\frac{\mu_0 p_0\omega}{4\pi r}\sin\theta\sin\omega(t-\tfrac{r}{c})}_{\cancel{\sim O(\frac{1}{r^2})}}\right\} = -\frac{\hat{\varphi}}{r}\frac{\mu_0 p_0\omega^2}{4\pi c}\sin\theta \cdot \cos\omega(t-r/c)$$

$$\Rightarrow \vec{E} = -\frac{\mu_0 p_0\omega^2}{4\pi}\frac{\sin\theta}{r}\hat{\theta}\cos\omega(t-\tfrac{r}{c})$$
$$\vec{B} = -\frac{\mu_0 p_0\omega^2}{4\pi c}\frac{\sin\theta}{r}\hat{\varphi}\cos\omega(t-\tfrac{r}{c})$$

} spherical wave traveling with the speed of light

Over small regions, this wave can be approximated by a plane wave with $\vec{k}\uparrow\uparrow\hat{r}$



$$E \sim \vec{A}\cos(\omega t - \vec{k}\vec{R})$$
$$\vec{B} = \frac{1}{c}\hat{r}\times\vec{E} = \frac{1}{c}\hat{k}\times\vec{E}$$

The energy radiated by this dipole is

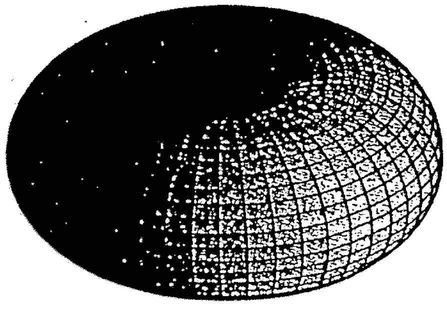
$$P = \int_{\substack{\text{large}\\\text{sphere}}} \vec{S}\cdot d\vec{a}$$

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{\mu_0}{c}\hat{r}\left(\frac{p_0\omega^2\sin\theta}{4\pi r}\cos\omega(t-\tfrac{r}{c})\right)^2$$

Time average

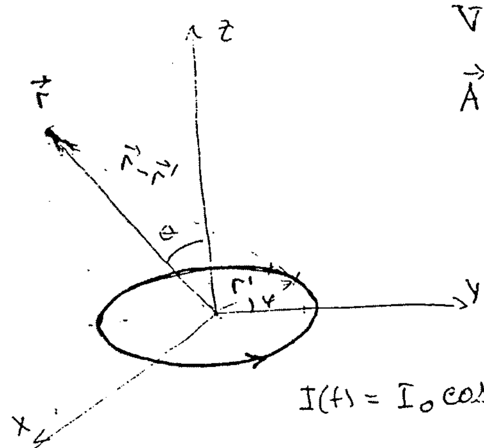
$$\langle S\rangle = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\frac{\sin^2\theta}{r^2}\hat{r}$$

$$\langle P\rangle = \int\langle S\rangle\cdot da = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\int dr\,d\theta\,d\varphi\, r^2\sin^3\theta = \frac{\mu_0 p_0^2\omega^4}{12\pi c}$$



← profile of $\langle S\rangle$

Magnetic dipole radiation



$$V(\vec{r},t) = \phi$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\oint \frac{d\vec{l}'\, I(t_r)}{|\vec{r}-\vec{r}'|} \quad \leftarrow \text{retarded potential}$$

$$t_r = t - \frac{|\vec{r}-\vec{r}'|}{c}$$

$$I(t) = I_0 \cos\omega t$$

$$\vec{m}(t) = \hat{e}_3 \pi b^2 I(t) = m_0 \cos\omega t\, \hat{e}_3 \qquad m_0 = \pi I_0 b^2$$

For simplicity, $r = r\sin\theta\, \hat{e}_1 + r\cos\theta\, \hat{e}_3$ ($\vec{r}$ lies in the xz plane).

$$\vec{r}' = b\cos\varphi\, \hat{e}_1 + b\sin\varphi\, \hat{e}_2 \Rightarrow (\vec{r}-\vec{r}')^2 = r^2\cos^2\theta + (r\sin\theta - b\cos\varphi)^2 + b^2\sin^2\varphi =$$
$$= r^2 + b^2 - 2br\sin\theta\cos\varphi$$

$$\left.\begin{aligned} \Rightarrow\ & |\vec{r}-\vec{r}'| = \sqrt{r^2+b^2-2br\sin\theta\cos\varphi} \\ & d\vec{l}' = dl\,\hat{\varphi} = dl\,(\hat{e}_2\cos\varphi - \hat{e}_1\sin\varphi) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \vec{A}(\vec{r},t) = \frac{\mu_0 I b}{4\pi}\int_0^{2\pi} d\varphi\, \frac{\cos\omega\left(t - \frac{1}{c}\sqrt{r^2+b^2-2br\sin\theta\cos\varphi}\right)}{\sqrt{r^2+b^2-2br\sin\theta\cos\varphi}}\,(\hat{e}_2\cos\varphi - \cancel{\hat{e}_1\sin\varphi})$$

$\sin\varphi$ is odd

$$= \frac{\mu_0 I b}{4\pi}\,\hat{e}_2 \int_0^{2\pi} d\varphi\, \cos\varphi\, \frac{\cos\omega\left(t - \frac{1}{c}\sqrt{r^2+b^2-2br\sin\theta\cos\varphi}\right)}{\sqrt{r^2+b^2-2br\sin\theta\cos\varphi}}$$

Perfect dipole: $b \to 0 \qquad m_0 = \pi b^2 I_0 = const$

$$\int_0^{2\pi} d\varphi\, \cos\varphi\, \frac{\cos\omega\left(t - \frac{1}{c}\sqrt{r^2+b^2-2br\sin\theta\cos\varphi}\right)}{\sqrt{r^2+b^2-2br\sin\theta\cos\varphi}} \xrightarrow[b\to 0]{} ?$$

Zero-order approximation: $b = 0$

$$\int_0^{2\pi} d\varphi\, \cos\varphi\, \frac{\cos\omega(t-\frac{r}{c})}{r} = \phi \Rightarrow$$

First-order approximation

$$\sqrt{r^2+b^2-2br\sin\theta\cos\varphi} \simeq r\left(1 - \frac{b}{r}\sin\theta\cos\varphi\right) = r - b\sin\theta\cos\varphi$$

$$(r^2+b^2-2br\sin\theta\cos\varphi)^{-1/2} \simeq \frac{1}{r}\left(1 + \frac{b}{r}\sin\theta\cos\varphi\right)$$

$$\cos\omega\left(t-\frac{1}{c}\sqrt{r^2+b^2-2br\sin\theta\cos\varphi}\right) \simeq \cos\left(\omega\left(t-\frac{r}{c}\right)+\frac{\omega b}{c}\sin\theta\cos\varphi\right) = \cos\omega\left(t-\frac{r}{c}\right)\cdot$$

$$\cdot\cos\left(\frac{\omega b}{c}\sin\theta\cos\varphi\right)-\sin\omega\left(t-\frac{r}{c}\right)\sin\left(\frac{\omega b}{c}\sin\theta\cos\varphi\right) \simeq \cos\omega\left(t-\frac{r}{c}\right)-\frac{\omega b}{c}\sin\theta\cos\varphi\,\sin\omega\left(t-\frac{r}{c}\right)$$

$$\Rightarrow \vec{A}(\vec{r},t) = \frac{\mu_0 I_0 b}{4\pi r}\hat{e}_2\int_0^{2\pi} d\varphi\cos\varphi\left(\cos\omega\left(t-\frac{r}{c}\right)-\frac{\omega b}{c}\sin\theta\cos\varphi\,\sin\omega\left(t-\frac{r}{c}\right)\right)\left(1+\frac{b}{r}\sin\theta\cos\varphi\right)$$

$$\simeq \frac{\mu_0 I_0 b}{4\pi r}\hat{e}_2\int_0^{2\pi} d\varphi\cos\varphi\left(\cancel{\cos\omega\left(t-\frac{r}{c}\right)}-\frac{\omega b}{c}\sin\theta\cos\varphi\,\sin\omega\left(t-\frac{r}{c}\right)+\frac{b}{r}\sin\theta\cos\varphi\cos\omega\left(t-\frac{r}{c}\right)\right)$$

$$= \frac{\mu_0 I_0 b\,\hat{e}_2}{4r}\left(-\frac{\omega b}{c}\sin\theta\,\sin\omega\left(t-\frac{r}{c}\right)+\frac{b}{r}\sin\theta\cos\omega\left(t-\frac{r}{c}\right)\right).$$

For an arbitrary $\vec{r}$, $\hat{e}_2$ should be replaced by $\hat{\varphi}$ (in spherical coordinates)

$$\vec{A}(\vec{r},t) = \frac{\mu_0 m_0}{4\pi}\hat{\varphi}\left(-\frac{\omega}{cr}\sin\theta\,\sin\omega\left(t-\frac{r}{c}\right)+\frac{\sin\theta}{r^2}\cos\omega\left(t-\frac{r}{c}\right)\right)$$

Check: $\omega\to 0$ $\quad \vec{A}(\vec{r},t)=\frac{\mu_0 m_0}{4\pi r^2}\sin\theta\,\hat{\varphi}$ – potential of a static magnetic dipole

We need the potential in the radiation zone $r \gg \lambda = \frac{2\pi c}{\omega}$

$$\vec{A}(\vec{r},t) = -\frac{\mu_0 m_0\omega}{4\pi c r}\sin\theta\,\sin\omega\left(t-\frac{r}{c}\right)\hat{\varphi}$$

$$A_\varphi = -\frac{\mu_0 m_0\omega}{4\pi c r}\sin\theta\,\sin\omega\left(t-\frac{r}{c}\right)$$
$$A_\theta = A_r = 0$$

Correspondingly, the fields are

$$\vec{E} = -\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t} = -\frac{\partial\vec{A}}{\partial t} = \frac{\mu_0 m_0\omega^2}{4\pi c r}\sin\theta\cos\omega\left(t-\frac{r}{c}\right)\hat{\varphi}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\varphi)-\frac{\hat{\theta}}{r}\frac{\partial}{\partial r}(rA_\varphi) \simeq -\frac{\mu_0 m_0\omega^2}{4\pi c^2 r}\sin\theta\cos\omega\left(t-\frac{r}{c}\right)$$

$$\underbrace{-\frac{\hat{r}\omega\mu_0 m_0}{4\pi c r^2}2\cos\theta\cos\omega\left(t-\frac{r}{c}\right)}_{} \ll -\frac{\hat{\theta}\mu_0 m_0\omega^2}{4\pi c^2 r}\sin\theta\cos\omega\left(t-\frac{r}{c}\right)$$

$$\frac{1}{r}\ll\frac{\omega}{c}$$

$$\vec{E} = \frac{\mu_0 m_0\omega^2}{4\pi c}\frac{\sin\theta}{r}\hat{\varphi}\cos\omega\left(t-\frac{r}{c}\right)$$
$$\vec{B} = -\frac{\mu_0 m_0\omega^2}{4\pi c^2}\frac{\sin\theta}{r}\hat{\theta}\cos\omega\left(t-\frac{r}{c}\right) = \frac{\hat{r}}{c}\times\vec{E}$$

} spherical wave

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{\mu_0}{c}\left(\frac{m_0\omega^2}{4\pi c}\frac{\sin\theta}{r}\cos\omega\left(t-\frac{r}{c}\right)\right)^2\hat{r}$$

Intensity $\quad \langle S\rangle = \frac{\mu_0 m_0^2\omega^4}{32\pi^2 c^3}\frac{\sin^2\theta}{r^2}\hat{r} \Rightarrow$

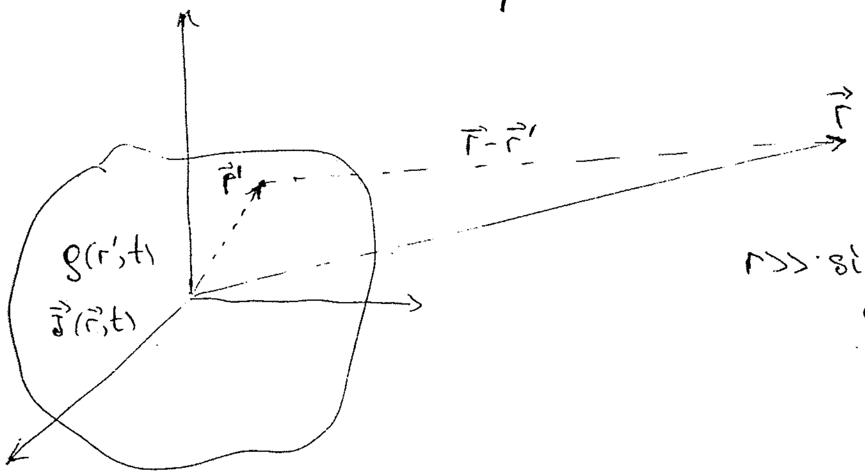
$$\Rightarrow P_{rad} = \frac{\mu_0 m_0^2\omega^4}{32\pi^2 c^3}\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\sin^3\theta = \frac{\mu_0 m_0^2\omega^4}{12\pi c^3}$$

} similar to radiation by electric dipole, but much smaller

$$\frac{P_{mag}}{P_{el}} = \left(\frac{m_0}{p_0 c}\right)^2 = \left(\frac{\pi b^2 I_0}{q_0 d c}\right)^2$$ In the case of electric dipole $I_0 = q_0\omega$

$$= \left(\frac{\pi b}{d}\frac{\omega b}{c}\right)^2 \quad \text{If } \frac{\pi b}{d}\sim 1 \quad \frac{P_{mag}}{P_{el}}\sim\left(\frac{\omega b}{c}\right)^2 \ll 1$$

Radiation from an arbitrary source



$r \gg$ size of the object

$$V(r,t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(r',t_r)}{|r-r'|} \qquad t_r = t - \frac{|r-r'|}{c}$$

$$r' \ll r \Rightarrow |r-r'| \simeq \sqrt{r^2 - 2\vec{r}\cdot\vec{r}' + r'^2} \simeq r\left(1 - \frac{\vec{r}\cdot\vec{r}'}{r^2}\right) = r - \frac{\vec{r}\cdot\vec{r}'}{r} = r - \hat{r}\cdot\vec{r}'$$

$$\frac{1}{|r-r'|} \simeq \frac{1}{r\left(1-\frac{\vec{r}\cdot\vec{r}'}{r^2}\right)} \simeq \frac{1}{r}\left(1 + \frac{\vec{r}\cdot\vec{r}'}{r^2}\right)$$

$$t_r = t - \frac{|\vec{r}-\vec{r}'|}{c} \simeq t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}$$

$$\rho(r',t_r) \simeq \rho(\vec{r}';\underbrace{t-\tfrac{r}{c}}_{t_0} + \tfrac{\hat{r}\cdot\vec{r}'}{c}) \simeq \rho(\vec{r}';t_0) + \frac{\hat{r}\cdot\vec{r}'}{c}\dot{\rho}(\vec{r}';t_0) + \dots$$

$\dot{\rho} \sim \omega_{char}\,\rho$

characteristic frequencies for the motion of charges

Approximation $\frac{c}{\omega_{char}} \gg r' \Leftrightarrow$ size of the object $\ll$ characteristic wavelength

$$V(\vec{r},t) \simeq \frac{1}{4\pi\epsilon_0 r}\int d^3x' \left(\rho(\vec{r}';t_0) + \frac{\hat{r}\cdot\vec{r}'}{c}\dot{\rho}(\vec{r}';t_0)\right)\left(1 + \frac{\hat{r}\cdot\vec{r}'}{r}\right) =$$

$$= \frac{1}{4\pi\epsilon_0 r}\Big[\underbrace{\int d^3x' \, \rho(\vec{r}';t_0)}_{=Q} + \frac{\hat{r}}{r}\cdot\underbrace{\int d^3x' \, \vec{r}'\rho(\vec{r}';t_0)}_{\vec{p}(t_0)} + \frac{\hat{r}}{c}\frac{d}{dt}\underbrace{\int d^3x' \, \vec{r}'\rho(\vec{r}';t_0)}_{\vec{p}(t_0)}\Big]$$

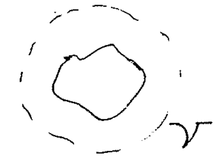
$$= \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\hat{r}\cdot\vec{p}(t_0)}{r^2} + \frac{\hat{r}\cdot\dot{\vec{p}}(t_0)}{rc}\right]$$

dipole potential

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{J(\vec{r}';t_r)}{|r-r'|} \simeq \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{r}';t_0)}{r} = \frac{\mu_0}{4\pi r}\int d^3x' \, \vec{J}(\vec{r}';t_0)$$

(first term is enough)

$\int d^3x' \, \vec J(\vec r',t) = \dot{\vec p}(t)$



Proof:

$\dot{\vec p}(t) = \int_{V\text{- large sphere}} d^3x' \, \vec r' \, \dot\rho(\vec r,t) =$

$\dot\rho(\vec r,t) = -\vec\nabla'\cdot\vec J(\vec r',t) = -\left(\frac{\partial}{\partial x_1'}J_1 + \frac{\partial}{\partial x_2'}J_2 + \frac{\partial}{\partial x_3'}J_3\right)$

$$= -\int_V d^3x' \, \vec r' \left(\frac{\partial}{\partial x_1'}J_1 + \frac{\partial}{\partial x_2'}J_2 + \frac{\partial}{\partial x_3'}J_3\right) = \text{by parts} = \int d^3x' \Big(J_1(\vec r',t)\underbrace{\frac{\partial}{\partial x_1'}\vec r'}_{\hat e_1} + J_2(\vec r',t)\underbrace{\frac{\partial}{\partial x_2'}\vec r'}_{\hat e_2} + J_3\underbrace{\frac{\partial}{\partial x_3'}\vec r'}_{\hat e_3}\Big) = \int d^3x' \, \vec J(\vec r',t)$$

Surface terms vanish since there is no current outside of the body.

Thus

$$V(\vec r,t) \simeq \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{r\cdot\vec p(t_0)}{r^2} + \frac{\hat r\cdot\dot{\vec p}(t_0)}{cr}\right]$$

$$\vec A(\vec r,t) \simeq \frac{\mu_0}{4\pi}\frac{\dot{\vec p}(t_0)}{r}$$

Next, we calculate $\vec E$ and $\vec B$ discarding terms $\sim \frac{1}{r^2}$

$$\vec\nabla V(r,t) = \underbrace{\cancel{\frac{1}{4\pi\epsilon_0}\frac{Q\hat r}{r^2}}}_{\sim O(\frac{1}{r^2})} + \frac{1}{4\pi\epsilon_0}\Big\{\underbrace{\cancel{\vec\nabla\left(\frac{\hat r}{r^2}\cdot\vec p(t_0)\right)}}_{\sim \frac{1}{r^2}} + \vec\nabla\left(\frac{\hat r}{rc}\cdot\dot{\vec p}(t_0)\right)\Big\}$$

$\frac{\partial r}{\partial x_1} = \frac{x_1}{r}$ $\quad \frac{\partial 1/r}{\partial x_1} = -\frac{x_1}{r^3}$

$\nabla\left(\frac{\hat r}{r^2}\cdot\dot{\vec p}(t_0)\right) = ?$

$$\frac{\partial}{\partial x_1}\left(\frac{\vec r\cdot\dot{\vec p}(t_0)}{r^2}\right) = \left(\frac{\partial}{\partial x_1}\frac{\vec r}{r^2}\right)\cdot\dot{\vec p}(t_0) + \frac{\vec r}{cr^2}\cdot\frac{\partial}{\partial x_1}\dot{\vec p}(t_0) =$$

$$= \underbrace{\cancel{\left(\frac{\hat e_1}{r^2} - \frac{2\vec r x_1}{r^4}\right)\cdot\frac{\dot{\vec p}(t_0)}{c}}}_{\sim O(\frac{1}{r^2})} + \frac{\vec r}{cr^2}\cdot\ddot{\vec p}(t_0)\frac{\partial t_0}{\partial x_1} = \frac{\vec r\cdot\ddot{\vec p}(t_0)}{cr^2}\left(-\frac{x_1}{cr}\right)$$

$\frac{\partial t_0}{\partial x_1} = \frac{\partial}{\partial x_1}\left(t - \frac{r}{c}\right) = -\frac{1}{c}\frac{x_1}{r} = -\frac{x_1}{cr}$

$$\Rightarrow \vec\nabla\frac{r\cdot\dot{\vec p}(t_0)}{r} = \left(\hat e_1\frac{\partial}{\partial x_1} + \hat e_2\frac{\partial}{\partial x_2} + \hat e_3\frac{\partial}{\partial x_3}\right)\frac{\hat r\cdot\dot{\vec p}(t_0)}{cr} = -\frac{x_1\hat e_1 + x_2\hat e_2 + x_3\hat e_3}{c^2 r}\frac{\vec r\cdot\ddot{\vec p}(t_0)}{r^2}$$

$$= -\frac{\hat r}{c^2}\frac{\hat r\cdot\ddot{\vec p}(t_0)}{r} \Rightarrow \vec\nabla V = -\frac{1}{4\pi\epsilon_0 c^2}\frac{\hat r}{r}\left(\hat r\cdot\ddot{\vec p}(t_0)\right)$$

$$\frac{\partial\vec A}{\partial t} = \frac{\mu_0}{4\pi}\ddot{\vec p}(t)$$

$$\vec\nabla\times\vec A = \frac{\mu_0}{4\pi}\vec\nabla\times\left(\dot{\vec p}(t_0)/r\right)$$

$$(\nabla\times A)_1 = \frac{\mu_0}{4\pi}\left(\frac{\partial}{\partial x_2}\frac{\dot p_3(t_0)}{r} - \frac{\partial}{\partial x_3}\frac{\dot p_2(t_0)}{r}\right) = \frac{\mu_0}{4\pi}\Big(\frac{\ddot p_3(t_0)}{r}\overbrace{\frac{\partial t_0}{\partial x_2}}^{-\frac{x_2}{cr}} - \frac{\ddot p_2(t_0)}{r}\overbrace{\frac{\partial t_0}{\partial x_3}}^{-\frac{x_3}{cr}} + \underbrace{\cancel{\dot p_3\frac{\partial}{\partial x_2}\frac{1}{r} - \dot p_2\frac{\partial}{\partial x_3}\frac{1}{r}}}_{\sim O(\frac{1}{r^2})}\Big) = \frac{\mu_0}{4\pi cr^2}\left(\ddot p_2 x_3 - \ddot p_3 x_2\right) = \frac{\mu_0}{4\pi cr^2}\left(\ddot{\vec p}\times\vec r\right)_1$$

$$\Rightarrow \vec\nabla\times\vec A = \frac{\mu_0}{4\pi cr}\ddot{\vec p}(t_0)\times\hat r = -\frac{\mu_0}{4\pi cr}\hat r\times\ddot{\vec p}(t_0)$$

Thus, the fields are

$$\vec{E}(\vec{r},t) = \frac{\hat{r}}{4\pi\epsilon_0 c^2 r}\,\hat{r}\cdot\ddot{\vec{p}}(t_0) - \frac{\mu_0}{4\pi}\ddot{\vec{p}}(t_0) = \frac{\mu_0}{4\pi r}\left(\hat{r}(\hat{r}\cdot\ddot{\vec{p}}(t_0)) - \ddot{\vec{p}}(t_0)\right) = \frac{\mu_0}{4\pi r}\hat{r}\times(\hat{r}\times\ddot{\vec{p}})$$

$$\vec{B}(\vec{r},t) = -\frac{\mu_0}{4\pi c r}\,\hat{r}\times\ddot{\vec{p}}(t_0) \qquad \vec{B} = \frac{\hat{r}}{c}\times\vec{E}$$

Let us choose the spherical polar coordinates with $Oz \uparrow\uparrow \ddot{\vec{p}}(t_0)$

$\hat{e}_3 = \hat{r}\cos\theta - \hat{\theta}\sin\theta$

$$\vec{E}(r,\theta,\varphi) \simeq \frac{\mu_0 \ddot{p}(t_0)}{4\pi}\frac{\sin\theta}{r}\hat{\theta}$$

$$\vec{B}(r,\theta,\varphi) \simeq \frac{\mu_0 \ddot{p}(t_0)}{4\pi c}\frac{\sin\theta}{r}\hat{\varphi}$$

$\vec{B} = \frac{\hat{r}}{c}\times\vec{E}$, as always for the radiation fields

The Poynting vector

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{\mu_0}{16\pi^2 c}(\ddot{p}(t_0))^2\frac{\sin^2\theta}{r^2}\hat{r} \Rightarrow$$

$\Rightarrow$ the radiated power is

$$P_{rad} = \int \vec{S}\cdot d\vec{a} = \int \frac{\mu_0}{16\pi^2 c}(\ddot{p}(t_0))^2\frac{\sin^2\theta}{r^2}da = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \frac{\mu_0 \ddot{p}^2}{16\pi^2 c}\sin^3\theta =$$

$$= \frac{\mu_0}{6\pi c}(\ddot{p}(t_0))^2$$

Examples

1. Oscillating electric dipole $p(t) = p_0\cos\omega t$

$\ddot{p}(t) = -\omega^2 p_0 \cos\omega t \Rightarrow \vec{E}(r,\theta,\varphi) = -\frac{\mu_0\omega^2 p_0}{4\pi}\frac{\sin\theta}{r}\hat{\theta}\cos\omega\left(t-\frac{r}{c}\right)$

$\vec{B}(r,\theta,\varphi) = \frac{\hat{r}}{c}\times\vec{E}$

– old formulas for the electric dipole radiation

2. A single point charge q

$\vec{p}(t) = q\vec{d}(t) \Rightarrow \ddot{\vec{p}}(t) = q\vec{a}(t)$ acceleration

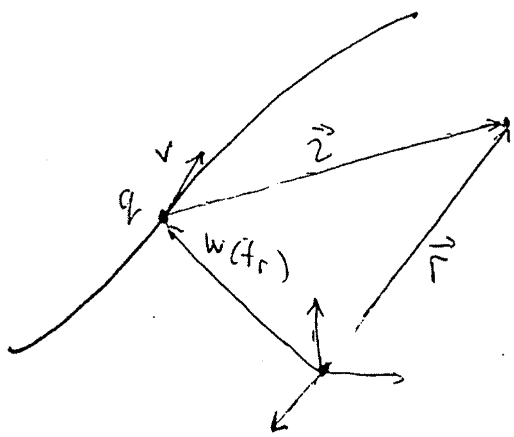
$$\Rightarrow P = \frac{\mu_0 q^2 a^2}{6\pi c} \qquad \text{Larmor formula}$$

Only dipole radiates, the monopole (charge) does not radiate due to the conservation of charge

If ~~not~~ there was the charge conservation

$$V_{mono} = \frac{1}{4\pi\epsilon_0}\frac{Q(t_0)}{r} \Rightarrow E_{mono} = \frac{1}{4\pi\epsilon_0 c}\frac{\dot{Q}(t_0)}{r}\hat{r}$$

Power radiated by a point charge



$\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$

$\vec{u} = c\hat{\mathscr{r}} - \vec{v}$

$t_r: \quad c(t - t_r) = \mathscr{r}$

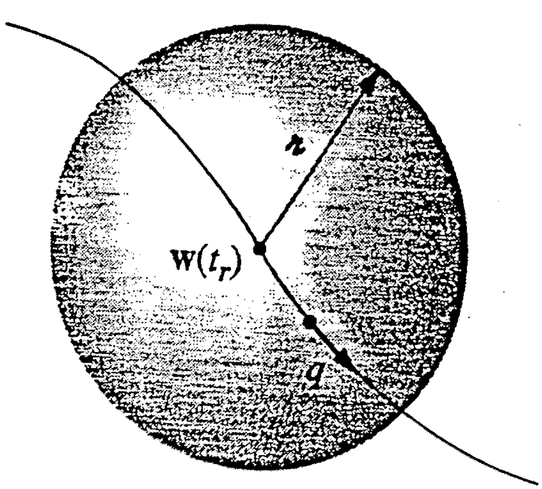
$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3} \left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}} \times (\vec{u} \times \vec{a}) \right] \Big|_{t=t_r}$$

"velocity field" "acceleration field"

$$B(\vec{r},t) = \frac{\hat{\mathscr{r}}}{c} \times \vec{E}(\vec{r},t)$$

$$S = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0 c} \vec{E} \times (\hat{\mathscr{r}} \times \vec{E}) = \frac{1}{\mu_0 c} \left(E^2 \hat{\mathscr{r}} - (\hat{\mathscr{r}} \cdot \vec{E})\vec{E} \right)$$

Some of the energy is radiation; another part is just field energy carried along by the particle as it moves. To calculate the power radiated by the particle at time t_r, we draw a large sphere, wait for $t \quad t_r = \frac{\mathscr{r}}{c}$ and integrate Poynting vector over the surface.



Velocity field $\sim \frac{1}{\mathscr{r}^3} \Rightarrow$

$\Rightarrow P_{rad}$ due to velocity field $=$

Acceleration field $\sim \frac{1}{\mathscr{r}^2} \Rightarrow$

$\Rightarrow P_{rad}$ is finite

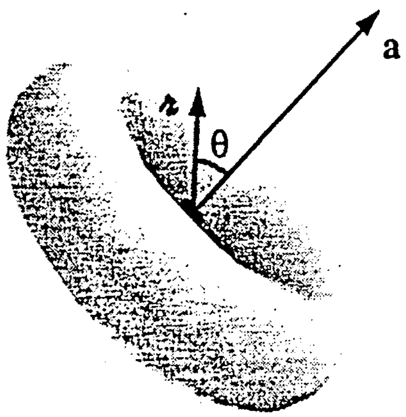
$$\vec{E}_{accel} \equiv \vec{E}_{rad} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\,\vec{\mathscr{r}}\times(\vec{u}\times\vec{a}) \Rightarrow \hat{\mathscr{r}}\cdot\vec{E}_{rad} = \emptyset \Rightarrow$$

$$\Rightarrow \vec{S}_{rad} = \frac{\hat{\mathscr{r}}}{\mu_0 c}E^2_{rad}$$

For simplicity, consider at first the charge which is instantaneously at rest at time t_r

$$\vec{v}(t_r) = 0 \Rightarrow \vec{u}(t_r) = c\hat{\mathscr{r}} \Rightarrow \vec{E}_{rad} = \frac{q}{4\pi\epsilon_0 c^2 \mathscr{r}}\hat{\mathscr{r}}\times(\hat{\mathscr{r}}\times\hat{a}) = \frac{q}{4\pi\epsilon_0 c^2\mathscr{r}}\cdot\left[(\hat{\mathscr{r}}\cdot\vec{a})\hat{\mathscr{r}} - \vec{a}\right] \Rightarrow$$

$$\Rightarrow \vec{S}_{rad} = \frac{\hat{\mathscr{r}}}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi\mathscr{r}}\right)^2\left(a^2 - (\hat{\mathscr{r}}\cdot\vec{a})\right)^2 = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin\theta}{\mathscr{r}^2}\hat{\mathscr{r}}$$



The total power

$$P_{rad} = \oint_{sphere} \vec{S}_{rad}\cdot d\vec{a} =$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c}\int \frac{\sin^2\theta}{\mathscr{r}^2}\mathscr{r}^2 \sin\theta\, d\theta\, d\varphi$$

$$= \frac{\mu_0 q^2 a^2}{8\pi c}\int_0^\pi d\theta\,\sin^3\theta = \frac{\mu_0 q^2 a^2}{6\pi c}$$

Larmor formula

$P_{rad} = \frac{\mu_0 q^2 a^2}{6\pi c}$ was derived under the assumption $v=0$ but it holds true as long as $v \ll c$.

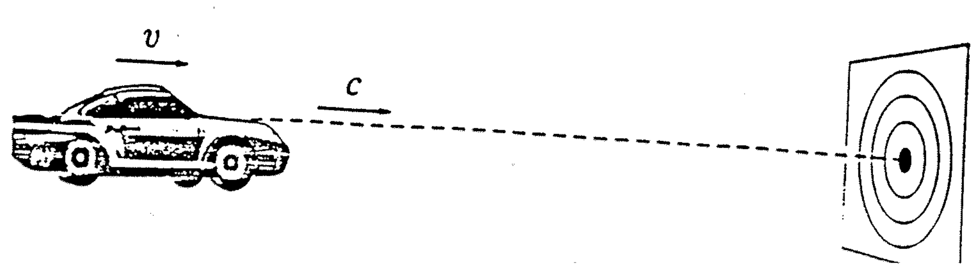
In the general case, the formula is modified

$$= \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right)$$

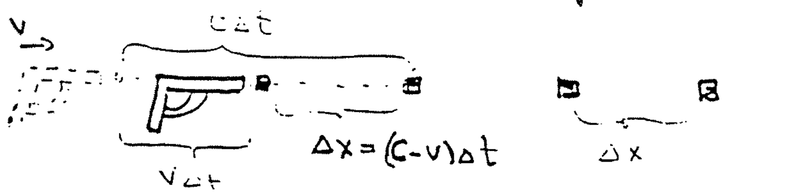
Lienard's formula

$$\left(\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\right)$$

To prove it, consider at first the mechanical problem



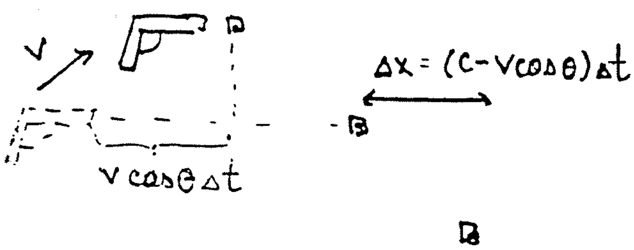
= ▪ N_g bullets per unit time $\Delta t = \frac{\ }{N_g}$



$$N_t = \frac{c}{\Delta x} = \frac{c}{c-v} N_g$$

$$\Rightarrow N_t = \frac{1}{1 - v/c} N_g$$

For an arbitrary angle



$$N_t = \frac{c}{\Delta x} = \frac{c\,N_g}{c - v\cos\theta} = \frac{N_g}{1 - \frac{v}{c}\cos\theta}$$

$$\Rightarrow N_t = \frac{1}{1 - \frac{\vec{v}\cdot\hat{\imath}}{c}} N_g$$

In our case $N_g = \frac{dW}{dt}$ = the rate at which energy passes thru the sphere at radius $\imath$

The rate at which energy left the charge was

$$\frac{dW}{dt_r} = \frac{dW}{dt}\frac{\partial t}{\partial t_r} = \frac{dW/dt}{\partial t_r/\partial t}$$

$$= \frac{\vec{\imath}\cdot\vec{u}}{\imath c}\frac{dW}{dt} = \frac{1}{1 - \frac{\vec{v}\cdot\hat{\imath}}{c}}\frac{dW}{dt}$$

$$\underbrace{\frac{dW}{dt_r}}_{\text{analog of } N_t} = \frac{1}{1 - \frac{\vec{v}\cdot\hat{\imath}}{c}}\underbrace{\frac{dW}{dt}}_{\text{analog of } N_g}$$

$$c(t - t_r) = |\vec{r} - \vec{w}(t_r)| \Rightarrow$$

$$c\left(1 - \frac{\partial t_r}{\partial t}\right) = \frac{-\dot{\vec{w}}(t_r)\cdot(\vec{r} - \vec{w}(t_r))}{|\vec{r} - \vec{w}(t_r)|}$$

$$\Rightarrow c\left(1 - \frac{\partial t_r}{\partial t}\right) = -\hat{\imath}\cdot\vec{v}\,\frac{\partial t_r}{\partial t} \Rightarrow$$

$$\Rightarrow \frac{\partial t_r}{\partial t} = \frac{1}{1 - \frac{\hat{\imath}\cdot\vec{v}}{c}} = \frac{\imath c}{\vec{\imath}\cdot\vec{u}}$$

The power radiated by the particle into an area $\imath^2\sin\theta\,d\theta\,d\varphi$ of the large sphere is

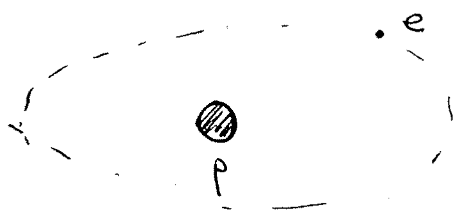
$$dP = \frac{\vec{\imath}\cdot\vec{u}}{\imath c}\frac{1}{\mu_0 c}E_{rad}^2\,\imath^2\underbrace{\sin\theta\,d\theta\,d\varphi}_{d\Omega \text{ - solid angle}}$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0}\frac{|\hat{\imath}\times(\vec{u}\times\vec{a})|^2}{(\hat{\imath}\cdot\vec{u})^5}$$

Integral over θ and φ gives $P_{rad} = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right)$

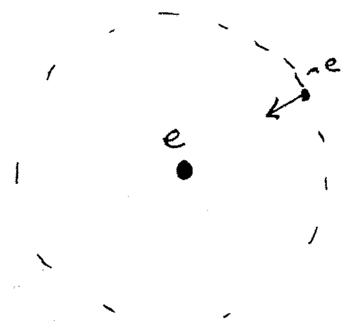
Hydrogen atom in classical physics

Rutherford's experiment $\Rightarrow$



However, the electron in circular motion radiates ($a \neq 0$) $\Rightarrow$
$\Rightarrow$ the atom eventually collapses.

Let us calculate the lifespan of the classical hydrogen atom (Problem 11.14)



$$F = ma = \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} \quad \Rightarrow \quad v^2 = \frac{1}{4\pi\epsilon_0}\frac{e^2}{mR}$$
$$ma = \frac{mv^2}{R}$$

$$\frac{v^2}{c^2} = \frac{1}{4\pi\epsilon_0 c^2}\frac{e^2}{mR} = \frac{\mu_0}{4\pi}\frac{e^2}{mR} \quad \Rightarrow \quad \frac{v}{c} = e\sqrt{\frac{\mu_0}{4\pi m R}} \simeq$$

$R \simeq 5 \cdot 10^{-11}\,m \Rightarrow \quad \simeq 3.5 \cdot 10^{-3} \ll 1 \Rightarrow$ we can use Larmor's formula

Assuming approximately circular motion

$$\frac{dE}{dt} = P = \frac{\mu_0 e^2 a^2}{6\pi c}$$

$$E = E_{kin} + E_{pot} \qquad \left.\begin{aligned} E_{kin} &= \frac{mv^2}{2} = \frac{1}{8\pi\epsilon_0}\frac{e^2}{R} \\ E_{pot} &= -\frac{1}{4\pi\epsilon_0}\frac{e^2}{R}\end{aligned}\right\} \; E = -\frac{1}{8\pi\epsilon_0}\frac{e^2}{R}$$

$$\Rightarrow \frac{e^2}{8\pi\epsilon_0}\frac{dR/dt}{R^2} = \frac{\mu_0 e^2}{6\pi c}\left(\frac{1}{4\pi\epsilon_0}\frac{e^2}{mR^2}\right)^2 \Rightarrow$$

$$\Rightarrow \frac{dR}{dt} = \frac{\mu_0 e^4}{128\pi^2 c\epsilon_0 m^2}\frac{1}{R^2} \Rightarrow R^2 dR = \frac{\mu_0 e^4}{128\pi^2 c\epsilon_0 m^2}\,dt \Rightarrow$$

$$\Rightarrow \frac{R_0^3}{3} - \frac{R^3(t)}{3} = \frac{\mu_0 e^4}{128\pi^2 c\epsilon_0 m^2}\,t \Rightarrow T = R_0^3 \cdot \frac{128\pi^2 c\epsilon_0 m^2}{3\mu_0 e^4} \simeq 1.4 \times 10^{-10}\,\text{sec.}$$

$\Rightarrow$ smth is wrong with the classical model.

In quantum mechanics, electron in the ground state does <u>not</u> radiate