HW 10 solution.

The generalized coordinate is the angle θ. The kinetic and potential energies are

$$\begin{aligned} T &= \frac{m}{2}\alpha^2 + \frac{m}{2}(l-\alpha t)^2\dot{\theta}^2 \\ V &= -mg(l-\alpha t)\cos\theta \end{aligned}$$

The Lagrangian is

$$L = \frac{m}{2}\alpha^2 + \frac{m}{2}(l-\alpha t)^2\dot{\theta}^2 + mg(l-\alpha t)\cos\theta$$

The canonical momentum is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m(l-\alpha t)^2\dot{\theta}$$

so the Hamiltonian is

$$H = p_\theta\dot{\theta} - L = \frac{m}{2}(l-\alpha t)^2\dot{\theta}^2 - mg(l-\alpha t)\cos\theta - \frac{m}{2}\alpha^2$$

It is obviously not equal to $T + V = \frac{m}{2}(l-\alpha t)^2\dot{\theta}^2 - mg(l-\alpha t)\cos\theta + \frac{m}{2}\alpha^2$

To check conservation of energy we can use Eq. (6.74) from the lecture notes

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + [H,H] = \frac{\partial H}{\partial t} = -m\alpha(l-\alpha t)\dot{\theta}^2 + mg\alpha\cos\theta$$

so the energy is not conserved. This agrees with common sense: the person who is pulling up the string is doing some work.