Phys. 807 — Statistical Mechanics

Solution.

**(a)**

Suppose the wheel moves with velocity $v$ to the right and consider the point on the rim which was passing origin at $t = 0$. At time $t$ the wheel rotates at the angle $\theta = \omega t = \frac{v}{a} t$ clockwise so in whee's frame the coordinates of the point would be

$$x' \;=\; -\,a \sin \omega t, \qquad y' \;=\; a(1 - \cos \omega t)$$

However, during time $t$ the center of the wheel shifts to the right at $x_0 = vt = \omega a$ so the coordinates of the point in rest frame are

$$x \;=\; x' + \omega t \;=\; x' \;=\; -\,\omega t - a \sin \omega t, \qquad y' \;=\; a(1 - \cos \omega t)$$

Eliminating $t$ in favor of $\theta = \omega t$ we get

$$x \;=\; a(\theta - \sin \theta), \qquad y \;=\; a(1 - \cos \theta)$$

The cycloid turned upside down is

$$x \;=\; a(\theta - \sin \theta), \qquad y \;=\; a(-1 + \cos \theta) \;=\; -\,2a \sin^2 \frac{\theta}{2} \tag{1}$$

Next, using

$$\begin{aligned} dx \;&=\; a(1 - \cos \theta) d\theta \;= 2a \sin^2 \frac{\theta}{2} d\theta \\ dy \;&=\; -\,a \sin \theta \;=\; -\,2a \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta \end{aligned}$$

we get the element of the length of the cycloid

$$ds \;=\; \sqrt{dx^2 + dy^2} \;=\; 2a \sin \frac{\theta}{2} d\theta$$

so the distance measured along the cycloid from the lowest point is

$$s \;=\; \int_0^s ds' \;=\; 2a \int_0^\theta d\theta' \sin \frac{\theta'}{2} \;=\; 4a \cos \frac{\theta}{2} \tag{2}$$

**(b)**

The equation for brachistochrone is (3.78) from the lecture notes

$$y'' \;=\; -\frac{1+y'^2}{2y}$$

From Eq. (2) we get

$$y' \;=\; \frac{dy}{dx} \;=\; -\cot\frac{\theta}{2} \tag{3}$$

$$y'' \;\equiv\; \frac{d}{dx}y' \;=\; -\frac{d}{dx}\cot\frac{\theta}{2} \;=\; \frac{\frac{d\theta}{dx}}{2\sin^2\frac{\theta}{2}} \;=\; \frac{1}{2\sin^2\frac{\theta}{2}\frac{dx}{d\theta}} \;=\; \frac{1}{4a\sin^4\frac{\theta}{2}}$$

and it is easy to see that our $y(x)$ from Eq. (1) and $y'(x)$ and $y''(x)$ from Eq. (3) satisfy the above brachistochrone equation.

**(c)**

Using $s$ from Eq. (2) as a generalized coordinate we get the Lagrangian in the form

$$L \;=\; \frac{m}{2}\dot{s}^2 - mgy \;=\; \frac{m}{2}\dot{s}^2 + 2amg\sin^2\frac{\theta}{2} \;=\; \frac{m}{2}\dot{s}^2 + 2amg\Big(1-\frac{s^2}{16a^2}\Big)$$

The constant $2amg$ is just a shift of potential energy so we can use

$$L \;=\; \frac{m}{2}\dot{s}^2 - mg\frac{s^2}{8a}$$

instead. This is the Lagrangian for harmonic oscillator with $\omega \;=\; \sqrt{\frac{g}{4a}}$