807 Midterm (20 points). 10/25/16, 10:50 a.m. - 12:20 p.m.

Problem 1.

A particle of mass m_1 moving from infinity with velocity v_0 and impact parameter b scatters off a target particle of mass m_2 initially at rest. Find the minimum distance between the particles for the repulsive interaction potential

$$U(r) \ = \ \frac{\alpha}{r^2}$$

Solution

The problem is equivalent to the scattering of a particle with effective mass $m = \frac{m_1 m_2}{m_1+m_2}$ in the central field with $U(r) \ = \ \frac{\alpha}{r^2}$.

The kinetic energy of the "effective particle" is $\frac{\mu\dot{r}^2}{2}$ and the effective potential is

$$U_{\text{eff}} \ = \ \frac{\alpha}{r^2} + \frac{L^2}{2\mu r^2} \ =$$

where $L = \mu v b$ is the angular momentum. At minimal distance r_{min} there is no kinetic energy so the conservation of energy gives

$$\frac{\mu v^2}{2} \ = \ \frac{\alpha}{r_{\text{min}}^2} + \frac{L^2}{2\mu r_{\text{min}}^2} \qquad \Rightarrow \quad r_{\text{min}} \ = \ b\sqrt{1 + \frac{2\alpha}{\mu v^2 b^2}}$$

Solution #2

Let us solve this problem in c.m. frame $m_1\vec{r}_1 + m_2\vec{r}_2 = 0$, $\vec{r} = \vec{r}_1 - \vec{r}_2$. We have a scattering of two particles: particle "1" at impact parameter $b_1 \ = \ \frac{m_2}{M}b$ with velocity at ∞ $v_{10} \ = \ \frac{m_2}{M}v_0$ and particle "2" with $b_2 \ = \ -\frac{m_1}{M}$ and $v_{20} \ = \ -\frac{m_1}{M}v_0$

$$\vec{R} \ = \ \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \ = \ \frac{m_1\vec{r}_1}{m_1+m_2} \quad \dot{\vec{R}} \ = \ \frac{m_1\vec{v}}{m_1+m_2}$$

where $M \equiv m_1 + m_2$. Conservation of energy reads

$$E \ = \ \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} + V(r)$$

Let us express it in terms of μ and $\vec{r}$ (and $\vec{v} = \dot{\vec{r}}$). Since

$$\vec{r}_1 = \frac{m_2}{M}\vec{r}, \quad \vec{v}_1 = \frac{m_2}{M}\vec{v}, \qquad \vec{r}_2 = -\frac{m_1}{M}\vec{r}, \quad v_2 = -\frac{m_2}{M}\vec{v},$$
$$L_1 = m_1 v_{10} b_1 = \frac{m_2}{M}\mu v b, \qquad L_2 = m_2 v_{20} b_2 = \frac{m_1}{M}\mu v b$$

we get

$$E \ = \ \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} \ = \ \frac{\mu v^2}{2} + V(r) \ = \ \frac{\mu\dot{r}^2}{2} + \frac{\mu r^2\dot{\phi}^2}{2} + V(r)$$

Conservation of angular momentum reads

$$L_1 + L_2 \;=\; m_1 r_1^2 \dot\phi + m_2 r_2^2 \dot\phi \;=\; \mu r^2 \dot\phi \;=\; \mu v b \quad \Rightarrow \quad \dot\phi \;=\; \frac{vb}{r^2}$$

so we get

$$E \;=\; \frac{\mu \dot r^2}{2} + \frac{\mu r^2 \dot\phi^2}{2} + V(r) \;=\; \frac{\mu \dot r^2}{2} + \frac{\mu v^2 b^2}{2r^2} + \frac{\alpha}{r^2}$$

At r_{min} we have $\dot r = 0$ which again results in

$$\frac{\mu v^2}{2} \;=\; \frac{\alpha}{r_{\text{min}}^2} + \frac{\mu v^2 b^2}{2 r_{\text{min}}^2} \qquad \Rightarrow \qquad r_{\text{min}} \;=\; b\sqrt{1 + \frac{2\alpha}{\mu v^2 b^2}}$$

Problem 2.

Imagine that you are on a large rotating disk, with the angular velocity $\vec\omega$ perpendicular to the disk. Assume that you know that the center of the disk is fixed, no real forces act on anything on the disk, and $\vec\omega$ changes only in magnitude. Suppose that at time $t = 0$ you release a point mass m and observe that it moves away in the radial direction. If the angular velocity at $t = 0$ was ω_0, what is the magnitude and direction of Coriolis force at a later time t?

Solution # 1

If the point mass is moving radially the sum of forces in $\hat\phi$ direction is zero:

$$2m(\vec\omega \times \dot{\vec r}) + m(\dot{\vec\omega} \times \vec r)$$

Since $\vec\omega \perp \vec r$

$$2\omega \dot r \;=\; -\,\dot\omega r \quad \Rightarrow \quad \frac{d}{dt}\omega r^2 \;=\; 0 \quad \Rightarrow \quad \omega r^2 \;=\; \omega_0 a^2 \quad \Rightarrow \quad r(t) \;=\; a\sqrt{1 + \omega_0^2 t^2}$$

so Newton's 2nd law (2.28) turns to

$$m\ddot r \;=\; m\omega^2 r \;=\; m\omega_0^2 \frac{a^4}{r^3} \quad \Rightarrow \quad \frac{d}{dt}\dot r^2 \;=\; -\frac{d}{dt}\frac{\omega_0^2 a^4}{r^2} \quad \Rightarrow \quad \dot r^2 \;=\; -\frac{\omega_0^2 a^4}{r^2} + \text{const}$$

At $t = 0$ $\dot r = 0$ so const $= \omega_0^2 a^2$ and

$$\dot r \;=\; \omega_0^2 a^2 \frac{r^2 - a^2}{r^2} \quad \Rightarrow \quad 2\omega_0^2 a^2 dt \;=\; \frac{dr^2}{\sqrt{r^2 - a^2}} \quad \Rightarrow \quad r(t) \;=\; a\sqrt{1 + \omega_0^2 t^2}$$

and the Coriolis force at time t is

$$2m(\vec\omega \times \dot{\vec r}) \;=\; 2m\omega \dot r \hat\phi \;=\; 2m\omega_0 \frac{a^2}{r^2} \dot r \hat\phi \;=\; \frac{4m\omega_0^3 a t}{(1 + \omega_0^2 t^2)^{3/2}}$$

Solution # 2

In the inertial frame the point mass moves with constant velocity $v = \omega_0 a$. Suppose at $t = 0$ the point was at $x = a, y = 0$, then at later time t it will be at the coordinates $x = a, y = \omega_0 a t$ in Cartesian or $r = a\sqrt{1 + \omega_0^2 t^2}$, $\phi = \arccos \frac{1}{\sqrt{1+\omega_0^2 t^2}}$. Since the disk moves in such a way that the point is always moving in radial direction, the angular velocity of the disk is

$$\omega(t) \;=\; \dot{\phi}(t) \;=\; \frac{d}{dt}\arccos\frac{1}{\sqrt{1+\omega_0^2t^2}} \;=\; \frac{\omega_0}{1+\omega_0^2t^2}$$

and the Coriolis force at time t is

$$2m(\vec{\omega}\times\dot{\vec{r}}) \;=\; 2m\omega\dot{r}\hat{\phi} \;=\; \frac{4m\omega_0^3at}{(1+\omega_0^2t^2)^{3/2}}\hat{\phi}$$

Problem 3.

Two point masses M and m are connected by the (massless) rope of length l. Th rope is suspended from a small hole in the frictionless table as shown below. The mass M can move (freely) only up or down while the mass m is free to move on the table surface.

1. Write the Lagrangian and Euler-Lagrange equations for this system.

2. At time $t = 0$ the mass m is at distance $r_0 < l$ from the hole and its velocity is v_0 in the direction orthogonal to the rope. Find at which v_0 the motion is circular

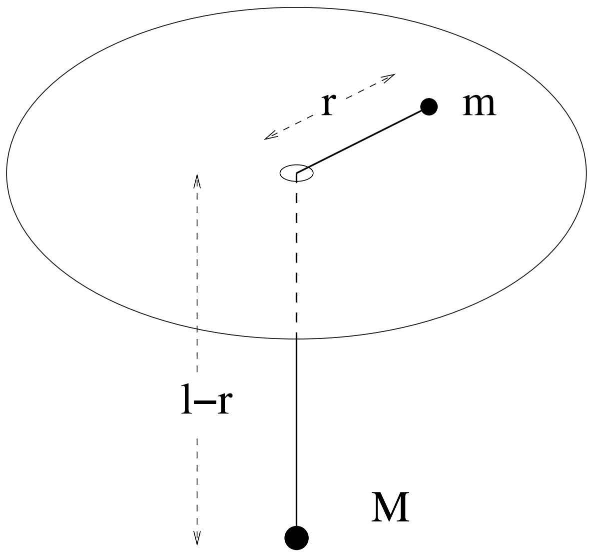

Figure 1. Projectile motion.

Solution

1. Let us choose r and ϕ as generalized coordinates. The kinetic and potential energies are

$$T \;=\; \frac{m}{2}(\dot{r}^2+r^2\dot{\phi}^2)+\frac{M}{2}\dot{r}^2, \quad V = Mg(r-l)$$

$\Rightarrow$

$$L \;=\; T-V \;=\; \frac{m}{2}(\dot{r}^2+r^2\dot{\phi}^2)+\frac{M}{2}\dot{r}^2+Mg(l-r)$$

The generalized momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = (M+m)\dot{r}, \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

Since $\frac{\partial L}{\partial r} = mr\dot{\phi}^2 - Mg$, $\frac{\partial L}{\partial \phi} = 0$ the Euler-Lagrange equations take the form

$$\begin{aligned} \dot{p}_r &= mr\dot{\phi}^2 - Mg \qquad \Rightarrow \qquad (m+M)\ddot{r} \;=\; mr\dot{\phi}^2 - Mg \\ \dot{p}_\phi &= 0 \end{aligned}$$

3. From the last equation we see that $p_\phi = mr^2\dot{\phi} = L$ is constant $\Rightarrow \dot{\phi} = \frac{L}{mr^2}$ and the first Euler-Lagrange eqn. takes the form

$$(m+M)\ddot{r} \;=\; \frac{L^2}{mr^3} - Mg$$

Thus, in the radial direction the particle moves under the influence of the "effective force" $F_r = \frac{L^2}{mr^3} - Mg$ which corresponds to the "effective potential"

$$V_{\text{eff}} = \frac{L^2}{2mr^2} + Mgr$$

The condition for the circular orbit is $F_r = \frac{L^2}{mr^3} - Mg = 0 \Rightarrow v_0 = \sqrt{\frac{Mgr_0}{m}}$