Contents

Chapter 4

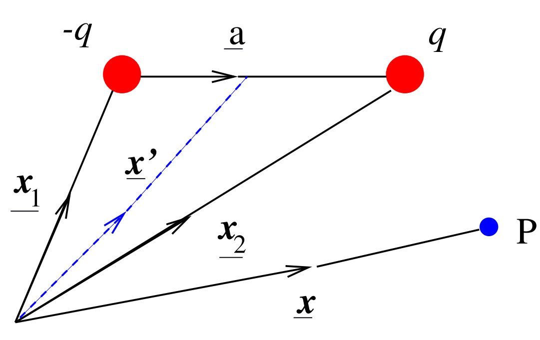
Multipoles and the Electrostatics of Macroscopic Media

The simplest source for an electrostatic field is a **point charge**; such a source is sometimes known as a **pole**. The arrangement of *two* point charges, of equal but opposite sign, is known as a **dipole**. The concept of a dipole plays a crucial rôle in electrostatics:

- Even in the case of a neutral atom or molecule, the positive and negative charges can become *separated*, e.g. by an applied external electric field. In that case, the atom or molecule gives rise to an electrostatic field that can be approximated by a **dipole.**
- The concept of dipoles, and, more generally, **multipoles**, leads to an important method for obtaining the electrostatic field and potential far from a charge distribution, the **multipole expansion**.

4.1 Introduction and Revision: Electric Dipoles

Consider two charges $-q$ and q at $\mathbf{x_1}$ and $\mathbf{x_2}$ respectively, and let $\mathbf{a}$ be the position vector of q relative to $-q$.



Let $\mathbf{x}'$ be the mid point of the dipole, so that

$$\begin{aligned}\mathbf{x_1} &= \mathbf{x}' - \mathbf{a}/2 \\ \mathbf{x_2} &= \mathbf{x}' + \mathbf{a}/2\end{aligned}$$

Then the potential at the point P is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mid \mathbf{x}-\mathbf{x_2}\mid} + \frac{-q}{\mid \mathbf{x}-\mathbf{x_1}\mid}\right) = \frac{1}{4\pi\epsilon_0} q\left(\frac{1}{\mid \mathbf{x}-\mathbf{x}'-\mathbf{a}/2\mid} - \frac{1}{\mid \mathbf{x}-\mathbf{x}'+\mathbf{a}/2\mid}\right) \tag{4.1.1}$$

We will now consider the case where the separation between the charges is much less than the distance of the point P from the charges, i.e. $|\mathbf{a}| \ll \mid \mathbf{x}-\mathbf{x}' \mid$. Then we have

$$\begin{aligned}|\mathbf{x}-\mathbf{x}' \pm \mathbf{a}/2|^{-1} &= \left\{|\mathbf{x}-\mathbf{x}'|^2 + \frac{|\mathbf{a}|^2}{4} \pm \mathbf{a}\cdot(\mathbf{x}-\mathbf{x}')\right\}^{-1/2} \\ &= |\mathbf{x}-\mathbf{x}'|^{-1}\left\{1 \pm \frac{\mathbf{a}\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2} + \frac{|\mathbf{a}|^2}{4|\mathbf{x}-\mathbf{x}'|^2}\right\}^{-1/2}.\end{aligned}$$

Expanding as a series in $|\mathbf{a}|^2/|\mathbf{x}-\mathbf{x}'|^2$ using the *binomial expansion*, we obtain

$$|\mathbf{x}-\mathbf{x}' \pm \mathbf{a}/2|^{-1} = |\mathbf{x}-\mathbf{x}'|^{-1}\left\{1 + \left(-\frac{1}{2}\right)\left[\pm\frac{\mathbf{a}\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}\right] + O\left(\frac{|\mathbf{a}|^2}{|\mathbf{x}-\mathbf{x}'|^2}\right)\right\} \tag{4.1.2}$$

Thus

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{q\mathbf{a}\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{4.1.3}$$

We now take the limit $|\mathbf{a}| \to 0$, $q \to \infty$, with $\mathbf{a}q = \mathbf{p}$ fixed and finite. This defines a **simple** or **ideal dipole** and we have

$$\boxed{\Phi_D(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}} \tag{4.1.4}$$

- $\mathbf{p}$ is the **vector moment** or **dipole moment** of the dipole.
- $\Phi_D(\mathbf{x})$ is the potential at $\mathbf{x}$ due to a dipole of moment $\mathbf{p}$ at $\mathbf{x}'$.

We can obtain the **electrostatic field** due to a dipole by applying $\mathbf{E}(\mathbf{x}) = -\nabla\Phi_D(\mathbf{x})$, and obtain

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{3(\mathbf{p}\cdot\mathbf{x})\,\mathbf{x} - r^2\mathbf{p}}{r^5} \tag{4.1.5}$$

for a dipole at the origin (useful relations: $\nabla r = \mathbf{x}/r$, $\nabla(\mathbf{p}\cdot\mathbf{x}) = \mathbf{p}$).

4.1.1 Dipole in External Electrostatic Field

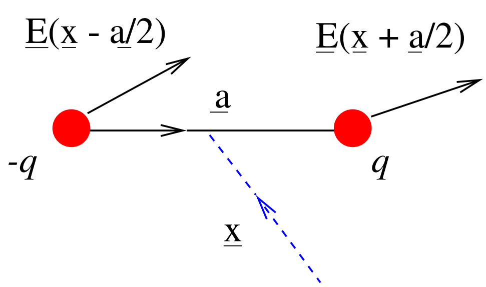
In this subsection, we will consider not the field due to a dipole, but rather the energy and forces on a dipole in an **external** field $\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x})$.

Potential Energy of Dipole in External Electrostatic Field

Recall from Section 3.5 that for a charge q in an **electrostatic potential** $\Phi(\mathbf{x})$, the **potential energy** is

$$U(\mathbf{x}) = q\Phi(\mathbf{x}) \tag{4.1.6}$$

Let us now apply this to the case of a **dipole** in an external field; once again, $\mathbf{a}$ is the separation of the charge q from $-q$.



The potential energy of the dipole is

$$U_D(\mathbf{x}) = (-q)\Phi(\mathbf{x} - \mathbf{a}/2) + q\Phi(\mathbf{x} + \mathbf{a}/2). \tag{4.1.7}$$

If the separation between the charges is small, we can expand about $\mathbf{x}$ to obtain

$$\begin{aligned} \Phi(\mathbf{x} \pm \mathbf{a}/2) = & \\ & \Phi(\mathbf{x}) \pm \frac{1}{2} a_i \frac{\partial}{\partial x_i}\Phi(\mathbf{r}) + \frac{1}{2!}\frac{a_i a_j}{4}\frac{\partial^2}{\partial x_i\,\partial x_j}\Phi(\mathbf{x}) + \ldots \\ = \; & \Phi(\mathbf{x}) \pm \frac{1}{2}\,\mathbf{a}\cdot\nabla\Phi(\mathbf{x}) + O(a^2) \end{aligned}$$

Thus we have

$$\begin{aligned} U_D(\mathbf{x}) &= q\,[\Phi(\mathbf{x}) + \frac{1}{2}\,\mathbf{a}\cdot\nabla\Phi(\mathbf{x}) - \Phi(\mathbf{x}) + \frac{1}{2}\,\mathbf{a}\cdot\nabla\Phi(\mathbf{x}) + O(a^3)] \\ &= q\,\mathbf{a}\cdot\nabla\Phi(\mathbf{x})[1 + O(a^2)] \end{aligned}$$

Now take the **point dipole limit**, $a \to 0$, $q \to \infty$, $\mathbf{a}q = \mathbf{p}$ fixed. Then

$$U_D(\mathbf{x}) = \mathbf{p}\cdot\nabla\Phi(\mathbf{x}) \tag{4.1.8}$$

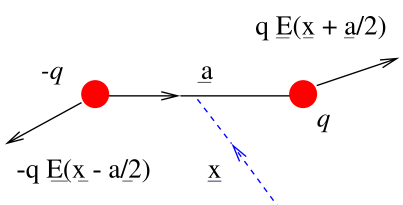
Aside: why did we take $\mathbf{x}$ to be at the *mid point* of the dipole? Because for a simple dipole, all the corrections to the formula above involving *even* derivatives of $\Phi(\mathbf{x})$ vanish. It just makes the expansion *neater*, but of course we could have performed the expansion about any point between the charges.
Recalling that $\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x})$, we have

$$U_D(\mathbf{x}) = -\mathbf{p}\cdot\mathbf{E}(\mathbf{x}) \qquad (4.1.9)$$

Note that the potential energy of a dipole has a **minimum** when $\mathbf{E}$ and $\mathbf{p}$ are **parallel**

Force on Dipole in External Electrostatic Field

We will now consider the force on an electric dipole.



The force on the dipole is

$$\mathbf{F_D}(\mathbf{x}) = -q\mathbf{E}(\mathbf{x}-\mathbf{a}/2) + q\mathbf{E}(\mathbf{x}+\mathbf{a}/2) \qquad (4.1.10)$$

Once again, we can expand about $\mathbf{r}$:

$$\mathbf{E}(\mathbf{x}\pm\mathbf{a}/2) = \mathbf{E}(\mathbf{x}) \pm \frac{1}{2}(\mathbf{a}\cdot\nabla)\mathbf{E} + O(a^2). \qquad (4.1.11)$$

We thus obtain

$$\mathbf{F_D}(\mathbf{x}) = q(\mathbf{a}\cdot\nabla)\mathbf{E}(\mathbf{x}) = (\mathbf{p}\cdot\nabla)\mathbf{E}(\mathbf{x}) \qquad (4.1.12)$$

Now since $\mathbf{E}(\mathbf{x})$ is an electrostatic field, it is *irrotational*:

$$\nabla\times\mathbf{E}(\mathbf{x}) = 0. \qquad (4.1.13)$$

Recall the identity for a double vector product

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \qquad (4.1.14)$$

which (for $\mathbf{A}=\mathbf{p}$, $\mathbf{B}=\nabla$ and $\mathbf{C}=\mathbf{E}(\mathbf{x})$) gives

$$\nabla(\mathbf{p}\cdot\mathbf{E}(\mathbf{x})) = \mathbf{p}\times(\nabla\times\mathbf{E}(\mathbf{x})) + (\mathbf{p}\cdot\nabla)\mathbf{E}(\mathbf{x})\ . \qquad (4.1.15)$$

Since $\nabla\times\mathbf{E}(\mathbf{x}) = 0$, we have

$$(\mathbf{p}\cdot\nabla)\mathbf{E}(\mathbf{x}) = \nabla(\mathbf{p}\cdot\mathbf{E}(\mathbf{x}))\ , \qquad (4.1.16)$$

Applying this result to equation (4.1.12) gives

$$\mathbf{F_D}(\mathbf{x}) = \nabla(\mathbf{p}\cdot\mathbf{E}(\mathbf{x})) = -\nabla U_D(\mathbf{x}) \ . \tag{4.1.17}$$

Thus the force on a dipole is just minus the gradient of the potential energy, and furthermore for a *uniform* external field, independent of $\mathbf{x}$, the force is zero.

Torque on a Dipole in an External Field

We will now evaluate the **torque**, or moment of the force, τ on a simple dipole about its centre. This is just the moment of the forces acting on the two charges about the centre of the dipole:

$$\begin{aligned}\tau &= \left(\frac{1}{2}\mathbf{a}\right)\times(+q)\,\mathbf{E}(\mathbf{x}+\mathbf{a}/2) \ + \ \left(-\frac{1}{2}\mathbf{a}\right)\times(-q\,)\,\mathbf{E}(\mathbf{x}-\mathbf{a}/2) \qquad (4.1.18)\\ &= \left(\frac{1}{2}\mathbf{a}\right)\times q\left(\mathbf{E}(\mathbf{x}) \ + \ \frac{1}{2}(\mathbf{a}\cdot\nabla)\,\mathbf{E}(\mathbf{x}) \ + \ \mathbf{E}(\mathbf{x}) \ - \ \frac{1}{2}(\mathbf{a}\cdot\nabla)\,\mathbf{E}(\mathbf{x}) + O(a^2)\right)\end{aligned}$$

i.e. $\tau = \mathbf{p}\times\mathbf{E}(\mathbf{x})$ in the **point dipole limit** .

- Note that the torque about some point other than the **centre** of the dipole will be different.
- $\tau = \mathbf{p}\times\mathbf{E}(\mathbf{x})$ is true for dipoles other than point dipoles if $\mathbf{E}(\mathbf{x})$ is constant over the dipole.

4.1.2 Force between Two Dipoles

Many materials are dipolar; the positive and negative materials are separated. Here we will consider the force between a dipole $\mathbf{p_1}$ at $\mathbf{x_1}$ and $\mathbf{p_2}$ at $\mathbf{x_2}$. The force $\mathbf{F_{21}}$ on the dipole at $\mathbf{x_2}$ due to the electrostatic field $\mathbf{E_1}$ produced by the dipole $\mathbf{p_1}$ is

$$\begin{aligned}\mathbf{F}_{21}(\mathbf{x}_2) &= (\mathbf{p}_2\cdot\nabla_2)\,\mathbf{E}_1(\mathbf{x}_2)\\ &= (\mathbf{p}_2\cdot\nabla_2)\,\frac{1}{4\pi\epsilon_0}\left\{\frac{3\,(\mathbf{p}_1\cdot(\mathbf{x}_2-\mathbf{x}_1))\ (\mathbf{x}_2-\mathbf{x}_1) \ - \ \mathbf{p}_1\,|\mathbf{x}_2-\mathbf{x}_1|^2}{|\mathbf{x}_2-\mathbf{x}_1|^5}\right\} \qquad (4.1.19)\end{aligned}$$

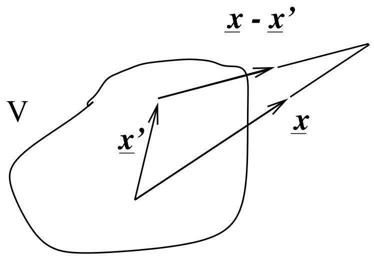
where ∇_2 means that we take derivatives with respect to $\mathbf{x}_2$ (the position vector of dipole $\mathbf{p}_2$), and we have used Eq. (4.1.5). As discussed above, we can express this also as

$$\begin{aligned}\mathbf{F_{21}}(\mathbf{x_2}) &= \nabla_2(\mathbf{p_2}\cdot\mathbf{E_1}(\mathbf{x_2}))\\ &= \frac{1}{4\pi\epsilon_0}\nabla_2\left\{\frac{3\,(\mathbf{p}_1\cdot(\mathbf{x}_2-\mathbf{x}_1))\ (\mathbf{p}_2\cdot(\mathbf{x}_2-\mathbf{x}_1)) \ - \ (\mathbf{p}_1\cdot\mathbf{p}_2)\,|\mathbf{x}_2-\mathbf{x}_1|^2}{|\mathbf{x}_2-\mathbf{x}_1|^5}\right\} \ . \qquad (4.1.20)\end{aligned}$$

From this representation, it is evident that the force $\mathbf{F_{12}}(\mathbf{x_1})$ is *equal and opposite* to $\mathbf{F_{21}}(\mathbf{x_2})$.

4.2 Multipole Expansion

In this section, we will see why the concept of dipoles, and more generally multipoles, is so important in electrostatics. Consider the case of a charge distribution, localized to some volume V. For convenience we will take the origin for our vectors inside V.



We have that the potential due to the charge distribution within V at a point P outside the volume is:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{x}')\,dV'}{|\mathbf{x}-\mathbf{x}'|} \tag{4.2.1}$$

For $r \equiv |\mathbf{x}|$ much larger than the extent of V, i.e. $r \gg r' \equiv |\mathbf{x}'|$ for all $\mathbf{x}'$ such that $\rho(\mathbf{x}') \neq 0$, we can expand the denominator

$$\begin{aligned}
|\mathbf{x}-\mathbf{x}'|^{-1} &= \{r^2 - 2\mathbf{x}\cdot\mathbf{x}' + r'^2\}^{-1/2} \\
&= r^{-1}\left\{1 - 2\frac{\mathbf{x}\cdot\mathbf{x}'}{r^2} + \frac{r'^2}{r^2}\right\}^{-1/2} \\
&= \frac{1}{r}\left\{1 + \frac{\mathbf{x}\cdot\mathbf{x}'}{r^2} - \frac{r'^2}{2r^2} + \frac{3}{2}\left(\frac{\mathbf{x}\cdot\mathbf{x}'}{r^2}\right)^2 + \mathcal{O}(r'^3/r^3)\right\}
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \frac{1}{r} + \frac{\mathbf{x}\cdot\mathbf{x}'}{r^3} + \frac{3(\mathbf{x}\cdot\mathbf{x}')^2 - r^2r'^2}{2r^5} + \mathcal{O}(r'^3/r^4) \\
&= \frac{1}{r} + \frac{1}{r^3}\sum_{i=1}^{3} x_i x_i' + \frac{x_i x_j}{2r^5}\sum_{i,j=1}^{3}(3x_i'x_j' - r'^2\delta_{ij}) + \mathcal{O}(r'^3/r^4)\ .
\end{aligned} \tag{4.2.2}$$

Hence we can write

$$\boxed{\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{P}\cdot\mathbf{x}}{r^3} + \frac{1}{2}\sum_{i,j=1}^{3} Q_{ij}\frac{x_i x_j}{r^5} + O(1/r^4)\right) \qquad (4.2.3)}$$

where

$Q = \int_V \rho(\mathbf{x}')dV'$ is the **total charge** within V

$\mathbf{P} = \int_V \rho(\mathbf{x}')\mathbf{x}'dV'$ is the **dipole moment** of the charge distribution about the origin

$Q_{ij} = \int_V \rho(\mathbf{x}')(3x_i'x_j' - r'^2\delta_{ij})dV'$ is the **quadrupole moment** of the charge distribution.

- We have defined the moments with respect to a particular point, e.g. the **dipole moment** is the integral of the **displacement** $\mathbf{x}'$ times the charge density $\rho(\mathbf{x}')$. In general, the moments depend on the choice of "origin". *What about the total dipole moment when the total charge is zero?*
- At large distances from the charge distribution, only the first few moments (Q, $\mathbf{P}$, **quadrupole moment**, ...) are important.
- For a **neutral** charge distribution, the leading behaviour is given by the dipole moment.

Example:

The region inside the sphere: $r < a$, contains a charge density

$$\rho(x, y, z) = f\, z\, (a^2 - r^2) \tag{4.2.4}$$

where f is a constant. Show that at large distances from the origin the potential due to the charge distribution is given approximately by

$$\Phi(\mathbf{x}) = \frac{2f\, a^7}{105\epsilon_0} \frac{z}{r^3} \tag{4.2.5}$$

Use the multipole expansion in SI units:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{r} + \frac{\mathbf{P}\cdot\mathbf{x}}{r^3} + O\left(\frac{1}{r^3}\right)\right) \tag{4.2.6}$$

In **spherical polars** (r, θ, φ),

$$x = r\sin\theta\cos\varphi \quad ; \quad y = r\sin\theta\sin\varphi \quad ; \quad z = r\cos\theta\ . \tag{4.2.7}$$

The **total charge** Q is

$$Q = \int_V \rho(\mathbf{x})\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \Big(fr\cos\theta\,(a^2 - r^2)\Big)\, r^2 \sin\theta\, dr\, d\theta\, d\varphi = 0. \tag{4.2.8}$$

The integral vanishes because

$$\int_0^{\pi} \cos\theta\sin\theta\, d\theta = -\int_{\theta=0}^{\theta=\pi} \frac{1}{2}\, d(\cos^2\theta)) = -\frac{1}{2}\cos^2\theta\Big|_{\theta=0}^{\theta=\pi} = 0. \tag{4.2.9}$$

The **total dipole moment P** about the origin is

$$\begin{aligned}\mathbf{P} &= \int_V \mathbf{x}\,\rho(\mathbf{x})\, dV = \int_V r\,\hat{\mathbf{e}}_{\mathbf{r}}\,\rho(\mathbf{x})\, dV \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r\,(\sin\theta\cos\varphi\,\mathbf{i} + \sin\theta\sin\varphi\,\mathbf{j} + \cos\theta\,\mathbf{k}) \\ &\qquad\qquad \Big(fr\cos\theta\,(a^2 - r^2)\Big)\, r^2\sin\theta\, dr\, d\theta\, d\varphi.\end{aligned}$$

The x and y components of the φ integral vanish. The z component factorizes:

$$P_z \;=\; f \;\times \int_0^{2\pi} d\varphi \times \int_0^{\pi} \sin\theta \cos^2\theta \, d\theta \times \int_0^{a} r^4 \, (a^2 \;-\; r^2) \, dr \;=\; f \times 2\pi \times \frac{2}{3} \times \frac{2a^7}{35}. \tag{4.2.10}$$

Putting it all together, we obtain

$$\Phi(\mathbf{x}) \;=\; \frac{1}{4\pi\epsilon_0} \, \frac{8\pi a^7 f}{105} \, \frac{\mathbf{k}\cdot\mathbf{x}}{r^3} \;=\; \frac{2f\,a^7}{105\epsilon_0} \, \frac{z}{r^3}. \tag{4.2.11}$$

4.2.1 Multipole Expansion using Spherical Harmonics

To proceed further, we go back to our expansion of a pole in *spherical harmonics*

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\varphi') Y_{lm}(\theta,\varphi). \tag{4.2.12}$$

We assume that the charge is confined to a sphere of radius a, and take the centre of the sphere to be the origin for our vectors. Then for the case $r > a$, we have

$$\begin{aligned} r_< &= r' \\ r_> &= r, \end{aligned}$$

and we have

$$\Phi(\mathbf{x}) = \frac{1}{\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}} \int d\Omega' \int dr'\, r'^2 Y_{lm}^*(\theta',\varphi') r'^l \rho(\mathbf{x}'). \tag{4.2.13}$$

We now write

$$q_{lm} = \int d\Omega' dr' r'^2 Y_{lm}^*(\theta',\varphi') r'^l \rho(\mathbf{x}') \tag{4.2.14}$$

so that the expansion may be written

$$\Phi(\mathbf{x}) = \frac{1}{\epsilon_0} \sum_{l,m} \frac{1}{2l+1} q_{lm} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}. \tag{4.2.15}$$

This is the multipole expansion using spherical harmonics. Up to $l = 2$, the combinations $r'^l Y_{lm}(\theta', \varphi')$ are given by

$$\begin{aligned}
Y^*_{00}(\theta', \varphi') =& \frac{1}{\sqrt{4\pi}} \\
r'\, Y^*_{11}(\theta', \varphi') =& - r'\sqrt{\frac{3}{8\pi}}\, \sin\theta'\, e^{-i\varphi'} = -\sqrt{\frac{3}{8\pi}}\,(r'\sin\theta'\, \cos\varphi' - ir'\sin\theta'\, \sin\varphi') \\
=& -\sqrt{\frac{3}{8\pi}}\,(x' - iy') \\
r'\, Y^*_{1,-1}(\theta', \varphi') =& - r'\sqrt{\frac{3}{8\pi}}\, \sin\theta'\, e^{i\varphi'} = -\sqrt{\frac{3}{8\pi}}\,(r'\sin\theta'\, \cos\varphi' + ir'\sin\theta'\, \sin\varphi') \\
=& -\sqrt{\frac{3}{8\pi}}\,(x' + iy') \\
r' Y^*_{10}(\theta', \varphi') =& r'\sqrt{\frac{3}{4\pi}}\, \cos\theta' = \sqrt{\frac{3}{4\pi}} z' \\
r'^2\, Y^*_{22}(\theta', \varphi') =& r'^2\, \frac{1}{4}\sqrt{\frac{15}{2\pi}}\, \sin^2\theta'\, e^{-2i\varphi'} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}(x' - iy')^2 \\
r'^2\, Y^*_{21}(\theta', \varphi') =& - r'^2\, \sqrt{\frac{15}{8\pi}}\, \cos\theta' \sin\theta'\, e^{-i\varphi'} = -\sqrt{\frac{15}{8\pi}} z'(x' - iy') \\
r'^2\, Y^*_{20}(\theta', \varphi') =& r'^2\sqrt{\frac{5}{4\pi}}\, P^0_2(\cos\theta) = r'^2\sqrt{\frac{5}{4\pi}}\, \frac{1}{2}(3\cos^2\theta - 1) = \frac{1}{2}\sqrt{\frac{5}{4\pi}}(3z'^2 - r'^2)\ .
\end{aligned}$$

To make the connection with our previous expansion, it is useful to consider the few terms in Cartesian coordinates

$$\begin{aligned}
q_{00} &= \frac{1}{\sqrt{4\pi}} \int d^3x' \rho(\mathbf{x}') = \frac{1}{\sqrt{4\pi}} Q \\
q_{11} &= -\sqrt{\frac{3}{8\pi}} \int d^3x'\, \rho(\mathbf{x}')(x' - iy') = -\sqrt{\frac{3}{8\pi}}(P_x - iP_y) \\
q_{10} &= \sqrt{\frac{3}{4\pi}} \int d^3x'\, \rho(\mathbf{x}') z' = \sqrt{\frac{3}{4\pi}} P_z \\
q_{22} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int d^3x' \rho(\mathbf{x}')(x' - iy')^2 = \frac{1}{12}\sqrt{\frac{15}{2\pi}}(Q_{11} - 2iQ_{12} - Q_{22}) \\
q_{21} &= -\sqrt{\frac{15}{8\pi}} \int d^3x' \rho(\mathbf{x}') z'(x' - iy') = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}(Q_{13} - iQ_{23}) \\
q_{20} &= \frac{1}{2}\sqrt{\frac{5}{4\pi}} \int d^3x'\, \rho(\mathbf{x}')(3z'^2 - r'^2) = \frac{1}{2}\sqrt{\frac{5}{4\pi}} Q_{33}.
\end{aligned}$$

Note that the components for negative m can be trivially obtained using

$$q_{l,-m} = (-1)^m q^*_{lm}. \tag{4.2.16}$$

In general, for the l-th multipole moment, there are $(l+1)(l+2)/2$ components in Cartesian coordinates, while only $2l+1$ components using spherical harmonics. There is no inconsistency here – the Cartesian tensors are **reducible** under rotations (i.e., under rotations, they mix with tensors having a fewer number of indices) whilst the tensor moments expressed in spherical harmonics are **irreducible** (i.e. the q_{lm} for fixed l mix only amongst themselves under rotations); that is why we *remove the trace* in the quadrupole moment Q_{ij}, to give us 5 irreducible components.

We can express the electric field components trivially in spherical harmonics. In particular, the contribution of definite l, m is

$$\begin{aligned} E_r = -\frac{\partial}{\partial r}\Phi &\Rightarrow \frac{1}{\epsilon_0}\frac{l+1}{2l+1}Y_{lm}(\theta,\varphi)q_{lm}\frac{1}{r^{l+2}} \\ E_\theta = -\frac{1}{r}\frac{\partial}{\partial\theta}\Phi &\Rightarrow -\frac{1}{\epsilon_0}\frac{1}{2l+1}q_{lm}\frac{1}{r^{l+2}}\frac{\partial}{\partial\theta}Y_{lm}(\theta,\varphi) \\ E_\varphi = -\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\Phi &\Rightarrow -\frac{1}{\epsilon_0}\frac{1}{2l+1}q_{lm}\frac{1}{r^{l+2}}\frac{im}{\sin\theta}Y_{lm}(\theta,\varphi). \end{aligned}$$

If we now consider the case of an ideal dipole $\mathbf{p}$ along the z-axis, then

$$\begin{aligned} q_{10} &= \sqrt{\frac{3}{4\pi}}p \\ q_{11} &= q_{1,-1} = 0 \ , \end{aligned}$$

so that only $Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$ is involved, and we have

$$\Phi(\mathbf{x}) = \frac{1}{\epsilon_0}\sum_{l,m}\frac{1}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}} = \frac{1}{\epsilon_0}\frac{1}{3}\sqrt{\frac{3}{4\pi}}\,p\,\sqrt{\frac{3}{4\pi}}\frac{\cos\theta}{r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} \tag{4.2.17}$$

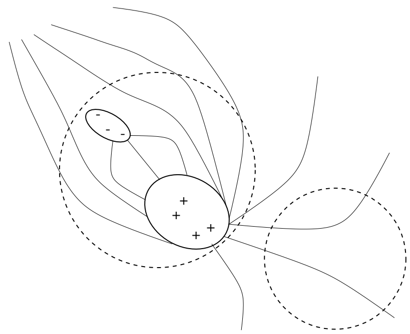
and

$$\begin{aligned} E_r &= \frac{2p\cos\theta}{4\pi\epsilon_0 r^3} \\ E_\theta &= \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \\ E_\varphi &= 0, \end{aligned}$$

which reduces to the expression we derived earlier for an ideal dipole, Eq. (4.1.5).

4.2.2 Point Dipole vs. Dipole Moment

There is a danger in using the expression for the electrostatic field due to an ideal, or point dipole. To see this, consider the electrostatic field $\mathbf{E}(\mathbf{x})$ due to a localized charge distribution $\rho(\mathbf{x})$. In partciular, consider the integral of $\mathbf{E}$ over some sphere of radius R, the center of which we will take as the origin of our vectors.



We have

$$\int_{r<R} d^3x\,\mathbf{E} = -\int_{r<R} d^3x\,\nabla\Phi = -R^2\int d\Omega\;\Phi(\mathbf{x})\mathbf{n} \tag{4.2.18}$$

where $\mathbf{n} = \mathbf{x}/R$ is a unit normal outward from the surface of the sphere, and we have used the generalization of the divergence theorem.

Using Coulomb's law for an extended charge distribution, we may write

$$\int d^3x\,\mathbf{E} = -\frac{R^2}{4\pi\epsilon_0}\int d^3x'\,\rho(\mathbf{x}')\int d\Omega\,\frac{\mathbf{n}}{|\mathbf{x}-\mathbf{x}'|}. \tag{4.2.19}$$

Now we can evaluate the x integration by writing the vector $\mathbf{n} = \mathbf{i}\sin\theta\cos\varphi + \mathbf{j}\sin\theta\sin\varphi + \mathbf{k}\cos\theta$, and then expressing these terms in spherical harmonics as

$$\begin{aligned}
\sin\theta\cos\varphi &= -\sqrt{\frac{8\pi}{3}}\left(\frac{Y_{11}(\theta,\varphi)+Y_{1,-1}(\theta,\varphi)}{2}\right)\\
\sin\theta\sin\varphi &= -\sqrt{\frac{8\pi}{3}}\left(\frac{Y_{11}(\theta,\varphi)-Y_{1,-1}(\theta,\varphi)}{2i}\right)\\
\cos\theta &= \sqrt{\frac{4\pi}{3}}Y_{10}(\theta,\varphi)\ .
\end{aligned}$$

As a result, we can write

$$n^i(\theta,\varphi) = \sum_{M=-1}^{1} n^i_M Y_{1M}(\theta,\varphi) \tag{4.2.20}$$

with known projection coefficients n^i_M for the unit vector whose direction is characterized by angles θ,φ.

Thus only the $l = 1$ terms contribute, and using the orthogonality property of spherical harmonics we have

$$
\begin{aligned}
\int d\Omega \, \frac{n^i}{|\mathbf{x}-\mathbf{x}'|} =& 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta',\varphi') \int d\Omega \, n^i \, Y^*_{lm}(\theta,\varphi) \\
&= 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta',\varphi') \sum_{M=-1}^{1} n^i_M \int d\Omega \, \underbrace{Y_{1M}(\theta,\varphi) Y^*_{lm}(\theta,\varphi)}_{\delta_{l1}\delta_{mM}} \\
&= \frac{4\pi}{3} \frac{r_<}{r_>^2} \sum_{M=-1}^{1} Y_{1M}(\theta',\varphi') n^i_M = \frac{4\pi}{3} \frac{r_<}{r_>^2} \, n^i(\theta',\varphi') \, , \qquad (4.2.21)
\end{aligned}
$$

i.e., the result is proportional to $\mathbf{n}'$, a unit vector in the direction of $\mathbf{x}'$. Hence we have

$$
\begin{aligned}
\int d^3x \, \mathbf{E} &= -\frac{R^2}{4\pi\epsilon_0} \int d^3x' \, \rho(\mathbf{x}') \frac{4\pi}{3} \frac{r_<}{r_>^2} \mathbf{n}' \\
&= -\frac{R^2}{3\epsilon_0} \int d^3x' \rho(\mathbf{x}') \frac{r_<}{r_>^2} \mathbf{n}' \, , \qquad (4.2.22)
\end{aligned}
$$

where $r_< = \min(r', R)$.
We now consider two cases

1. *Charge density completely outside the sphere.* Then we have $r_< = R$, $r_> = r'$, and we have

$$
\begin{aligned}
\int d^3x \mathbf{E} = -\frac{R^3}{3\epsilon_0} \int d^3x' \, \frac{\mathbf{n}'}{r'^2} \rho(\mathbf{x}') &= -\frac{R^3}{3\epsilon_0} \int d^3x' \, \frac{\mathbf{x}'}{r'^3} \rho(\mathbf{x}') \\
&= \frac{4\pi}{3} R^3 \, \mathbf{E}(\mathbf{0}).
\end{aligned}
$$

   Thus the average value of the electric field over a spherical volume containing no charge is just the value of the field at the centre of the sphere.

2. *Sphere completely encloses the charge density.* Then we have $r_< = r'$, and $r_> = R$, and we have, from Eq. (4.2.22),

$$
\int d^3x \, \mathbf{E} = -\frac{R^2}{3\epsilon_0} \int d^3x' \, \frac{r'}{R^2} \, \mathbf{n}' \, \rho(\mathbf{x}') = -\frac{1}{3\epsilon_0} \int d^3x' \, \mathbf{x}' \, \rho(\mathbf{x}') = -\frac{\mathbf{P}}{3\epsilon_0}, \qquad (4.2.23)
$$

   where $\mathbf{P}$ is the electric dipole moment. Note that this expression is independent of the size of the sphere, provided it completely encloses the dipole.

Let us now consider the corresponding expression for the integrated $\mathbf{E}$ in the case of an *ideal dipole*, Eq. (4.1.5):

$$
\int_{r<R} d^3x \mathbf{E}(\mathbf{x}) = \int_{r<R} d^3x \frac{1}{4\pi\epsilon_0} \frac{3(\mathbf{p}\cdot\mathbf{n})\mathbf{n} - \mathbf{p}}{r^3} = 0 \, . \qquad (4.2.24)
$$

To get this result, we first rewrite the integral in component notation

$$\int_{r<R} d^3x E_i(\mathbf{x}) \;=\; \frac{1}{4\pi\epsilon_0}\sum_j p_j \int_{r<R} \frac{d^3x}{r^3}(3n_j n_i - \delta_{ij}) \;.$$

Note that the integral

$$I_{ij} \equiv \int_{r<R} \frac{d^3x}{r^3}(3n_j n_i - \delta_{ij})$$

should be proportional to δ_{ij}, since there is no external vector in the integrand. Convoluting the equation $I_{ij} = A\delta_{ij}$ with δ_{ij} on both sides (with summation over i and j), we receive $A = 0$. One can also get this result by taking dipole in z-direction, $\mathbf{p} = p\mathbf{k}$, and work in spherical polars.
For Eq. (4.2.24) to be consistent with Eq. (4.2.23), our expression for the electrostatic field due to a dipole at $\mathbf{x_0}$ must be modified

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n}) - \mathbf{p}}{|\mathbf{x}-\mathbf{x_0}|^3} - \frac{4\pi}{3}\mathbf{p}\,\delta^3(\mathbf{x}-\mathbf{x_0})\right]. \tag{4.2.25}$$

This expression only changes the electric field *at the position of the dipole*, and we can then, with some care, use the expression as if we were using ideal, or point, dipoles. The δ-function contains information about the finite distribution of the charge lost in the multipole expansion.

4.3 Energy of Charge Distribution in External Electrostatic Field

The energy is given by

$$W = \int d^3x\, \rho(\mathbf{x})\Phi(\mathbf{x}). \tag{4.3.1}$$

We now suppose that Φ is slowly varying, so that

$$\begin{aligned} \Phi(\mathbf{x}) &= \Phi(0) + \mathbf{x}\cdot\nabla\Phi + \frac{1}{2}\sum_{i,j} x_i x_j \frac{\partial^2\Phi}{\partial x_i \partial x_j} + \dots \\ &= \Phi(\mathbf{0}) - \mathbf{x}\cdot\mathbf{E}(\mathbf{0}) - \frac{1}{2}\sum_{i,j} x_i x_j \frac{\partial E_i}{\partial x_j}. \end{aligned}$$

Now in the case of an *external* electrostatic field, we have $\nabla \cdot \mathbf{E} = 0$, and thus we may write

$$\begin{aligned}\Phi(\mathbf{x}) &= \Phi - \mathbf{x}\cdot\mathbf{E} - \frac{1}{2}\sum_{i,j} x_i x_j \left\{\frac{\partial E_i}{\partial x_j} - \frac{1}{3}\delta_{ij}\nabla\cdot\mathbf{E}\right\} \\ &= \Phi - \mathbf{x}\cdot\mathbf{E} - \frac{1}{6}\sum_{i,j}\left[3x_i x_j - \delta_{ij}r^2\right]\frac{\partial E_i}{\partial x_j}\end{aligned}$$

where the derivatives are evaluated at $\mathbf{0}$. Thus we have

$$\begin{aligned}W &= \int d^3x\,\rho(\mathbf{x})\left\{\Phi(0) - \mathbf{x}\cdot\mathbf{E} - \frac{1}{6}\sum_{i,j}\left[3x_i x_j - \delta_{ij}r^2\right]\frac{\partial E_i}{\partial x_j}\right\} \\ &= \Phi(0)Q - \mathbf{E}(\mathbf{0})\cdot\mathbf{P} - \frac{1}{6}\sum_{i,j} Q_{ij}\frac{\partial E_i}{\partial x_j}(\mathbf{0})\end{aligned}$$

4.4 Electrostatics with Ponderable Media

So far we have only considered the case of electrostatics in free space. We will now consider the case of macroscopic materials in the presence of electric fields. Such materials are classified according to whether or not electrons, or charges, can flow over long distances. In the case of conductors, charges can move freely about the material, and, as we have already seen, generate an induced field that exactly cancels any applied external field.

In this chapter we consider the case of **dielectics**. Here the electrons are bound to atoms, and have only limited freedom to move. The material might have an inherent dipole moment, or a dipole moment might be generated by the presence of an external electric field. The crucial property of a dielectric is that

$$\nabla \times \mathbf{E} = 0. \tag{4.4.1}$$

Thus

- We have a *conservative* electric force
- We can express the field as the gradient of a potential

In the following, we will assume the applied field induces a dipole moment, but no higher moments. Now consider the potential at $\mathbf{x}$ due to the charge, and dipole moment, of a volume ΔV at $\mathbf{x}'$:

$$\Delta\Phi(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi\epsilon_0}\left[\frac{\rho(\mathbf{x}')\Delta V}{|\mathbf{x}-\mathbf{x}'|} + \frac{\mathbf{P}(\mathbf{x}')\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\Delta V\right], \tag{4.4.2}$$

where $\mathbf{x}$ is outside the volume ΔV. The dipole moment per unit volume is called *polarization*. We now pass to an integral in the usual way, and obtain

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int_V d^3x' \left[\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \mathbf{P}(\mathbf{x}')\cdot \underbrace{\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}}_{\nabla'(1/|\mathbf{x}-\mathbf{x}'|)}\right] \quad (\textit{integ. by parts}) \\ &= \frac{1}{4\pi\epsilon_0}\int_V d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}\left[\rho(\mathbf{x}') - \nabla'\cdot\mathbf{P}(\mathbf{x}')\right] + \frac{1}{4\pi\epsilon_0}\int_{S=\partial V} dS' \frac{\mathbf{P}(\mathbf{x}')\cdot\mathbf{n}}{|\mathbf{x}-\mathbf{x}'|}\end{aligned}$$

This expression can be rewritten as follows

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x' \frac{\rho_f(\mathbf{x}') + \rho_b(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{1}{4\pi\epsilon_0}\int_{S=\partial V} dS' \frac{\sigma_b(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

where $\sigma_b \equiv \mathbf{P}\cdot\mathbf{n}$ is the surface density of the bound charge, $\rho_b \equiv -\nabla\cdot\mathbf{P}$ is the volume density of the bound charge, and the "old" charge density ρ is called the free charge density ρ_f to distinguish from the density of the bound charge.
Thus Maxwell's equation becomes

$$\nabla\cdot\mathbf{E} = -\nabla^2\Phi(\mathbf{x}) = \frac{1}{\epsilon_0}[\rho_f - \nabla\cdot\mathbf{P}]. \tag{4.4.3}$$

(we use here $-\nabla^2(1/|\mathbf{x}-\mathbf{x}'|) = 4\pi\delta^3(\mathbf{x}-\mathbf{x}')$) which we can write as

$$\nabla\cdot\mathbf{D} = \rho_f \tag{4.4.4}$$

where

$$\mathbf{D} \equiv \epsilon_0\mathbf{E} + \mathbf{P} \tag{4.4.5}$$

is the **electric displacement**. Note that $-\nabla\cdot\mathbf{P}$ is the **polarization charge density**.
We now suppose that the media is *isotropic*, i.e. no preferred direction. Then the induced dipole moment must be aligned with $\mathbf{E}$, and we set

$$\mathbf{P} = \epsilon_o\chi_e\mathbf{E} \tag{4.4.6}$$

where χ_e is the **electric susceptibility**. Thus we have

$$\mathbf{D} = \epsilon_0\mathbf{E} + \epsilon_0\chi_e\mathbf{E} = \epsilon\mathbf{E} \tag{4.4.7}$$

where $\epsilon = \epsilon_0(1+\chi_e)$. Note that ϵ/ϵ_0 is the **dielectric constant**. Finally, if the material is **uniform**, then χ_e does not depend on position, and we have

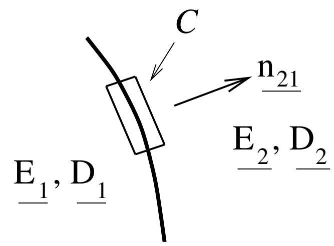
$$\mathbf{D} = \epsilon\mathbf{E}, \quad \text{with} \quad \nabla\cdot\mathbf{E} = \rho_f/\epsilon. \tag{4.4.8}$$

Thus, the potential $\Phi_q(\mathbf{x}, \mathbf{x}')$ of a charge q located at $\mathbf{x}'$ inside a uniform material having the dielectric constant ϵ is given by

$$\Phi_q(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi\epsilon} \frac{q}{|\mathbf{x} - \mathbf{x}'|} \; .$$

4.4.1 Boundary Conditions at Boundary between Materials

We will now consider the boundary conditions at the boundary between two materials, of permittivities ϵ_1 and ϵ_2, and with electric fields $\mathbf{E}_1, \mathbf{D}_1$ and $\mathbf{E}_2, \mathbf{D}_2$ respectively.



Tangential condition

We have that $\nabla \times \mathbf{E} = 0$, and thus, applying Stoke's theorem to the closed curve C shown above, we have

$$\int_C \mathbf{E} \cdot \mathbf{dl} = 0, \tag{4.4.9}$$

yielding

$$\mathbf{E}_1^{||} = \mathbf{E}_2^{||} \tag{4.4.10}$$

which we can express as

$$\boxed{(\mathbf{E}_2 - \mathbf{E}_1) \times \mathbf{n}_{21} = \mathbf{0} \qquad (4.4.11)}$$

where $\mathbf{n}_{21}$ is the normal from 1 to 2.

Normal condition

Applying Gauss' law to the usual elementary pill-box we have

$$\nabla \cdot \mathbf{D} = \rho_f \quad \Rightarrow \int \mathbf{D} \cdot \mathbf{dS} = \int \rho_f \, dV \tag{4.4.12}$$

from which we find

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n}_{21} = \sigma_f \qquad (4.4.13)$$

where σ is the macroscopic *free* surface charge density at the interface.
To summarise, at the interface between two dielectrics:

- The **tangential component** of $\mathbf{E}$ is **continuous**.
- The **normal component** of $\mathbf{D}$ has a discontinuity given by

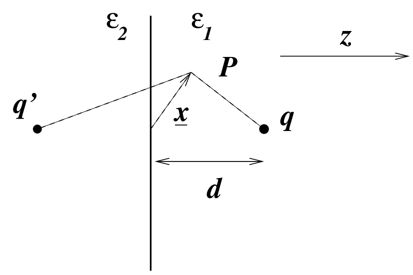
$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n}_{21} = \sigma_f \qquad (4.4.14)$$

4.5 Boundary-value Problems with Dielectrics

The method we adopt here essentially follows that of the solution of boundary-value problems *in vacua*, with the boundaries given by conducting surfaces. The method is best illustrated by examples.

Example:

A point charge q in a material of permittivity ϵ_1 a distance d from the interface with a charge-free region of permittivity ϵ_2.



The boundary conditions at the interface $z = 0$ are

$$\begin{aligned} \epsilon_1 E_z(0_+) &= \epsilon_2 E_z(0_-) \quad (\textit{normal on } \mathbf{D}) \\ E_x(0_+) &= E_x(0_-) \quad (\textit{tangential}) \\ E_y(0_+) &= E_y(0_-) \quad (\textit{tangential}). \end{aligned}$$

In order to determine the potential in the region $z > 0$, let us try an image charge q' at $z = -d$. Then the potential at $\mathbf{x}$ is

$$\Phi(\mathbf{x})|_{z>0} = \frac{1}{4\pi\epsilon_1}\left[\frac{q}{|\mathbf{x} - d\mathbf{e}_z|} + \frac{q'}{|\mathbf{x} + d\mathbf{e}_z|}\right]. \qquad (4.5.1)$$

We know that the potential in the region $z < 0$ must satisfy Laplace's equation in that region, and therefore, in particular, there cannot be any poles in the region $z < 0$. Therefore, let us try the potential due to a charge q'' at the position of our original charge q:

$$\Phi(\mathbf{x})|_{z<0} = \frac{1}{4\pi\epsilon_2}\frac{q''}{|\mathbf{x} - d\mathbf{e}_z|}. \tag{4.5.2}$$

We now introduce cylindrical polar coordinates, so that

$$\Phi(\rho, \theta, z) = \begin{cases} \dfrac{1}{4\pi\epsilon_2}\dfrac{q''}{\{\rho^2 + (z-d)^2\}^{1/2}} & z < 0 \\ \dfrac{1}{4\pi\epsilon_1}\left\{\dfrac{q}{\{\rho^2 + (z-d)^2\}^{1/2}} + \dfrac{q'}{\{\rho^2 + (z+d)^2\}^{1/2}}\right\} & z > 0 \end{cases} \tag{4.5.3}$$

We have two unknowns, q' and q'', which we determine by imposing the boundary conditions at $z = 0$. We begin with the *tangential* condition. We have that $E_\rho = -\partial\Phi/\partial\rho$, and thus

$$E_\rho = \begin{cases} \dfrac{1}{4\pi\epsilon_2}\dfrac{q''\rho}{(\rho^2 + d^2)^{3/2}} & z = 0_- \\ \dfrac{1}{4\pi\epsilon_1}\left\{\dfrac{q\rho}{(\rho^2 + d^2)^{3/2}} + \dfrac{q'\rho}{(\rho^2 + d^2)^{3/2}}\right\} & z = 0_+ \end{cases} \tag{4.5.4}$$

Thus the tangential boundary condition is

$$\frac{1}{\epsilon_2}q'' = \frac{1}{\epsilon_1}[q + q'] \quad \Rightarrow \quad q''\epsilon_1 = (q + q')\epsilon_2. \tag{4.5.5}$$

To impose the normal boundary condition, we note that

$$E_z = \begin{cases} \dfrac{1}{4\pi\epsilon_2}\dfrac{-d}{(\rho^2 + d^2)^{3/2}}q'' & z = 0_- \\ \dfrac{1}{4\pi\epsilon_1}\dfrac{-d}{(\rho^2 + d^2)^{3/2}}(q - q') & z = 0_+ \end{cases}, \tag{4.5.6}$$

from which we find

$$q'' = q - q' \tag{4.5.7}$$

or $q'' + q' = q$. To solve for q' and q'' we combine Eq. (4.5.5) $q''\epsilon_1 = (q + q')\epsilon_2$ with $q''\epsilon_1 = (q - q')\epsilon_1$ which is Eq.(4.5.7) multiplied by ϵ_1. This gives

$$(q + q')\epsilon_2 = (q - q')\epsilon_1 \qquad \Rightarrow \qquad q' = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}q \tag{4.5.8}$$

and

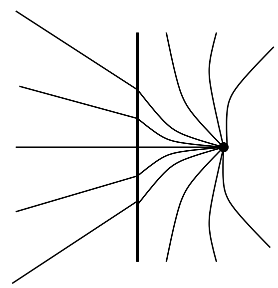
$$q'' = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2}q. \tag{4.5.9}$$

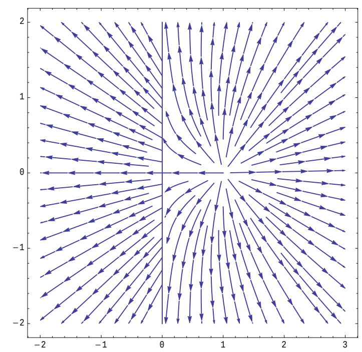
Thus we have a solution that satisfies the Laplace's equation in $z < 0$, and Poisson's equation in $z > 0$, and the correct boundary conditions at $z = 0$. Thus, by our uniqueness theorem, it is *the* solution.

To see the form of the field lines we consider two cases, $\epsilon_1 > \epsilon_2$ and $\epsilon_1 < \epsilon_2$; in both cases the field lines for $z < 0$ are those of a point charge, of magnitude q'', at q.

1. $\epsilon_1 > \epsilon_2$.

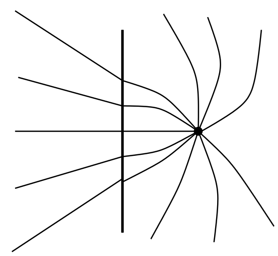
   Then q' is *same sign* as q.

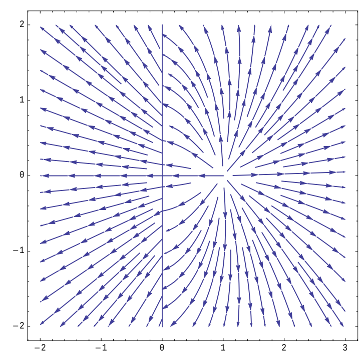




2. $\epsilon_2 > \epsilon_1$.

   Then q' and q have different signs.





In order to compute the polarization (bound) charge density, $\sigma_{\text{pol}} = -\nabla \cdot \mathbf{P}$, we observe first that $\mathbf{P}_i = \epsilon_0 \chi_i \mathbf{E}_i$, $i = 1, 2$, where $\epsilon_i = \epsilon_0(1 + \chi_i)$. Thus we have

$$\mathbf{P}_i = (\epsilon_i - \epsilon_0)\mathbf{E}_i. \tag{4.5.10}$$

Clearly the polarization charge density vanishes, except at the point charge q, and at the interface between the two materials. At the interface, there is a discontinuity in $\mathbf{P}$, and integrating over the discontinuity we obtain

$$\sigma_b = -(\mathbf{P}_1 - \mathbf{P}_2) \cdot \mathbf{n}_{21}, \tag{4.5.11}$$

where $\mathbf{n}_{21}$ is the unit normal from region 2 to region 1 (in our case, $\mathbf{n}_{21} = \mathbf{k}$), and $\mathbf{P}_2$ and $\mathbf{P}_1$ are the polarizations at $z = 0_-$ and $z = 0_+$ respectively. Thus we have

$$\sigma_b = P_{2z} - P_{1z} = (\epsilon_2 - \epsilon_0)E_{2z} - (\epsilon_1 - \epsilon_0)E_{1z} \tag{4.5.12}$$

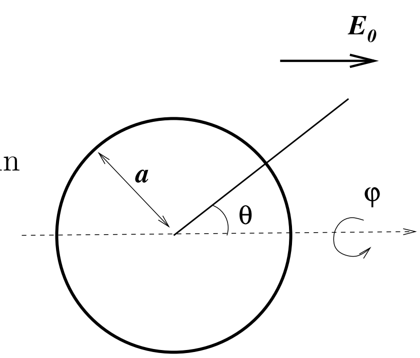
or (using $q - q' = q''$ at an intermediate step and then $q'' = 2q\epsilon_2/(\epsilon_1+\epsilon_2)$)

$$\begin{aligned}
\sigma_b &= \left\{(\epsilon_2-\epsilon_0)\frac{1}{4\pi\epsilon_2}\frac{-d\,q''}{(\rho^2+d^2)^{3/2}} - (\epsilon_1-\epsilon_0)\frac{1}{4\pi\epsilon_1}\frac{-d}{(\rho^2+d^2)^{3/2}}\underbrace{(q-q')}_{q''}\right\} \\
&= \frac{-d}{4\pi(\rho^2+d^2)^{3/2}}\left\{\frac{\epsilon_2-\epsilon_0}{\epsilon_2} - \frac{\epsilon_1-\epsilon_0}{\epsilon_1}\right\}\underbrace{q''}_{2q\epsilon_2/(\epsilon_1+\epsilon_2)} \\
&= \frac{d}{4\pi(\rho^2+d^2)^{3/2}}\epsilon_0\left\{\frac{1}{\epsilon_2}-\frac{1}{\epsilon_1}\right\}\frac{2q\epsilon_2}{\epsilon_1+\epsilon_2} \\
&= -\frac{q}{2\pi}\frac{\epsilon_0(\epsilon_2-\epsilon_1)}{\epsilon_1(\epsilon_2+\epsilon_1)}\frac{d}{(\rho^2+d^2)^{3/2}}\ .
\end{aligned}$$

Note that in the limit $\epsilon_2/\epsilon_1 \gg 1$, the electric field in region $z < 0$ becomes very small, and the polarization charge density approaches the value of the induced surface charge density for a conductor at $z = 0$, up to the factor of ϵ_0/ϵ_1. In that sense, the material in $z < 0$ behaves as a conductor in the $\epsilon_2 \to \infty$ limit.

Example

Dielectric sphere, radius a, dielectric constant ϵ/ϵ_0, in uniform field along z-axis.



We will work in spherical polar coordinates, and express our solution as an expansion in Legendre polynomials:

$$\Phi(r,\theta,\varphi) = \begin{cases} \sum_l A_l r^l P_l(\cos\theta) & r < a \\ \sum_l [B_l r^l + C_l r^{-l-1}] P_l(\cos\theta) & r > a \end{cases}\ , \tag{4.5.13}$$

where we have noted that the potential must be finite at $r = 0$.
To determine the coefficients, we impose the boundary conditions. At large distances, the potential is that for a uniform field along the z axis, and thus our boundary condition at infinity is

$$\Phi(\rho,\theta,\varphi) \longrightarrow -E_0 r\cos\theta \quad \text{as} \quad r \longrightarrow \infty\ . \tag{4.5.14}$$

We now impose the boundary conditions at the surface of the sphere

$$\begin{aligned} E_\theta(a_-) &= E_\theta(a_+) \quad (\textit{tangential condition}) \\ \epsilon_0 E_r(a_+) &= \epsilon E_r(a_-) \quad (\textit{normal condition}) \end{aligned}$$

The boundary condition at infinity tells us

$$\begin{aligned} B_1 &= -E_0 \\ B_l &= 0 \quad l \neq 1 \end{aligned}$$

To impose other boundary conditions, we evaluate the components of the electric field, beginning with E_θ:

$$E_\theta = \begin{cases} -\sum_l A_l r^{l-1} \dfrac{d}{d\theta} P_l(\cos\theta) & r < a \\ -\sum_l C_l r^{-l-2} \dfrac{d}{d\theta} P_l(\cos\theta) - B_1 \dfrac{d}{d\theta} P_1(\cos\theta) & r > a \end{cases}, \tag{4.5.15}$$

From the generalized Rodrigues' formula, we have

$$\begin{aligned} P_l^1(x) &= (-1)^1 (1-x^2)^{1/2} \frac{d}{dx} P_l(x) \\ \Rightarrow P_l^1(\cos\theta) &= -\sin\theta \frac{d}{d\cos\theta} P_l(\cos\theta) \\ &= \frac{d}{d\theta} P_l(\cos\theta), \end{aligned}$$

whence

$$E_\theta = \begin{cases} -\sum_l A_l r^{l-1} P_l^1(\cos\theta) & r < a \\ -\sum_l C_l r^{-l-2} P_l^1(\cos\theta) - B_1 P_1^1(\cos\theta) & r > a \end{cases}. \tag{4.5.16}$$

The radial component is straightforward,

$$E_r = \begin{cases} -\sum_l A_l \cdot l \cdot r^{l-1} P_l(\cos\theta) & r < a \\ \sum_l C_l (l+1) r^{-l-2} P_l(\cos\theta) - B_1 P_1(\cos\theta) & r > a \end{cases}. \tag{4.5.17}$$

Thus imposing the tangential boundary condition we have

$$\sum_l A_l a^{l-1} P_l^1(\cos\theta) = \sum_l C_l a^{-l-2} P_l^1(\cos\theta) + B_1 P_1^1(\cos\theta). \tag{4.5.18}$$

Using the orthogonality property of the Legendre polynomials, we have, for $l \neq 1$,

$$A_l a^{l-1} = C_l a^{-l-2} \qquad \Rightarrow \qquad A_l \;=\; C_l a^{-2l-1}. \tag{4.5.19}$$

For the case $l = 1$, we have

$$A_1 = C_1 a^{-3} + B_1 = C_1 a^{-3} - E_0. \tag{4.5.20}$$

The normal boundary condition yields

$$\epsilon_0 \left\{ \sum_l C_l (l+1) a^{-l-2} P_l(\cos\theta) - B_1 P_1(\cos\theta) \right\} = -\epsilon \sum_l A_l\, l\, a^{l-1} P_l(\cos\theta). \tag{4.5.21}$$

Once again, there are two cases

$$\epsilon_0 [C_l (l+1) a^{-l-2}] \;=\; -\epsilon A_l\, l a^{l-1} \quad , \quad l \neq 1 \tag{4.5.22}$$

$$\epsilon_0 [2C_1 a^{-3} + E_0] \;=\; -\epsilon A_1 \quad , \quad l = 1 \tag{4.5.23}$$

Substituting Eq. (4.5.19) into Eq. (4.5.22), we find

$$\epsilon_0 [C_l (l+1) a^{-l-2}] = -\,\epsilon C_l a^{-2l-1}\, l a^{l-1} \Rightarrow C_l = 0 \Rightarrow A_l = 0 \quad , \quad l \neq 1. \tag{4.5.24}$$

Finally, we have $2A_1 = 2C_1 a^{-3} - 2E_0$ from Eq. (4.5.20) and $A_1 \epsilon/\epsilon_0 = -2C_1 a^{-3} - E_0$ from Eq. (4.5.23). This gives

$$\begin{aligned} A_1 &= \frac{-3E_0}{2 + \epsilon/\epsilon_0} \ , \\ C_1 &= a^3 (A_1 + E_0) = \left(\frac{\epsilon/\epsilon_0 - 1}{2 + \epsilon/\epsilon_0} \right) a^3 E_0 \ . \end{aligned}$$

Thus we have

$$\Phi(r, \theta, \varphi) = \begin{cases} -\dfrac{3}{2 + \epsilon/\epsilon_0} E_0 r \cos\theta \quad , & r < a \\ -E_0 r \cos\theta + \left(\dfrac{\epsilon/\epsilon_0 - 1}{2 + \epsilon/\epsilon_0} \right) \dfrac{a^3}{r^2} E_0 \cos\theta \quad , & r > a \end{cases} . \tag{4.5.25}$$

- *Inside* the sphere, the field is parallel to the field at infinity,

$$\mathbf{E}_{\text{in}} = \frac{3}{2 + \epsilon/\epsilon_0} \mathbf{E}_0, \tag{4.5.26}$$

  with $|\mathbf{E}_{\text{in}}| < E_0$ if $\epsilon > \epsilon_0$.

- *Outside* the sphere, the field is equivalent to that of the applied field, together with that due to a point dipole at the origin, of moment

$$p = 4\pi\epsilon_0 \left(\frac{\epsilon/\epsilon_0 - 1}{2+\epsilon/\epsilon_0}\right) a^3 E_0 = \frac{4\pi a^3}{3}\frac{3}{2+\epsilon/\epsilon_0}(\epsilon-\epsilon_0)E_0 \qquad (4.5.27)$$

  oriented in the direction of the applied field.

The polarization $\mathbf{P} = (\epsilon - \epsilon_0)\mathbf{E}$ is constant throughout the sphere,

$$\mathbf{P} = \frac{3(\epsilon-\epsilon_0)}{2+\epsilon/\epsilon_0}\mathbf{E}_0. \qquad (4.5.28)$$

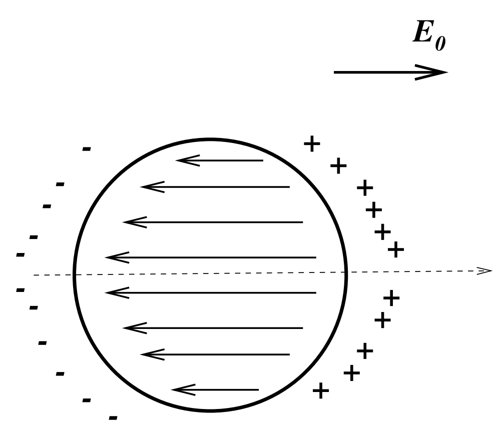
We can evaluate the *volume integral* of $\mathbf{P}$, to obtain

$$\begin{aligned} \int_{r<a} dV\,\mathbf{P} &= \frac{4}{3}\pi a^3 \frac{3(\epsilon-\epsilon_0)}{2+\epsilon/\epsilon_0}\mathbf{E}_0 \\ &= 4\pi\epsilon_0 \left(\frac{\epsilon/\epsilon_0 - 1}{2+\epsilon/\epsilon_0}\right) a^3\mathbf{E}_0, \end{aligned}$$

which is just the dipole moment we obtained in Eq. (4.5.27). Thus the dipole moment is just the volume integral of the polarization.

Because $\mathbf{P}$ is constant throughout the sphere, the polarization charge density $-\nabla\cdot\mathbf{P}$ vanishes throughout the interior. However, because of the discontinuity in $\mathbf{P}$ at the surface, we have a surface polarization charge density, whose magnitude we can obtain from Eq. (4.5.11):

$$\begin{aligned} \sigma_b &= \mathbf{P}\cdot\mathbf{e}_r \quad (\mathbf{P}\text{ vanishes outside sphere}) \\ &= 3\epsilon_0 \left(\frac{\epsilon/\epsilon_0 - 1}{2+\epsilon/\epsilon_0}\right) E_0 \cos\theta \end{aligned}$$



4.6 Electrostatic Energy in Dielectric Media

Back in the introduction, we computed the energy of a system of charges in free space:

$$W = \frac{1}{2}\int d^3x\,\rho(\mathbf{x})\Phi(\mathbf{x}). \tag{4.6.1}$$

We obtained this expression by assembling the charges, one-by-one, from infinity under the potential of the charges already assembled. In the case of dielectrics, work is done not only in assembling the charges, but also in polarising the medium.
To see how to perform the calculation in this case, consider the change in energy due to a macroscopic charge density $\delta\rho(\mathbf{x})$,

$$\delta W = \int d^3x\,\delta\rho_f(\mathbf{x})\Phi(\mathbf{x}). \tag{4.6.2}$$

We now use recall that $\nabla\cdot\mathbf{D} = \rho_f$, enabling us to write $\nabla\cdot\delta\mathbf{D} = \delta\rho_f$. Thus we have

$$\begin{aligned}\delta W &= \int d^3x\,\Phi(\mathbf{x})\,\nabla\cdot\delta\mathbf{D}\\ &= \int d^3x\,\mathbf{E}\cdot\delta\mathbf{D},\end{aligned}$$

where we have integrated by parts, assuming that the charge is *localized* so that the surface term vanishes. Thus the total energy in constructing the system is

$$W = \int d^3x\int_0^D \mathbf{E}\cdot\delta\mathbf{D}. \tag{4.6.3}$$

We now make the critical assumption of a *linear, isotropic* constitutive relation between **E** and **D**,

$$\mathbf{D}(\mathbf{x}) = \epsilon(\mathbf{x})\mathbf{E}(\mathbf{x}). \tag{4.6.4}$$

Then we have $\mathbf{E}\cdot\delta\mathbf{D} = \frac{1}{2}\delta(\mathbf{E}\cdot\mathbf{D})$, and thus

$$W = \int d^3x\int_0^D \frac{1}{2}\delta(\mathbf{E}\cdot\mathbf{D}) \tag{4.6.5}$$

yielding

$$W = \frac{1}{2}\int d^3x\,\mathbf{E}\cdot\mathbf{D}. \tag{4.6.6}$$

We can recover our expression Eq. (4.6.1) by the substitution $\mathbf{E} = \nabla\Phi$ and using $\nabla\cdot\mathbf{D} = \rho_f$. The crucial observation is that the expression Eq. (4.6.6) is valid *only if the relation between* **D** *and* **E** *is linear.*

4.6.1 Energy of Dielectric in an Electric Field with Fixed Charges

As an important application of this formula, we will consider the case of a dielectric medium introduced into an electric field $\mathbf{E}_0(\mathbf{x})$ arising from a **fixed** charge distribution $\rho_f = \rho_0(\mathbf{x})$. Initially, the energy of the system is

$$W_0 = \frac{1}{2}\int d^3x\, \mathbf{E}_0 \cdot \mathbf{D}_0 \tag{4.6.7}$$

with $\mathbf{D}_0 = \epsilon_0 \mathbf{E}_0$; here ϵ_0 is the initial permittivity of the dielectric, not necessarily the permittivity of free space.

We now introduce the medium, of volume V_1, with permittivity

$$\epsilon(\mathbf{x}) = \begin{cases} \epsilon_1(\mathbf{x}) & \mathbf{x} \in V_1 \\ \epsilon_0(\mathbf{x}) & \mathbf{x} \notin V_1 \end{cases}, \tag{4.6.8}$$

noting that the charge distribution is unaltered. Then the new energy is

$$W_1 = \frac{1}{2}\int d^3x\, \mathbf{E}(\mathbf{x}) \cdot \mathbf{D}(\mathbf{x}) \tag{4.6.9}$$

and the *change* in energy is

$$\delta W = \frac{1}{2}\int d^3x\, \mathbf{E} \cdot \mathbf{D} - \frac{1}{2}\int d^3x\, \mathbf{E}_0 \cdot \mathbf{D}_0. \tag{4.6.10}$$

With a little juggling, we can write this as

$$\delta W = \frac{1}{2}\int d^3x\, (\mathbf{E} \cdot \mathbf{D}_0 - \mathbf{E}_0 \cdot \mathbf{D}) + \frac{1}{2}\int d^3x\, (\mathbf{E} + \mathbf{E}_0) \cdot (\mathbf{D} - \mathbf{D}_0). \tag{4.6.11}$$

To evaluate the second term, we note that both $\nabla \times \mathbf{E} = 0$ and $\nabla \times \mathbf{E}_0 = 0$, and thus we may write $\mathbf{E} + \mathbf{E}_0 = -\nabla\Psi(\mathbf{x})$. Hence the second integral may be written

$$I = -\frac{1}{2}\int d^3x\, \nabla\Psi \cdot (\mathbf{D} - \mathbf{D}_0) = \frac{1}{2}\int d^3x\, \Psi\, \nabla \cdot (\mathbf{D} - \mathbf{D}_0) \tag{4.6.12}$$

where we assume the integrand falls off sufficiently rapidly at infinity.

Now $\nabla \cdot (\mathbf{D} - \mathbf{D}_0) = \rho_f(\mathbf{x}) - \rho_{0f}(\mathbf{x}) = \mathbf{0}$, since we required that the free charge distribution be unaltered by the introduction of the dielectric. Thus the integral vanishes, and we have

$$\delta W = \frac{1}{2}\int d^3x\, (\mathbf{E} \cdot \mathbf{D}_0 - \mathbf{E}_0 \cdot \mathbf{D}). \tag{4.6.13}$$

We now spilt the region of integration into V_1 and the remainder,

$$\delta W = \frac{1}{2}\int_{x\in V_1} d^3x\, (\mathbf{E} \cdot \mathbf{D}_0 - \mathbf{E}_0 \cdot \mathbf{D}) + \frac{1}{2}\int_{x\notin V_1} d^3x\, (\mathbf{E} \cdot \mathbf{D}_0 - \mathbf{E}_0 \cdot \mathbf{D}). \tag{4.6.14}$$

For $x \notin V_1$ we have $\mathbf{D}_0 = \epsilon_0 \mathbf{E}_0$ and $\mathbf{D} = \epsilon_0 \mathbf{E}$, and the integrand vanishes, so that

$$\begin{aligned} \delta W &= \frac{1}{2}\int_{V_1} d^3x\,(\epsilon_0 \mathbf{E}\cdot\mathbf{E}_0 - \epsilon_1 \mathbf{E}_0\cdot\mathbf{E}) \\ &= -\frac{1}{2}\int_{V_1} d^3x\,(\epsilon_1 - \epsilon_0)\mathbf{E}\cdot\mathbf{E}_0. \end{aligned}$$

We now specialize to the case where the original dielectric is indeed the vacuum, and ϵ_0 the permittivity of free space, and write

$$(\epsilon_1 - \epsilon_0)\mathbf{E} = \mathbf{P}, \tag{4.6.15}$$

yielding

$$\boxed{\delta W = -\frac{1}{2}\int_{V_1} d^3x\,\mathbf{P}\cdot\mathbf{E}_0.}$$

We can interpret $w = -\frac{1}{2}\mathbf{P}\cdot\mathbf{E}_0$ as the **energy density** of the dielectric. The expression can be likened to that for the energy of a dipole distribution derived at the end of Section 4.3. There we were considering a *permanent* dipole, whilst here energy is expended in polarizing the dielectric, and this is reflected in the factor of 1/2.

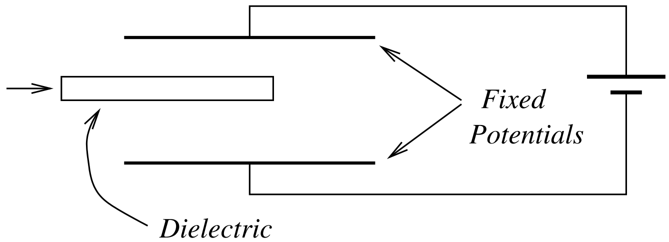
Note that the energy tends to *decrease* if the dielectric moves to a region of increasing $\mathbf{E}_0$, provided $\epsilon_1 > \epsilon_0$. Since the charges are held fixed, the total energy is *conserved*, and we can interpret the change in field energy W due to a displacement of the dielectric body ξ as producing a corresponding change in potential energy, and hence a force on the body of magnitude

$$F = -\left(\frac{\partial W}{\partial \xi}\right)_Q, \tag{4.6.16}$$

where the subscript Q denotes at fixed charge.

4.6.2 Energy of Dielectric Body at Fixed Potentials

We will conclude this section by considering the contrasting case where we introduce a dielectric body into a system where the **potentials**, rather than charges, are kept fixed. A paradigm is the introduction of a dielectric between the plates of a capacitor connected to a battery, and hence at a fixed potential difference.



In this case, charges can flow to or from the conducting plates as the dielectric is introduced to maintain the potentials, and hence the total energy can change. Again, we will assume that the media are **linear**.

It is sufficient to consider *small* changes to the potential $\delta\Phi$ and to the charge distribution $\delta\Phi$, for which the change in energy δW, from Eq. 4.6.1, is

$$\delta W = \frac{1}{2}\int d^3x\,(\rho_f \delta\Phi + \Phi\delta\rho_f). \tag{4.6.17}$$

For the case of linear media, these two terms are equal if the dielectric properties are unaltered. However, in the case where the dielectric properties are altered during the change, $\epsilon(\mathbf{x}) \rightarrow \epsilon(\mathbf{x}) + \delta\epsilon(\mathbf{x})$, this is no longer true, because of a polarization charge density generated in the dielectric. We have already considered this problem for fixed charges, $\delta\rho_f = 0$. In order to compute the change of energy at fixed potentials, we study the problem in two stages;

1. The battery is disconnected, so that the distribution of charges is fixed, $\delta\rho_f = 0$, and the dielectric is introduced. Then there is a change in potential $\delta\Phi_1$, and the corresponding change in energy is

$$\delta W_1 = \frac{1}{2}\int d^3x\,\rho_f \delta\Phi_1 = -\frac{1}{2}\int d^3x\,(\epsilon_1 - \epsilon_0)\mathbf{E}\cdot\mathbf{E}_0, \tag{4.6.18}$$

   using the result of the previous subsection.

2. We now reconnect the battery. The potential on the conductors, where the only macroscopic charges reside, must regain its original value, i.e. $\delta\Phi_2 = -\delta\Phi_1$, and there is a corresponding change in charge density $\delta\rho_{2f}$, yielding

$$\delta W_2 = \frac{1}{2}\int d^3x\,(\rho_f \delta\Phi_2 + \Phi_2\delta\rho_{2f}). \tag{4.6.19}$$

   In this step, the dielectric properties are unaltered and the two terms are equal, so we have

$$\begin{aligned}\delta W_2 &= \int d^3x\,\rho_f\delta\Phi_2 \\ &= -\int d^3x\,\rho_f\delta\Phi_1 \\ &= -2\,\delta W_1\end{aligned} \tag{4.6.20}$$

Thus the *total* energy change

$$\delta W = \delta W_1 + \delta W_2 = -\delta W_1, \tag{4.6.21}$$

which we write as

$$\delta W_V = -\delta W_Q, \tag{4.6.22}$$

i.e. the change in energy *at fixed potential* is **minus** the change in energy *at fixed charges.* In this case, if a dielectric with $\epsilon_1 > \epsilon_0$ moves into a region at fixed potentials, the energy **increases**, and a mechanical force

$$F_\xi = +\left(\frac{\partial W}{\partial \xi}\right)_V \tag{4.6.23}$$

acts on the body.