Contents

Chapter 5

Magnetostatics

5.1 Introduction

The crucial difference between electric and magnetic phenomena is the absence of isolated magnetic charges, or **magnetic monopoles**. Here the basic building blocks are **magnetic dipoles**. For a magnetic field, or flux density, $\mathbf{B}$, the torque $\boldsymbol{\tau}$ acting on a dipole of moment $\boldsymbol{\mu}$ is

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}. \tag{5.1.1}$$

The other concept we need in the study of magnetostatics is the electric current $\mathbf{J}$, defined as the flow of charge per unit time per unit area, with normal in the direction of $\mathbf{J}$.

$$\mathbf{J} = \frac{dQ}{da_\perp dt}\hat{J} = \frac{d\underline{I}}{da_\perp}\hat{J}\ . \tag{5.1.2}$$

For a line current I running in some direction specified by a unit vector $\hat{\mathbf{e}}$ we have

$$I = \int d\mathbf{a}_\perp \cdot \mathbf{J}\ , \tag{5.1.3}$$

with $d\mathbf{a}_\perp = da_\perp\,\hat{\mathbf{e}}$ being an elementary area with normal in the direction of $\hat{\mathbf{e}}$.

5.1.1 Current Conservation

Current conservation is represented by the **continuity equation**

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0, \tag{5.1.4}$$

where ρ is the charge density. This statement just states that the rate of change of charge in any volume V is (minus) the flux of charge across the surface of V, as you can see by applying the divergence theorem to charge Q_{in} inside volume V. Note that decrease of Q_{in}

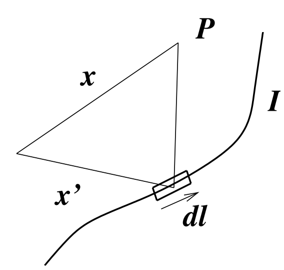
corresponds to increase of charge Q_{out} outside the volume V, which is due to the current $\mathbf{J}$ flowing from V to the outside space:

$$-\frac{dQ_{\text{in}}}{dt} = \frac{dQ_{\text{out}}}{dt} = \oint \frac{dQ_{\text{out}}}{da_\perp dt} da_\perp = \oint J da_\perp = \oint \mathbf{J} \cdot \mathbf{da} = \int d^3x \, \nabla \cdot \mathbf{J}$$

For steady currents we are considering in this chapter

$$\nabla \cdot \mathbf{J} = 0. \tag{5.1.5}$$

5.2 Biot-Savart Law



This describes the element of magnetic field $\mathbf{B}$ at some point $\mathbf{x}$ due to an element of current flow $I\mathbf{dl}$ at $\mathbf{x}'$:

$$d\mathbf{B} = kI \frac{\mathbf{dl} \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \,, \tag{5.2.1}$$

where, in SI units,

- I is the current (Ampères),
- $\mathbf{dl}$ is an element of length in the direction of the current flow,
- $k = \mu_0/4\pi$, where μ_0 is the **permeability of free space**.

For a point charge q moving with velocity $\mathbf{v}$, we can replace $I\mathbf{dl}$ by $q\mathbf{v}$, and we have

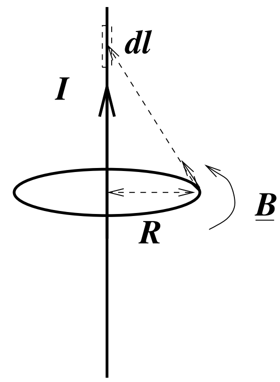
$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{q\mathbf{v} \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}, \tag{5.2.2}$$

providing $\mathbf{v}$ is constant, and small compared to the velocity of light.
We can apply the superposition principle to the magnetic field, and obtain for a general current density

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}. \tag{5.2.3}$$

Example

Consider the magnetic field due to straight wire carrying current I. Then the field a distance R from the wire is tangential, and can be written (using $l/R = \tan\theta$)



$$\begin{aligned}\mathbf{B} &= \frac{\mu_0 I}{4\pi} R \int_{-\infty}^{\infty} \frac{dl}{(l^2+R^2)^{3/2}} \mathbf{e}_\varphi \\ &= \frac{\mu_0 I}{4\pi} R \int_{-\pi/2}^{\pi/2} \frac{R d\theta}{\cos^2\theta} \frac{1}{R^3/\cos^3\theta}\, \mathbf{e}_\varphi \\ &= \frac{\mu_0 I}{4\pi R} \int_{-\pi/2}^{\pi/2} d\theta\, \cos\theta\, \mathbf{e}_\varphi = \frac{\mu_0 I}{2\pi R} \mathbf{e}_\varphi\end{aligned}$$

5.2.1 Force on a Current in Presence of Magnetic Field

The element of force on a current element $I\mathbf{dl}$ at $\mathbf{x}$ in a magnetic field $\mathbf{B}(\mathbf{x})$ is

$$\mathbf{dF} = I\mathbf{dl} \times \mathbf{B}. \tag{5.2.4}$$

Thus the force on a closed loop of current l_1 due to magnetic field from closed loop l_2 is

$$\begin{aligned}\mathbf{F}_{12} &= \frac{\mu_0}{4\pi} I_1 I_2 \oint \mathbf{dl}_1 \times \left\{ \oint \frac{\mathbf{dl}_2 \times (\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^3} \right\} \\ &= \frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{\mathbf{dl}_1 \times [\mathbf{dl}_2 \times (\mathbf{x}_1 - \mathbf{x}_2)]}{|\mathbf{x}_1 - \mathbf{x}_2|^3}.\end{aligned}$$

We can put this expression in a more symmetric form by writing

$$\mathbf{dl}_1 \times (\mathbf{dl}_2 \times \mathbf{x}_{12}) = (\mathbf{dl}_1 \cdot \mathbf{x}_{12})\mathbf{dl}_2 - (\mathbf{dl}_1 \cdot \mathbf{dl}_2)\mathbf{x}_{12}, \tag{5.2.5}$$

yielding

$$\mathbf{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \oint \oint \left\{ -\frac{\mathbf{dl}_1 \cdot \mathbf{dl}_2}{|\mathbf{x}_{12}|^3} \mathbf{x}_{12} + \mathbf{dl}_2 \frac{\mathbf{dl}_1 \cdot \mathbf{x}_{12}}{|\mathbf{x}_{12}|^3} \right\}. \tag{5.2.6}$$

We will now show that the second term vanishes. Consider the integration around loop 1, for fixed $\mathbf{x}_2$. Then under a change $\mathbf{x}_1 \to \mathbf{x}_1 + \mathbf{dl}_1$, we have

$$\mathbf{x}_{12} \to \mathbf{x}_{12} + \mathbf{dl}_1. \tag{5.2.7}$$

Now consider the change in $1/|\mathbf{x}_{12}|$:

$$\begin{aligned}\delta\left(\frac{1}{|\mathbf{x}_{12}|}\right) &= \frac{1}{|\mathbf{x}_{12} + \mathbf{dl}_1|} - \frac{1}{|\mathbf{x}_{12}|} \\ &= \frac{1}{|\mathbf{x}_{12}|}\left\{1 - \frac{\mathbf{x}_{12}\cdot\mathbf{dl}_1}{|\mathbf{x}_{12}|^2} - 1\right\} \\ &= -\frac{\mathbf{x}_{12}\cdot\mathbf{dl}_1}{|\mathbf{x}_{12}|^3}.\end{aligned} \tag{5.2.8}$$

Thus the integrand in the second term of Eq. (5.2.6) is an exact differential, and therefore the integrand around the closed loop vanishes, and we have

$$\mathbf{F}_{12} = -\frac{\mu_0 I_1 I_2}{4\pi} \int \frac{\mathbf{dl}_1 \cdot \mathbf{dl}_2}{|\mathbf{x}_{12}|^3} \mathbf{x}_{12} \; . \tag{5.2.9}$$

Now Newton's third law is satisfied explicity, and we have

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \; . \tag{5.2.10}$$

For a general current density $\mathbf{J}(\mathbf{x})$ in a magnetic field $\mathbf{B}(\mathbf{x})$, we have

$$\mathbf{F} = \int d^3x \, \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}) \tag{5.2.11}$$

$$\tau = \int d^3x \, \mathbf{x} \times (\mathbf{J} \times \mathbf{B}). \tag{5.2.12}$$

5.3 Laws of Magnetostatics in Differential Form

In analogy with electrostatics, our starting point is the expression for $\mathbf{B}$ due to general current density

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}. \tag{5.3.1}$$

We begin by recalling that $\nabla \times (\varphi \mathbf{a}) = \nabla \varphi \times \mathbf{a}$, where $\mathbf{a}$ is a constant vector. Thus

$$\begin{aligned} \nabla_x \times \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \mathbf{J}(\mathbf{x}') \right) &= \nabla_x \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \times \mathbf{J}(\mathbf{x}') \\ &= -\frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \times \mathbf{J}(\mathbf{x}') = \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \; . \end{aligned}$$

Thus we can write

$$\boxed{\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \nabla \times \int d^3x' \, \frac{1}{|\mathbf{x} - \mathbf{x}'|} \, \mathbf{J}(\mathbf{x}') \qquad (5.3.2)}$$

or

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A} \; , \tag{5.3.3}$$

where

$$\boxed{\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \qquad (5.3.4)}$$

is the **magnetic vector potential**.

From Eq. (5.3.4), using $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ we immediately see that

$$\boxed{\nabla \cdot \mathbf{B} = 0. \qquad (5.3.5)}$$

This is another of Maxwell's equations, and is just another statement that you cannot have isolated magnetic charges, and that the total flux of $\mathbf{B}$ through any closed surface vanishes

$$\int_{S=\partial V} \mathbf{dS} \cdot \mathbf{B} = 0\ . \tag{5.3.6}$$

To obtain another differential equation, we evaluate $\nabla \times \mathbf{B}$. We begin by recalling the vector identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}, \tag{5.3.7}$$

so that

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \nabla \int d^3x' \, \nabla_x \cdot \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} - \frac{\mu_0}{4\pi} \int d^3x' \, \mathbf{J}(\mathbf{x}') \nabla_x^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right). \tag{5.3.8}$$

Now

$$\begin{aligned} \nabla_x \cdot \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} &= -\mathbf{J}(\mathbf{x}') \cdot \nabla_{x'} \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \ , \\ \nabla_x^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) &= -4\pi \delta(\mathbf{x} - \mathbf{x}'), \end{aligned}$$

and thus

$$\begin{aligned} \nabla \times \mathbf{B} &= -\frac{\mu_0}{4\pi} \nabla \int d^3x' \, \mathbf{J}(\mathbf{x}') \cdot \nabla_{x'} \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) + \mu_0 \int d^3x' \, \mathbf{J}(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') \\ &= \frac{\mu_0}{4\pi} \nabla \int d^3x' \, \frac{1}{|\mathbf{x} - \mathbf{x}'|} \nabla_{x'} \cdot \mathbf{J} + \mu_0 \mathbf{J}(\mathbf{x}) \end{aligned}$$

For magnetostatics, we have $\nabla \cdot \mathbf{J} = 0$, and thus

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}\ . \qquad (5.3.9)}$$

This is the second fundamental differential equation. We can apply Stokes' theorem to a closed curve C spanned by a surface S to obtain

$$\int \nabla \times \mathbf{B} \cdot \mathbf{dS} = \int \mathbf{B} \cdot \mathbf{dl} = \mu_0 \int \mathbf{J} \cdot \mathbf{dS}. \tag{5.3.10}$$

5.4 Vector Potential

For static fields, the governing equations of magnetostatics are

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \end{aligned}$$

For the case $\mathbf{J} \equiv \mathbf{0}$, we have $\nabla \times \mathbf{B} = 0$, and we can introduce a magnetic scalar potential ϕ_M.

Much more interesting is the general case $\mathbf{J} \neq \mathbf{0}$. We can show that if $\nabla \cdot \mathbf{B} = 0$ in a **star-shaped** region,[1] then a **vector potential** $\mathbf{A}$ can be found such that

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{5.4.1}$$

In the case where $\mathbf{B}$ is the magnetic field, we call $\mathbf{A}$ the **magnetic vector potential**. We have already established this representation for $\mathbf{B}$ in case of unbounded space (see Eq. (5.3.4)).

5.4.1 Uniqueness of A and Gauge Transformations

If $\mathbf{A}$ is a solution of $\mathbf{B} = \nabla \times \mathbf{A}$, then $\mathbf{A}' = \mathbf{A} + \nabla f$, where f is an arbitrary, continuously differentiable scalar field, is also a solution, because

$$\nabla \times (\nabla f) = 0. \tag{5.4.2}$$

Transformation of this form are called **Gauge Transformations**; we say that $\mathbf{B}$ is invariant under gauge transformations. To simplify calculations, we often make a specific choice of gauge.

Examples

1. We could require $A_1(\mathbf{x}) = 0 \quad \forall \mathbf{x}$. In general, we could require $A_i(\mathbf{x}) = 0$ for any $i = 1, 2, 3$. Even more general, we could impose the **axial gauge** condition

$$\mathbf{n} \cdot \mathbf{A}(\mathbf{x}) = 0 \quad \forall \mathbf{x}\ , \tag{5.4.3}$$

where $\mathbf{n}$ is an arbitrary (non-zero) vector.

[1] A star shaped region is one in which there exists a point which can be connected to every other point by a straight line

2. We could require

$$\nabla \cdot \mathbf{A}(\mathbf{x}) = 0 \quad \forall \mathbf{x} \ . \tag{5.4.4}$$

   This is the **Coulomb Gauge**.

3. A more exotic condition

$$\mathbf{x} \cdot \mathbf{A}(\mathbf{x}) = 0 \quad \forall \mathbf{x} \tag{5.4.5}$$

   called **Fock-Schwinger gauge** is used in studies on quantum chromodynamics, which is an example of *non-Abelian gauge theories.*

Choosing, or fixing, the gauge reduces the number of degrees of freedom, clear in example (1) above. All the fundamental forces of nature are described by **Gauge Theories**, having the property of a gauge, or local, symmetry.

5.4.2 Solutions for the Vector Potential in Free Space

We will specify that we work in the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$. Then the second of our governing equation becomes

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \tag{5.4.6}$$

and thus

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}. \tag{5.4.7}$$

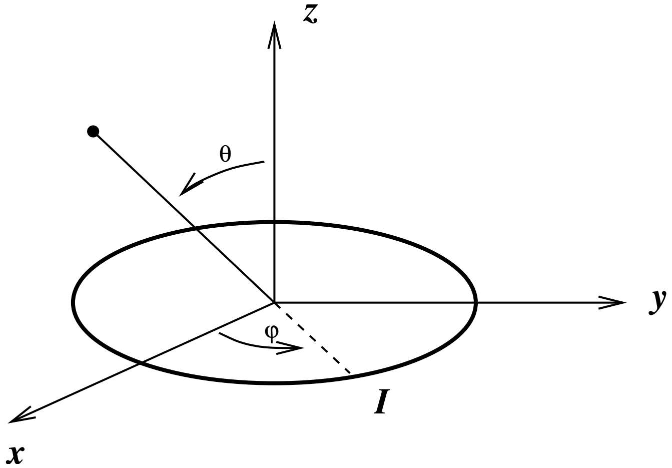
This is just Poisson's equation, applied to each of the Cartesian components of $\mathbf{A}$, and from our investigation of electrostatics has the solution

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \tag{5.4.8}$$

This expression for $\mathbf{A}(\mathbf{x})$ also follows from the general representation (5.3.1) for $\mathbf{B}(\mathbf{x})$ (which may be treated as a solution of Maxwell's equations).

Example

Potential due to a wire loop of radius a, carrying current I.



The current is purely in the azimuthal direction, and in spherical polars, we can write the current density as

$$J_\varphi = I\delta(\theta' - \pi/2)\frac{\delta(r' - a)}{a} = I \sin\theta' \delta(\cos\theta')\frac{\delta(r' - a)}{a} \ . \tag{5.4.9}$$

Normalization here is fixed by

$$I = \int da'_\perp J_\varphi = \int r' dr'\, d\theta' J_\varphi \ . \tag{5.4.10}$$

W.l.o.g. we will consider the case where the observation point is in the $x - z$ plane, so that, in Cartesian coordinates, the current density is

$$\mathbf{J} = -J_\varphi \sin\varphi' \mathbf{i} + J_\varphi \cos\varphi' \mathbf{j}. \tag{5.4.11}$$

Thus the vector potential, from Eq. (5.4.8), is given by

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d\Omega'\, r'^2 dr' \left\{-J_\varphi \sin\varphi' \mathbf{i} + J_\varphi \cos\varphi' \mathbf{j}\right\} \times \frac{1}{|\mathbf{x} - \mathbf{x}'|}. \tag{5.4.12}$$

The x component of $\mathbf{A}$ will vanish, since the expansion of $1/|\mathbf{x} - \mathbf{x}'|$ is symmetric under $\varphi' \leftrightarrow -\varphi'$. Thus the only non-vanishing component of $\mathbf{A}$ is in the y-direction, which coincides with $\mathbf{e}_\varphi$. Thus we have

$$A_\varphi = \frac{\mu_0}{4\pi} I \int d\Omega'\, dr' r'^2 \delta(\theta' - \pi/2)\frac{\delta(r' - a)}{a} \cos\varphi' \ \frac{1}{|\mathbf{x} - \mathbf{x}'|}. \tag{5.4.13}$$

Performing the integrations over r' and θ' yields

$$A_\varphi = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} d\varphi' \ \cos\varphi' \left\{a^2 + r^2 - 2ar\sin\theta\cos\varphi'\right\}^{-1/2}. \tag{5.4.14}$$

This is an elliptic integral, and its expression in elliptic functions is not particularly illuminating. Instead, we will perform an expansion in spherical harmonics:

$$A_\varphi = \frac{\mu_0 I}{4\pi}\Re\int d\Omega'\, dr'\, r'^2\delta(\theta'-\pi/2)\frac{\delta(r'-a)}{a}e^{i\varphi'}$$
$$\times 4\pi\sum_{l,m}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi),$$

where we write

$$\cos\varphi' = \Re e^{i\varphi'}. \tag{5.4.15}$$

Performing the delta-function integrations, we arrive at

$$A_\varphi = \mu_0 I a \Re\sum_{l,m}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,0)\frac{1}{2l+1}\int d\varphi'\, e^{i\varphi'}Y^*_{lm}(\pi/2,\varphi'). \tag{5.4.16}$$

We now use the orthogonality properties of the functions $\exp im\varphi$ to write (you see why we expressed $\cos\varphi'$ this way. . .):

$$\int d\varphi'\, e^{i\varphi'}Y^*_{lm}(\pi/2,\varphi') = \begin{cases} 2\pi Y_{l1}(\pi/2,0) & m=1 \\ 0 & \text{otherwise} \end{cases}, \tag{5.4.17}$$

and thus

$$A_\varphi = 2\pi\mu_0 I a\sum_{l=1}^{\infty}\frac{r_<^l}{r_>^{l+1}}Y_{l1}(\theta,0)Y_{l1}(\pi/2,0)\frac{1}{2l+1}. \tag{5.4.18}$$

Now we have that

$$Y_{l1}(\pi/2,0) = \sqrt{\frac{(l-1)!(2l+1)}{4\pi(l+1)!}}P_l^1(0) \tag{5.4.19}$$

which vanishes if l is even, since $P_l^1(x) = dP_l(x)/dx$ has the opposite parity to $P_l(x)$. The explicit evaluation of these integrals is performed in *Jackson*, so I leave it for you to look them up there. However, the important feature is that even when we have azimuthal symmetry, the vector potential and magnetic fields involve the P_l^1 Legendre polynomials; this reflects the vector nature of the source in magnetostatics, as opposed to the scalar nature of the source in electrostatics.

5.5 Magnetic Field Far from Current Distribution

Consider a localized current distribution $\mathbf{J}(\mathbf{x}')$, and the magnetic vector potential produced at a point $P(\mathbf{x})$ where $|\mathbf{x}| \gg |\mathbf{x}'|$. Then we can write

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3} + \ldots, \tag{5.5.1}$$

so that, in the Coulomb gauge

$$A_i(\mathbf{x}) = \frac{\mu_0}{4\pi}\left\{\frac{1}{|\mathbf{x}|}\int d^3x'\, J_i(\mathbf{x}') + \frac{\mathbf{x}}{|\mathbf{x}|^3}\cdot\int d^3x'\, J_i(\mathbf{x}')\,\mathbf{x}' + \ldots\right\} \tag{5.5.2}$$

We can rewrite the scalar product in the second term in components:

$$A_i(\mathbf{x}) = \frac{\mu_0}{4\pi}\left\{\frac{1}{|\mathbf{x}|}\int d^3x'\, J_i(\mathbf{x}') + \frac{1}{|\mathbf{x}|^3}\sum_{j=1}^{3} x_j \int d^3x'\, J_i(\mathbf{x}')\, x'_j + \ldots\right\}\ . \tag{5.5.3}$$

Thus we need to know the volume integrals of $J_i(\mathbf{x}')$ and $J_i(\mathbf{x}')\,x'_j$, with $J_i(\mathbf{x}')$ in principle being an arbitrary function. For magnetostatics, however, it satisfies $\nabla\cdot\mathbf{J}(\mathbf{x}) \equiv 0$. Let us use this property. Integrating $\nabla'\cdot\mathbf{J}(\mathbf{x}') \equiv 0$ with any function $F(\mathbf{x}')$, we should get zero,

$$0 = -\int d^3x'\, F(\mathbf{x}')\nabla'\cdot\mathbf{J}(\mathbf{x}') = \int d^3x'\, \mathbf{J}(\mathbf{x}')\cdot\nabla' F(\mathbf{x}')\ , \tag{5.5.4}$$

where in the second step we have integrated by part and used the fact that the surface integral vanishes for a localised current distribution. In components, we have

$$\sum_{k=1}^{3}\int d^3x'\, J_k(\mathbf{x}')\nabla'_k F(\mathbf{x}') = 0\ . \tag{5.5.5}$$

We now consider the first term in Eq. (5.5.2). To get $J_i(\mathbf{x}')$ in the integrand of Eq. (5.5.5), we take $F(\mathbf{x}') = x'_i$, which using $\nabla'_k x'_i = \delta_{ik}$ results in

$$\begin{aligned} \sum_{k=1}^{3}\int d^3x'\, J_k\delta_{ik} &= 0 \\ \Rightarrow \int d^3x'\, J_i &= 0. \end{aligned}$$

Thus the first term in Eq. (5.5.3) vanishes. This is just a further restatement that there is no “monopole” contribution to the multipole expansion for magnetic fields.

For the second term in Eq. (5.5.3) we have $J_i(\mathbf{x}')\, x'_j$ in the integrand. To get it, we apply the identity Eq. (5.5.5) to the case $F = x'_i x'_j$. Then, using the chain rule we have

$$\begin{aligned}
\sum_{k=1}^{3} \int d^3x'\, J_k \left[\frac{\partial x'_i}{\partial x'_k} x'_j + x'_i \frac{\partial x'_j}{\partial x'_k}\right] &= 0 \\
\Rightarrow \sum_{k=1}^{3} \int d^3x'\, J_k \left[\delta_{ik} x'_j + x'_i \delta_{jk}\right] &= 0 \\
\Rightarrow \int d^3x' \left[J_i x'_j + J_j x'_i\right] &= 0
\end{aligned}$$

or

$$\int d^3x'\, J_i x'_j = -\int d^3x'\, J_j x'_i = \frac{1}{2}\int d^3x' \left[J_i x'_j - J_j x'_i\right] .$$

Thus, going back to Eq. (5.5.3), we may write

$$\begin{aligned}
A_i(\mathbf{x}) &= \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\sum_{j=1}^{3} x_j \int d^3x'\, J_i x'_j \\
&= -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\sum_{j=1}^{3} x_j \int d^3x' [x'_i J_j - x'_j J_i] ,
\end{aligned}$$

or, in vector form,

$$\mathbf{A}(\mathbf{x}) = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\int d^3x' [\mathbf{x}'(\mathbf{x}\cdot\mathbf{J}) - (\mathbf{x}'\cdot\mathbf{x})\mathbf{J}] ,$$

which may be also written as

$$\boxed{\mathbf{A}(\mathbf{x}) = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\mathbf{x}\times\int d^3x'\, \mathbf{x}'\times\mathbf{J}\ .}$$

This result may be easily verified using $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$, but it is instructive to derive it using the *Levi-Civita* tensor notations.

Levi-Civita Tensor

We recall the definition of the **Levi-Civita tensor**

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any two if } i,j,k \text{ are equal} \\ 1 & \text{if } (ijk) \text{ is an } \textit{even} \text{ permutation of } (123) \\ -1 & \text{if } (ijk) \text{ is an } \textit{odd} \text{ permutation of } (123) \end{cases} \tag{5.5.6}$$

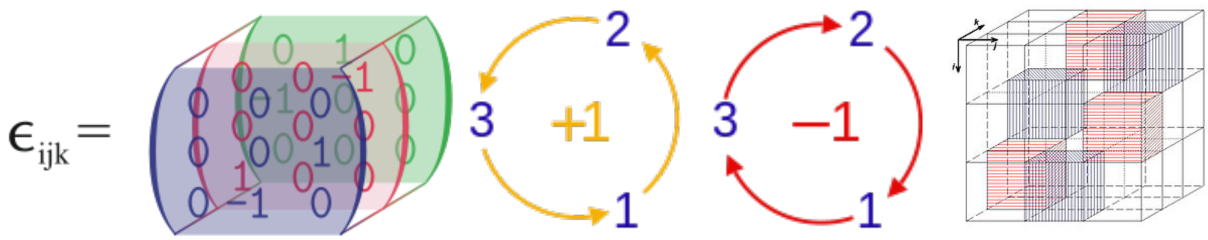


Figure 5.1: Visualizations of Levi-Civita symbol

This tensor is *isotropic*, and *totally anti-symmetric*. In particular, we have

$$\mathbf{A} \times \mathbf{B}|_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} A_j B_k. \tag{5.5.7}$$

There is the following well-known and easily shown identity

$$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}, \tag{5.5.8}$$

which we will now use to write

$$\begin{aligned} x_i' J_j - x_j' J_i &= \sum_{l,m=1}^{3} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) x_l' J_m \\ &= \sum_{k,l,m=1}^{3} \epsilon_{kij}\epsilon_{klm} x_l' J_m \\ &= \sum_{k=1}^{3} \epsilon_{ijk} (\mathbf{x}' \times \mathbf{J})_k \end{aligned}$$

and

$$\sum_{j=1}^{3} x_j \, [x_i' J_j - x_j' J_i] = \sum_{j=1}^{3} x_j \sum_{k=1}^{3} \epsilon_{ijk} (\mathbf{x}' \times \mathbf{J})_k = [\mathbf{x} \times (\mathbf{x}' \times \mathbf{J})]_i \ .$$

Thus we have

$$\boxed{A_i(\mathbf{x}) = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\left[\mathbf{x} \times \int d^3x' \, \mathbf{x}' \times \mathbf{J}\right]_i}$$

The vector

$$\mathbf{m} = \frac{1}{2}\int d^3x' \, \mathbf{x}' \times \mathbf{J} \tag{5.5.9}$$

is the **magnetic moment**, whilst

$$\boldsymbol{\mu} = \frac{1}{2}\mathbf{x}' \times \mathbf{J} \tag{5.5.10}$$

is the **magnetic moment density**. Thus we can write

$$\boxed{\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\mathbf{m} \times \mathbf{x}}$$

This is the lowest non-vanishing term in the multipole expansion of the magnetic vector potential for a localised current density. Applying $\mathbf{B} = \nabla \times \mathbf{A}$, we have

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\mathbf{x}\cdot\mathbf{m})\mathbf{x} - r^2\mathbf{m}}{r^5}\right], \tag{5.5.11}$$

exactly analogous to the electrostatic field due to a point dipole.[2]

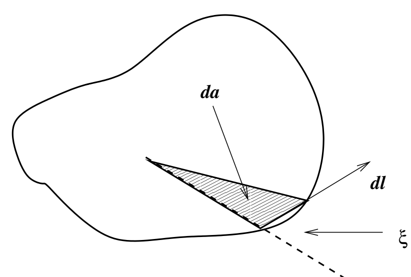
Example

For the case of a current confined to a loop, we have

$$\mathbf{m} = \frac{I}{2}\oint \mathbf{x} \times \mathbf{dl}. \tag{5.5.12}$$

Furthermore, if we have a planar loop, $\mathbf{x} \times \mathbf{dl}$ is normal to the plane of the loop, and we have

$$\begin{aligned} \frac{1}{2}\mathbf{x} \times \mathbf{dl} &= \mathbf{n}\,\frac{1}{2}x\,dl\,\sin\xi \\ &= da\,\mathbf{n} \end{aligned}$$



so that

$$\mathbf{m} = IA\,\mathbf{n} \tag{5.5.13}$$

where $\mathbf{n}$ is a normal to the plane of the loop, and A is the total area of the loop.

Example

We conclude this section by considering the case where the current distribution arises from the motion of a number of charged point-like particles:

$$\mathbf{J} = \sum_i q_i\mathbf{v}_i\delta(\mathbf{x} - \mathbf{x}_i), \tag{5.5.14}$$

[2]It is possible to introduce a vector potential to describe electric dipole fields

where $\mathbf{v}_i$ is the velocity of the i^{th} particle, which we assume is much less than the velocity of light. Then we have

$$\mathbf{m} = \frac{1}{2}\sum_i q_i \mathbf{x}_i \times \mathbf{v}_i. \tag{5.5.15}$$

Now the orbital angular momentum of a particle is given by

$$\mathbf{L}_i = M_i \mathbf{x}_i \times \mathbf{v}_i, \tag{5.5.16}$$

where M_i is the mass of the i^{th} particle. Thus we may write

$$\mathbf{m} = \sum_i \frac{q_i}{2M_i}\mathbf{L}_i. \tag{5.5.17}$$

In the case where all the particles have equal mass, we see that the magnetic moment is proportional to the **total angular momentum**.

5.6 Magnetostatics of Matter

5.6.1 Torques and forces on magnetic dipoles

First, consider a magnetic dipole in the uniform magnetic field $\mathbf{B}$. Let us visualize magnetic dipole $\mathbf{m}$ as a wire loop with area a carrying current I such as $m = Ia$.
The element of force on a current element $I\mathbf{dl}$ at $\mathbf{x}$ in a magnetic field $\mathbf{B}(\mathbf{x})$ is

$$\mathbf{dF} = I\mathbf{dl} \times \mathbf{B}. \tag{5.6.1}$$

The total force acting on the loop is zero:

$$\mathbf{F} \ = \ I\oint \mathbf{dl} \times \mathbf{B} \ = \ -I\mathbf{B} \times \oint \mathbf{dl} = 0 \ .$$

The torque acting on the loop is $\mathbf{m} \times \mathbf{B}$. Let us show this. Using $\mathbf{dx}' \equiv \mathbf{dl}$, we start with

$$\mathbf{N} \ = \oint \mathbf{x}' \times d\mathbf{F} \ = \oint \mathbf{x}' \times (I\mathbf{dx}' \times \mathbf{B}) \ = I\oint \mathbf{dx}'(\mathbf{x}' \cdot \mathbf{B}) - \frac{1}{2}\mathbf{B}I\oint d(x'^2) \ = I\oint \mathbf{dx}'(\mathbf{x}' \cdot \mathbf{B}) \ .$$

It is easy to prove that for an arbitrary constant vector $\mathbf{a}$

$$\oint \mathbf{dx}'(\mathbf{x}' \cdot \mathbf{a}) = -\frac{1}{2}\mathbf{a} \times \oint (\mathbf{x}' \times \mathbf{dx}') \ . \tag{5.6.2}$$

Indeed,

$$\mathbf{a} \times \oint (\mathbf{x}' \times \mathbf{dx}') = \oint [\mathbf{x}'(\mathbf{a} \cdot \mathbf{dx}') - \mathbf{dx}'(\mathbf{a} \cdot \mathbf{x}')] \ ,$$

furthermore

$$\oint \mathbf{x}'(\mathbf{a}\cdot\mathbf{dx}') = \oint [d(\mathbf{x}'(\mathbf{a}\cdot\mathbf{x}')) - \mathbf{dx}'(\mathbf{a}\cdot\mathbf{x}')] = -\oint \mathbf{dx}'(\mathbf{a}\cdot\mathbf{x}') \ , \tag{5.6.3}$$

and therefore

$$\mathbf{a}\times\oint(\mathbf{x}'\times\mathbf{dx}') = -2\oint \mathbf{dx}'(\mathbf{a}\cdot\mathbf{x}') \ .$$

Taking $\mathbf{a} = \mathbf{B}$ we get

$$\mathbf{N} = -\frac{I}{2}\mathbf{B}\times\oint(\mathbf{x}'\times\mathbf{dx}') \ = \ \left(\frac{I}{2}\oint \mathbf{x}'\times\mathbf{dx}'\right)\times\mathbf{B} \ = \ \mathbf{m}\times\mathbf{B} \tag{5.6.4}$$

so the torque in a uniform external field is a cross product of the magnetic moment and the field.

Let us now consider a small dipole in the non-uniform external field (the size of the dipole $\ll$ characteristic size of the field). The formula for the torque remains the same: $\mathbf{N} = \mathbf{m}\times\mathbf{B}$ where the magnetic field should be taken at the position of the dipole. However, the total force is no longer zero.

$$\mathbf{F} \ = \ I\oint \mathbf{dl}\times\mathbf{B} \ \neq 0 \ .$$

Since our dipole is small we can expand $\mathbf{B}(\mathbf{x}')$ in powers of $\mathbf{x}'$. For simplicity, suppose that the dipole is located at the origin. We get

$$\mathbf{B}(\mathbf{x}') = \mathbf{B}(0) + (\mathbf{x}'\cdot\nabla)\,\mathbf{B}(0) + ...$$

and therefore (using $\mathbf{dl}' \equiv \mathbf{dx}'$)

$$\mathbf{F} \ = \ I\oint \mathbf{dl}'\times\mathbf{B}(0) + I\oint \mathbf{dl}'\times(\mathbf{x}'\cdot\nabla)\mathbf{B} + O(x'^2) \ = \ I\oint \mathbf{dx}'(\mathbf{x}'\cdot\nabla)\times\mathbf{B} + O(x'^2) \ .$$

Next we use formula (5.6.2) with $\mathbf{a} = \nabla$ and obtain

$$I\oint \mathbf{dx}'(\mathbf{x}'\cdot\nabla) \ = \ \frac{I}{2}\oint(\mathbf{x}'\times\mathbf{dx}')\times\nabla \ = \ \mathbf{m}\times\nabla$$

so finally

$$\mathbf{F} \ = \ (\mathbf{m}\times\nabla)\times\mathbf{B} \ = \ \nabla(\mathbf{m}\cdot\mathbf{B}) - \mathbf{m}(\nabla\cdot\mathbf{B}) \ = \ \nabla(\mathbf{m}\cdot\mathbf{B})$$

because $\nabla\cdot\mathbf{B} = 0$.

Since $\mathbf{F} = -\nabla U$ we see that the potential energy of a (small) magnetic dipole in the external magnetic field is

$$U \ = \ -\,\mathbf{m}\cdot\mathbf{B}$$

(similarly to $U = -\mathbf{p}\cdot\mathbf{E}$ for the electric dipole).

5.6.2 Maxwell equations in matter

We could, in principle, attempt to describe the magnetostatics of a material in terms of the microscopic, or "vacuum", fields. As in the case of electrostatics, this approach is neither feasible nor desirable. At the microscopic level, the individual atoms have magnetic moments and eddy currents are generated that we cannot account for exactly. Rather, we discuss macroscopic quantities, including that part of the magnetic field arising from these microscopic currents. In the following, we will use the subscript *micro* to denote microscopic properties, with the remaining variables denoting macroscopic quantities.
At the microscopic level, we have $\nabla \cdot \mathbf{B}_{\rm micro} = 0$. We can average this to obtain

$$\nabla \cdot \mathbf{B} = 0 \tag{5.6.5}$$

and hence we know that we can write the *macroscopic* magnetic field in terms of a vector potential

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{5.6.6}$$

Suppose now that we have a collection of atoms of various types i, with magnetization $\mathbf{m}_i$. Then the macroscopic magnetization

$$\mathbf{M} = \sum_i N_i \langle \mathbf{m}_i \rangle, \tag{5.6.7}$$

where N_i is the numer of atoms of type i/unit volume, and $\langle \mathbf{m}_i \rangle$ is their average magnetic moment. Note the $\mathbf{M}$ is analogous to the polarization density of electrostatics.
We will now consider the contribution to the vector potential at $\mathbf{x}$ due to an infinitesimal volume ΔV at $\mathbf{x}'$. There are two contributions

$$\Delta \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{\mathbf{J}_f(\mathbf{x}')}{\mid \mathbf{x} - \mathbf{x}' \mid} \Delta V + \frac{\mu_0}{4\pi} \frac{1}{\mid \mathbf{x} - \mathbf{x}' \mid^3} \mathbf{M} \times (\mathbf{x} - \mathbf{x}') \Delta V, \tag{5.6.8}$$

where the first term arises from the "free" macroscopic current densities and the second is due to the macroscopic magnetization described above. We now sum over the volume elements ΔV and get

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\mathbf{J}_f(\mathbf{x}')}{\mid \mathbf{x} - \mathbf{x}' \mid} + \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{\mid \mathbf{x} - \mathbf{x}' \mid^3} \mathbf{M} \times (\mathbf{x} - \mathbf{x}'). \tag{5.6.9}$$

There is a way to rewrite the second term in a more illuminating way. First, note that

$$\int d^3x' \frac{\mathbf{M} \times (\mathbf{x} - \mathbf{x}')}{\mid \mathbf{x} - \mathbf{x}' \mid^3} = \int d^3x' \, \mathbf{M} \times \nabla' \left(\frac{1}{\mid \mathbf{x} - \mathbf{x}' \mid} \right) \tag{5.6.10}$$

Then we use the formula $\nabla \times (f\mathbf{M}) = f(\nabla \times \mathbf{M}) - \mathbf{M} \times (\nabla f)$ which follows from

$$\begin{aligned}\nabla \times (f\mathbf{M})_i &= \sum_{j,k} \epsilon_{ijk} \nabla_j (f M_k) = \sum_{j,k} \epsilon_{ijk} [(\nabla_j f) M_k + f(\nabla_j M_k)] \\ &= -\underbrace{\sum_{j,k} \epsilon_{ijk} M_j (\nabla_k f)}_{(M \times \nabla f)_i} + f \underbrace{\sum_{j,k} \epsilon_{ijk} \nabla_j M_k}_{(\nabla \times M)_i} \ . \end{aligned} \tag{5.6.11}$$

Rewriting it as $\mathbf{M} \times (\nabla f) = f(\nabla \times \mathbf{M}) - \nabla \times (f\mathbf{M})$ and substituting $f = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$ we obtain

$$\int d^3x' \, \mathbf{M} \times \nabla' \frac{1}{\mid \mathbf{x} - \mathbf{x}' \mid} = \int d^3x' \ \frac{1}{\mid \mathbf{x} - \mathbf{x}' \mid} \ \nabla' \times \mathbf{M}(\mathbf{x}') \ - \int d^3x' \ \nabla' \times \left(\frac{\mathbf{M}(\mathbf{x}')}{\mid \mathbf{x} - \mathbf{x}' \mid} \right) . \tag{5.6.12}$$

Using the divergence theorem for vector fields (see the cover of *Jackson*)

$$\int_V d^3x' \ \nabla' \times \mathbf{A}(\mathbf{x}') \ = \ \int_{S=\partial V} \mathbf{n} \times \mathbf{A} \ dS \tag{5.6.13}$$

the second term can be rewritten as a surface integral

$$\int_{S=\partial V} \frac{\mathbf{M}(\mathbf{x}') \times \mathbf{n}}{|\mathbf{x} - \mathbf{x}'|} \ dS \ . \tag{5.6.14}$$

Finally, we get

$$\frac{\mu_0}{4\pi} \int d^3x' \, \frac{\mathbf{M} \times (\mathbf{x} - \mathbf{x}')}{\mid \mathbf{x} - \mathbf{x}' \mid^3} = \frac{\mu_0}{4\pi} \int d^3x' \ \frac{\mathbf{J}_b}{\mid \mathbf{x} - \mathbf{x}' \mid} + \frac{\mu_0}{4\pi} \int_{S=\partial V} \frac{\mathbf{K}_b(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \ dS \tag{5.6.15}$$

where $J_b \equiv \nabla \times \mathbf{M}$ is called a bound volume current density and $K_b \equiv \mathbf{M} \times \mathbf{n}$ a bound surface current density.

If we take the surface to be an infinitely large sphere and assume that K vanishes at infinity, we get

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{dV'}{\mid \mathbf{x} - \mathbf{x}' \mid} \{\mathbf{J}_f(\mathbf{x}') + \nabla' \times \mathbf{M}'\} \, . \tag{5.6.16}$$

Comparing with the fundamental equation of magnetostatics *in vacua*,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_f , \tag{5.6.17}$$

we have

$$\nabla \times \mathbf{B} = \mu_0 \{\mathbf{J}_f + \nabla \times \mathbf{M}\}. \tag{5.6.18}$$

It is now conventional to introduce the **magnetic field H**, where

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}. \tag{5.6.19}$$

In the context of media, the field **B** is known as the **magnetic induction** or **magnetic flux density**. In terms of **H** and **B**, the fundamental equations of magnetostatics in matter are

$$\nabla \cdot \mathbf{B} = 0\ . \tag{5.6.20}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f\ . \tag{5.6.21}$$

Note that **H** is analogous to **D** in electrostatics; **E** and **B** are the fundamental fields, whilst **H** and **D** depend on the medium.

5.6.3 Constitutive relation

In the case of (isotropic) diamagnetic and paramagnetic materials, where the magnetic moment arises solely from the applied magnetic field, there is a simple linear relation between **H** and **B**

$$\mathbf{M} = \chi_m \mathbf{H}, \tag{5.6.22}$$

where χ_m is the **magnetic susceptibility**. Then we may write

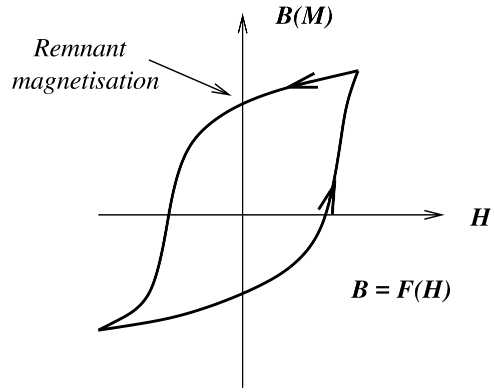
$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \chi_m \mathbf{H} \tag{5.6.23}$$

yielding

$$\mathbf{B} = \mu \mathbf{H} \tag{5.6.24}$$

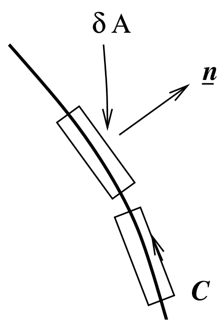
where $\mu \equiv \mu_0(1 + \chi_m)$ is the **magnetic permeability**.

For *ferromagnets*, the corresponding relation is **non-linear** and exhibits *hysteresis*, i.e. the material retains a *memory* of its preparation.



5.6.4 Boundary Conditions at Surface Between Media

We will now obtain boundary conditions for the normal and tangential components of the field at the boundary between two materials. Note that the following discussion is independent of whether or not there is a linear relation between the **H** and **B**.



Normal Condition

Apply Gauss' Law to the pillbox shown

$$0 = \int dV\, \nabla \cdot \mathbf{B} = \int \mathbf{B} \cdot \mathbf{n}\, dS = (\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n}\, \delta A \tag{5.6.25}$$

where $\mathbf{n}$ is a unit normal from medium 1 to medium 2, and δA is the surface area of the pillbox. Thus we have

$$\boxed{\mathbf{B}_1^{\perp} = \mathbf{B}_2^{\perp}}$$

Tangential Condition

To get the boundary conditions on the tangential components, we apply Stoke's theorem to a contour C having the size Δl along the boundary surface and lying in plane perpendicular to it

$$\oint_C \mathbf{H} \cdot \mathbf{dl} = \int_S (\nabla \times \mathbf{H}) \cdot \mathbf{da} = \int_S \mathbf{J}_f \cdot \mathbf{da} = \left(\int J_f dz \right) \Delta l \equiv K \Delta l, \tag{5.6.26}$$

where S is a surface spanning C and K is the **surface current density**.
Thus we have the tangential boundary condition $H_2^{\|} - H_1^{\|} = K$ or

$$\boxed{\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}\ .}$$

5.7 Methods of Solving Boundary Value Problems

We will now look at various methods of solving boundary value problems between different media. The method depends on nature of the constitutive relation between **B** and **H**, and on whether there is non-zero current density.

5.7.1 Vector Potential

The magnetic field is always solenoidal, and therefore we can essentially *always* introduce a vector potential **A** such that $\mathbf{B} = \nabla \times \mathbf{A}$.
The dynamical information for the magnetostatics of media is provided by the equation

$$\nabla \times \mathbf{H} = \mathbf{J}_f. \tag{5.7.1}$$

We will now specialise to the case where we have a **linear** constitutive relation, $\mathbf{B} = \mu \mathbf{H}$, enabling us to write

$$\nabla \times \left[\frac{\nabla \times \mathbf{A}}{\mu} \right] = \mathbf{J}_f. \tag{5.7.2}$$

This can be written

$$\nabla^2 \mathbf{A} - \nabla[\nabla \cdot \mathbf{A}] = -\mu \mathbf{J}_f, \tag{5.7.3}$$

which in Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) becomes

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}_f. \tag{5.7.4}$$

This is analogous to the case discussed in Section 5.4.2, and the solution is that of Eq. (5.4.8), with μ_0 replaced by μ.

5.7.2 Solution when $\mathbf{J_f} \equiv 0$

In this case we have $\nabla \times \mathbf{H} = 0$, and therefore we may admit introduce a **scalar potential** ϕ_M such that

$$\mathbf{H} = -\nabla\phi_M. \tag{5.7.5}$$

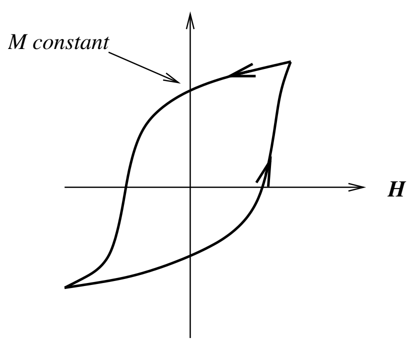
Once again, we will consider *linear* media, so that $\mathbf{B} = \mu\mathbf{H}$. Then we find that the scalar potential satisfies Laplace's equation

$$\nabla^2\phi_M = 0, \tag{5.7.6}$$

where we assume that μ is piecewise constant, i.e has a constant value in each of the different media we are considering.

5.7.3 Hard Ferromagnetic

In the case of a hard ferromagnet, we have $\mathbf{J}_f \equiv 0$, and the magnetization is non-zero, and essentially independent of the magnetic field $\mathbf{H}$ provided it is sufficiently small.



Since $\mathbf{J}_f \equiv 0$, we can solve this problem using either a scalar or a vector potential.

Solution using Scalar Potential

The governing equations are

$$\nabla \cdot \mathbf{B} = 0 \tag{5.7.7}$$
$$\nabla \times \mathbf{H} = 0 \tag{5.7.8}$$
$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \tag{5.7.9}$$

Since $\nabla \times \mathbf{H} = 0$, we can introduce a scalar potential for the magnetic field,

$$\mathbf{H} = -\nabla\phi_M. \tag{5.7.10}$$

Then from Eq. (5.7.9), we have $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ and from Eq. (5.7.7)

$$\nabla^2 \phi_M = -\rho_M, \tag{5.7.11}$$

where

$$\rho_M = -\nabla \cdot \mathbf{M}. \tag{5.7.12}$$

In the case where there are no boundaries, this equation has the solution

$$\begin{aligned} \phi_M &= \frac{1}{4\pi} \int d^3x' \, \frac{\rho_M(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = -\frac{1}{4\pi} \int d^3x' \, \frac{\nabla' \cdot \mathbf{M}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \qquad (\textit{integration by parts}) \\ &= \frac{1}{4\pi} \int d^3x' \, \mathbf{M}(\mathbf{x}') \, \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= -\frac{1}{4\pi} \nabla \cdot \int d^3x' \, \frac{1}{|\mathbf{x} - \mathbf{x}'|} \mathbf{M}(\mathbf{x}'). \end{aligned}$$

Note that if we are far away from a non-zero $\mathbf{M}$, i.e. $r \gg r'$, then we have

$$\phi_M \simeq -\frac{1}{4\pi} \nabla \left(\frac{1}{r} \right) \cdot \int d^3x' \, \mathbf{M}(\mathbf{x}') = \frac{1}{4\pi r^3} \mathbf{m} \cdot \mathbf{x}. \tag{5.7.13}$$

where

$$\mathbf{m} = \int d^3x' \, \mathbf{M}(\mathbf{x}'). \tag{5.7.14}$$

Suppose now that we had a hard ferromagnet confined to a volume V, with surface S. Then there is a contribution arising from the discontinuity in $\mathbf{M}$ at the surface, which we can express as a *surface magnetization density*,

$$\sigma_M = \mathbf{n} \cdot \mathbf{M}, \tag{5.7.15}$$

and apply Gauss' Law to obtain its contribution

$$\phi_M = -\frac{1}{4\pi} \int_V d^3x' \frac{\nabla' \cdot \mathbf{M}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \frac{1}{4\pi} \oint_S dS \frac{\sigma_M}{|\mathbf{x} - \mathbf{x}'|}. \tag{5.7.16}$$

Note that for a *uniform* magnetization, the bulk volume integral vanishes, and the only contribution arises from the surface term.

Solution using Vector Potential

We now write $\mathbf{B} = \nabla \times \mathbf{A}$, so that we have

$$\mathbf{H} = \frac{1}{\mu_0} \nabla \times \mathbf{A} - \mathbf{M}. \tag{5.7.17}$$

Thus Eq. (5.7.8) becomes

$$0 = \nabla \times \mathbf{H} = \frac{1}{\mu_0}\nabla \times (\nabla \times \mathbf{A}) - \nabla \times \mathbf{M}. \tag{5.7.18}$$

Introducing an effective magnetization current

$$\mathbf{J}_{\mathrm{M}} = \nabla \times \mathbf{M}, \tag{5.7.19}$$

we have, in Coulomb gauge,

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}_M. \tag{5.7.20}$$

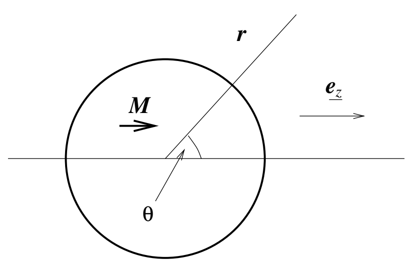
Thus again each component of **A** satisfies Poisson's equation, with solution

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\mathbf{J}_M}{|\mathbf{x}-\mathbf{x}'|}. \tag{5.7.21}$$

In the case where there is a sharp boundary between two media, we again have a surface contribution which we treat as for the case of a scalar potential, yielding

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V d^3x' \, \frac{\nabla' \times \mathbf{M}}{|\mathbf{x}-\mathbf{x}'|} + \frac{\mu_0}{4\pi}\oint_S dS \frac{\mathbf{M}(\mathbf{x}') \times \mathbf{n}'}{|\mathbf{x}-\mathbf{x}'|}. \tag{5.7.22}$$

Example: uniformly magnetized sphere in a vacuum



Consider a sphere of radius a, with uniform magnetization $\mathbf{M} = M_0\mathbf{e_z}$. We will consider the solution using a scalar potential.
Since the magnetization is constant throughout the body of the sphere, only the surface integral contributes in Eq. (5.7.16), and we have

$$\begin{aligned}\phi_M &= \frac{1}{4\pi}\oint_S dS' \, \frac{\mathbf{n}' \cdot \mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \\ &= \frac{M_0 a^2}{4\pi}\int d\Omega' \, \frac{\cos\theta'}{|\mathbf{x}-\mathbf{x}'|}.\end{aligned}$$

To proceed further, we expand $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ in terms of spherical harmonics

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi). \tag{5.7.23}$$

Noting that $\cos\theta' = P_1(\cos\theta') = \sqrt{4\pi} Y_{10}(\theta', \varphi')$, and using orthogonality, we can write

$$\phi_M(r, \theta) = \frac{1}{3} M_0 a^2 \frac{r_<}{r_>^2} \cos\theta, \tag{5.7.24}$$

where $r_{<\{>\}} = \min\{\max\}(r, a)$.
Inside the sphere, we have $r_< = r$ and $r_> = a$. Thus

$$\phi_M = \frac{1}{3} M_0 r \cos\theta = \frac{1}{3} M_0 z, \tag{5.7.25}$$

which gives

$$\left.\begin{aligned} \mathbf{H}_{\text{in}} = -\nabla\phi_M = -\tfrac{1}{3}\mathbf{M} \\ \mathbf{B}_{\text{in}} = \mu_0(\mathbf{H}+\mathbf{M}) = \tfrac{2}{3}\mu_0\mathbf{M} \end{aligned}\right\}, \tag{5.7.26}$$

and we have that **H** (**B**) is anti-parallel (parallel) to **M**.
Outside the sphere,

$$\phi_M = \frac{1}{3} M_0 \frac{a^3}{r^2} \cos\theta. \tag{5.7.27}$$

Since **M** is uniform inside the sphere, we can associate this with the potential due to a magnetic dipole of moment

$$\mathbf{m} = \frac{4\pi a^3}{3} \mathbf{M}. \tag{5.7.28}$$

The magnetic induction is parallel to the magnetic field, and given by

$$\mathbf{B}_{\text{out}} = \mu_0 \mathbf{H}_{\text{out}} = -\mu_0 \nabla\phi_M = \frac{2}{3} M_0 \mu_0 \frac{a^3}{r^3} (\mathbf{e_r} \cos\theta + \frac{\mathbf{e}_\theta}{2} \sin\theta) \tag{5.7.29}$$

Sphere in External Field

Suppose now we add a uniform magnetic induction $\mathbf{B_0} = \mu_0 \mathbf{H_0}$. Then by the principle of linear superposition, the resulting field inside the sphere is just the sum of the two solutions

$$\mathbf{B}_{\text{in}} = \mathbf{B_0} + \frac{2\mu_0}{3}\mathbf{M} \tag{5.7.30}$$

$$\mathbf{H}_{\text{in}} = \frac{1}{\mu_0}\mathbf{B_0} - \frac{1}{3}\mathbf{M} \tag{5.7.31}$$

Suppose now that the substance is *not* permanently magnetized, but rather has a linear relation between **B** and **H**,

$$\mathbf{B}_{\text{in}} = \mu \mathbf{H}_{\text{in}} \ . \tag{5.7.32}$$

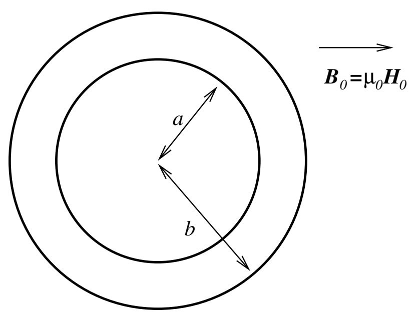
Then **M** is also linearly related, and from eqns. (5.7.30) and (5.7.31) we have

$$\mathbf{M} = \frac{3}{\mu_0}\left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right)\mathbf{B_0}. \tag{5.7.33}$$

For the case of ferromagnets described earlier, we do not have such a linear relation; indeed we have non-zero **M** for zero applied magnetic field. We can obtain one relation between $\mathbf{B}_{\text{in}}$ and $\mathbf{H}_{\text{in}}$ by eliminating **M** in Eqs. (5.7.30) and (5.7.31), whilst obtaining another from the hysteresis curve.

Example: spherical shell in uniform field

Consider a shell of permeability μ in a vacuum, as shown below.



Since the current density is zero, we can once again write $\mathbf{H} = -\nabla\phi_M$. Furthermore, $\mathbf{B} = \mu\mathbf{H}$, and thus $\nabla \cdot \mathbf{H} = 0$ so that the scalar potential satisfies

$$\nabla^2 \phi_M = 0, \tag{5.7.34}$$

subject to the boundary conditions at $r = a$ and $r = b$. We are now experts at writing down the solution in terms of Legendre polynomials.

$$\begin{aligned}
\phi_M &= -H_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta) \qquad r > b \\
\phi_M &= \sum_{l=0}^{\infty}\left[\beta_l r^l + \frac{\gamma_l}{r^{l+1}}\right] P_l(\cos\theta) \qquad a < r < b \\
\phi_M &= \sum_{l=0}^{\infty} \delta_l r^l P_l(\cos\theta) \qquad r < a
\end{aligned}$$

where we have imposed that there be a uniform field at infinity for the case $r > b$, and that the solution is regular as $r \to 0$.

We now impose the boundary conditions at the interfaces $r = a$ and $r = b$

$$\begin{array}{c}\mathbf{B}^{\perp} \text{ is continuous} \\ \mathbf{H}^{\parallel} \text{ is continuous}\end{array} \tag{5.7.35}$$

which become:

$$\begin{aligned}
\frac{\partial \phi_M}{\partial \theta}(b_+) &= \frac{\partial \phi_M}{\partial \theta}(b_-) \\
\mu_0 \frac{\partial \phi_M}{\partial r}(b_+) &= \mu \frac{\partial \phi_M}{\partial r}(b_-) \\
\frac{\partial \phi_M}{\partial \theta}(a_+) &= \frac{\partial \phi_M}{\partial \theta}(a_-) \\
\mu_0 \frac{\partial \phi_M}{\partial r}(a_-) &= \mu \frac{\partial \phi_M}{\partial r}(a_+)
\end{aligned}$$

We now use these equations to determine the coefficients $\alpha_l, \beta_l, \gamma_l$, noting that

$$\frac{\partial}{\partial \theta} P_l(\cos\theta) = P_l^1(\cos\theta). \tag{5.7.36}$$

All the coefficients vanish for $l > 1$ (*exercise*), and we have (see *Jackson*)

$$\begin{aligned}
\alpha_1 &= \left[\frac{(2\mu'+1)(\mu'-1)}{(2\mu'+1)(\mu'+2) - \frac{2a^3}{b^3}(\mu'-1)^2}\right](b^3 - a^3) H_0 \\
\delta_1 &= -\left[\frac{9\mu'}{(2\mu'+1)(\mu'+2) - \frac{2a^3}{b^3}(\mu'-1)^2}\right] H_0,
\end{aligned} \tag{5.7.37}$$

where $\mu' = \mu/\mu_0$.

For $r > b$, we have the uniform field together with a dipole of moment α_1, parallel to H_1:

$$\phi_M = -H_0 r \cos\theta + \frac{\alpha_1}{r^2}\cos\theta. \tag{5.7.38}$$

For $r < a$, there is a uniform magnetic field parallel to H_0, of magnitude $-\delta_1$:

$$\phi_M = -(-\delta_1) r \cos\theta. \tag{5.7.39}$$

From Eq. (5.7.37), we see that $\delta_1 \simeq 1/\mu'$ as $\mu' \to \infty$: the effect of a shell of high permeability is to shield the interior from the magnetic field.