1. Element of volume in spherical polar coordinates

Spherical polar coordinates

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = \cos\theta \tag{1}$$

Element of volume in spherical polars

$$\int dxdydz\; F(x,y,z) = \int \left|\frac{\partial x\;\partial y\;\partial z}{\partial r\;\partial\theta\;\partial\phi}\right| drd\theta d\phi\; F(r,\theta,\phi) \tag{2}$$

where the Jacobian is defined as

$$\frac{\partial x\;\partial y\;\partial z}{\partial r\;\partial\theta\;\partial\phi} \equiv \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta} & \frac{\partial z}{\partial\theta} \\ \frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi} & \frac{\partial z}{\partial\phi} \end{pmatrix} = \det\begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\sin\phi & r\cos\theta\cos\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix} = r^2\sin\theta \tag{3}$$

so that

$$\int dxdydz\; F(x,y,z) = \int drd\theta d\phi\; r^2\sin\theta\; F(r,\theta,\phi) \tag{4}$$

2. Unit vectors in spherical polar coordinates

Unit vectors

$$\begin{aligned} \hat{r} &= \hat{e}_1\sin\theta\cos\phi + \hat{e}_2\sin\theta\sin\phi + \hat{e}_3\cos\theta \\ \hat{\theta} &= \hat{e}_1\cos\theta\cos\phi + \hat{e}_2\cos\theta\sin\phi - \hat{e}_3\sin\theta \\ \hat{\phi} &= -\hat{e}_1\sin\phi + \hat{e}_2\cos\phi \end{aligned} \tag{5}$$

Set of partial derivatives

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{r} = \sin\theta\cos\phi & \frac{\partial r}{\partial y} &= \frac{y}{r} = \sin\theta\sin\phi & \frac{\partial r}{\partial z} &= \frac{z}{r} = \cos\theta \\ \frac{\partial\theta}{\partial x} &= \frac{\cos\theta\cos\phi}{r} & \frac{\partial\theta}{\partial y} &= \frac{\cos\theta\sin\phi}{r} & \frac{\partial\theta}{\partial z} &= -\frac{\sin\theta}{r} \\ \frac{\partial\phi}{\partial x} &= -\frac{\sin\phi}{r\sin\theta} & \frac{\partial\phi}{\partial y} &= \frac{\cos\phi}{r\sin\theta} & \frac{\partial\phi}{\partial z} &= 0 \end{aligned}$$

3. Divergence of vector field $\vec{v}(\vec{r})$ is spherical polars

Definition: if $\vec{v}(x,y,z) = \hat{e}_1 v_1(x,y,z) + \hat{e}_2 v_2(x,y,z) + \hat{e}_3 v_3(x,y,z)$

$$\vec{\nabla}\cdot\vec{v}(x,y,z) \equiv \frac{\partial v_1(x,y,z)}{\partial x} + \frac{\partial v_2(x,y,z)}{\partial y} + \frac{\partial v_3(x,y,z)}{\partial z} \tag{6}$$

Rewrite $\vec{v}$ in spherical polars

$$\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3 = v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi} \tag{7}$$

From Eqs. (5) and (6) one gets

$$\begin{aligned} v_r &= \vec{v}\cdot\hat{r} = v_1\sin\theta\cos\phi + v_2\sin\theta\sin\phi + v_3\cos\theta \\ v_\theta &= \vec{v}\cdot\hat{\theta} = v_1\cos\theta\cos\phi + v_2\cos\theta\sin\phi - v_3\sin\theta \\ v_\phi &= \vec{v}\cdot\hat{\phi} = -v_1\sin\phi + v_2\cos\phi \end{aligned} \tag{8}$$

and

$$\begin{aligned} v_1 &= \vec{v}\cdot\hat{e}_1 = v_r\sin\theta\cos\phi + v_\theta\cos\theta\cos\phi - v_\phi\sin\phi \\ v_2 &= \vec{v}\cdot\hat{e}_2 = v_r\sin\theta\sin\phi + v_\theta\cos\theta\sin\phi + v_\phi\cos\phi \\ v_3 &= \vec{v}\cdot\hat{e}_3 = v_r\cos\theta - v_\theta\sin\theta \end{aligned} \tag{9}$$

By chain rule

$$\begin{aligned} \frac{\partial v_1}{\partial x} &= \frac{\partial v_1}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v_1}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial v_1}{\partial \phi}\frac{\partial \phi}{\partial x} = \sin\theta\cos\phi\frac{\partial v_1}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial v_1}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial v_1}{\partial \phi} \\ \frac{\partial v_2}{\partial y} &= \frac{\partial v_2}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v_2}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial v_2}{\partial \phi}\frac{\partial \phi}{\partial y} = \sin\theta\sin\phi\frac{\partial v_2}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial v_2}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial v_2}{\partial \phi} \\ \frac{\partial v_3}{\partial z} &= \frac{\partial v_3}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial v_3}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial v_3}{\partial \phi}\frac{\partial \phi}{\partial z} = \cos\theta\frac{\partial v_3}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v_3}{\partial \theta} \end{aligned} \tag{10}$$

$\Rightarrow$

$$\begin{aligned} \frac{\partial v_1}{\partial x} &= \sin\theta\cos\phi\Big[\frac{\partial v_r}{\partial r}\sin\theta\cos\phi + \frac{\partial v_\theta}{\partial r}\cos\theta\cos\phi - \frac{\partial v_\phi}{\partial r}\sin\phi\Big] \\ &+ \frac{\cos\theta\cos\phi}{r}\Big[\frac{\partial v_r}{\partial \theta}\sin\theta\cos\phi + \frac{\partial v_\theta}{\partial \theta}\cos\theta\cos\phi - \frac{\partial v_\phi}{\partial \theta}\sin\phi + v_r\cos\theta\cos\phi - v_\theta\sin\theta\cos\phi\Big] \\ &- \frac{\sin\phi}{r\sin\theta}\Big[\frac{\partial v_r}{\partial \phi}\sin\theta\cos\phi + \frac{\partial v_\theta}{\partial \phi}\cos\theta\cos\phi - \frac{\partial v_\phi}{\partial \phi}\sin\phi - v_r\sin\theta\sin\phi - v_\theta\cos\theta\sin\phi - v_\phi\cos\phi\Big], \\ \frac{\partial v_2}{\partial y} &= \sin\theta\sin\phi\Big[\frac{\partial v_r}{\partial r}\sin\theta\sin\phi + \frac{\partial v_\theta}{\partial r}\cos\theta\sin\phi + \frac{\partial v_\phi}{\partial r}\cos\phi\Big] \\ &+ \frac{\cos\theta\sin\phi}{r}\Big[\frac{\partial v_r}{\partial \theta}\sin\theta\sin\phi + \frac{\partial v_\theta}{\partial \theta}\cos\theta\sin\phi + \frac{\partial v_\phi}{\partial \theta}\cos\phi + v_r\cos\theta\sin\phi - v_\theta\sin\theta\sin\phi\Big] \\ &+ \frac{\cos\phi}{r\sin\theta}\Big[\frac{\partial v_r}{\partial \phi}\sin\theta\sin\phi + \frac{\partial v_\theta}{\partial \phi}\cos\theta\sin\phi + \frac{\partial v_\phi}{\partial \phi}\cos\phi + v_r\sin\theta\cos\phi + v_\theta\cos\theta\cos\phi - v_\phi\sin\phi\Big], \\ \frac{\partial v_3}{\partial z} &= \cos\theta\Big[\frac{\partial v_r}{\partial r}\cos\theta - \frac{\partial v_\theta}{\partial r}\sin\theta\Big] - \frac{\sin\theta}{r}\Big[\frac{\partial v_r}{\partial \theta}\cos\theta - \frac{\partial v_\theta}{\partial \theta}\sin\theta - v_r\sin\theta - v_\theta\cos\theta\Big] \end{aligned} \tag{11}$$

In the sum colored terms cancel and we get

$$\begin{aligned} &\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \\ =\ & \sin\theta\cos\phi\frac{\partial v_r}{\partial r}\sin\theta\cos\phi + \frac{\cos\theta\cos\phi}{r}\Big[\frac{\partial v_\theta}{\partial \theta}\cos\theta\cos\phi + v_r\cos\theta\cos\phi\Big] + \frac{\sin\phi}{r\sin\theta}\Big[\frac{\partial v_\phi}{\partial \phi}\sin\phi + v_r\sin\theta\sin\phi + v_\theta\cos\theta\sin\phi\Big] \\ +\ & \sin\theta\sin\phi\frac{\partial v_r}{\partial r}\sin\theta\sin\phi + \frac{\cos\theta\sin\phi}{r}\Big[\frac{\partial v_\theta}{\partial \theta}\cos\theta\sin\phi + v_r\cos\theta\sin\phi\Big] + \frac{\cos\phi}{r\sin\theta}\Big[\frac{\partial v_\phi}{\partial \phi}\cos\phi + v_r\sin\theta\cos\phi + v_\theta\cos\theta\cos\phi\Big] \\ +\ & \cos\theta\frac{\partial v_r}{\partial r}\cos\theta + \frac{\sin\theta}{r}\Big[\frac{\partial v_\theta}{\partial \theta}\sin\theta + v_r\sin\theta\Big] \\ =\ & \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + 2\frac{v_r}{r} + \frac{v_\theta\cos\theta}{\sin\theta} = \frac{1}{r^2}\frac{\partial}{\partial dr}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \end{aligned} \tag{12}$$