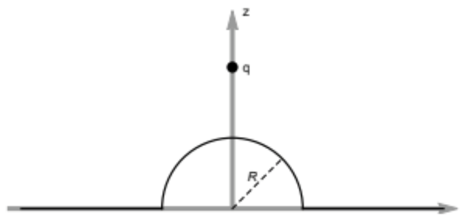
604 Final Exam (40 points). 12/15/11, 15:45 - 18:45

Problem 1. A conducting surface grounded at infinity consists of a plane with a hemispherical bump of radius R (see the figure below). A charge q sits a distance $r > R$ above the center of the hemispherical bump.



Calculate the force on the charge. (Hint: Use image charges. Note that you may need more than one image charge.)

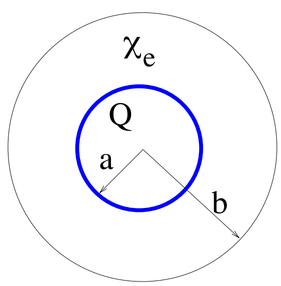
Solution:
By symmetry

$$\Phi(\vec{x}) \;=\; \frac{q}{4\pi\epsilon_0}\Big[\frac{1}{|\vec{x}-\hat{z}r|} - \frac{R/r}{|\vec{x}-\hat{z}\frac{R^2}{r}|} + \frac{R/r}{|\vec{x}+\hat{z}\frac{R^2}{r}|} - \frac{1}{|\vec{x}+\hat{z}r|}\Big]$$

which vanishes on the sphere and on $z = 0$ plane. The force due to image charges is

$$\vec{F} \;=\; \hat{z}\frac{q}{4\pi\epsilon_0}\Big[\frac{qR/r}{\big(r-\frac{R^2}{r}\big)^2} - \frac{qR/r}{\big(r+\frac{R^2}{r}\big)^2} + \frac{q}{4r^2}\Big]$$

Problem 2.
A conducting sphere of radius a carries charge Q. It is surrounded by linear dielectric material of susceptibility χ_e, out to radius b. Find the energy of this configuration.



Solution
The problem has spherical symmetry $\Rightarrow$ from Gauss' law we get $\vec{D}(r) = \frac{Q}{4\pi r^2}\hat{r} \;\Rightarrow\; \vec{E}(r) = \frac{Q}{4\pi\epsilon r^2}\hat{r}$ inside the dielectric and $\vec{D}(r) = \frac{Q}{4\pi r^2}\hat{r}$, $\vec{E}(r) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$ outside the dielectric.
The energy is

$$W \;=\; \frac{1}{2}\int d^3x \vec{D}\cdot\vec{E} \;=\; 2\pi\int_a^b r^2 dr\frac{D^2}{\epsilon} + 2\pi\int_b^\infty r^2 dr\frac{D^2}{\epsilon_0} \;=\; \frac{Q}{8\pi^2\epsilon_0}\Big[\frac{1}{\chi_e}\big(\frac{1}{a}-\frac{1}{b}\big) + \frac{1}{b}\Big]$$

Problem 3.
A grounded infinite conducting cylinder of radius a is placed in an otherwise uniform electric field E orthogonal to the axis of the cylinder. Find the potential outside the cylinder.

Solution
A general solution of Laplace equation has the form

$$\phi(s,\varphi) \;=\; a_0 + b_0 \ln s + \sum_{n=1}^{\infty} [s^n(a_n \cos n\varphi + \tilde{a}_n \sin n\varphi) + s^{-n}(b_n \cos n\varphi + \tilde{b}_n \sin n\varphi)]$$

As $s \to \infty$ $\phi(s,\varphi) \to -E_0 s \cos\varphi$ so $a_0 = -E_0$, $\tilde{a}_1 = 0$ $b_0 = 0$, and $a_n = \tilde{a}_n = 0$ for $n \geq 2$. Thus,

$$\phi(s,\varphi) \;=\; a_0 - E_0 s \cos\varphi + \sum_{n=1}^{\infty} s^{-n}(b_n \cos n\varphi + \tilde{b}_n \sin n\varphi)$$

Let us take the center of the cylinder as a reference point for the potential, then $\phi = 0$ throughout the cylinder so

$$\phi(a,\varphi) \;=\; 0 \;=\; a_0 - E_0 a \cos\varphi + \sum_{n=1}^{\infty} a^{-n}(b_n \cos n\varphi + \tilde{b}_n \sin n\varphi)$$

Due to the orthogonality of the set $\{\cos n\varphi, \sin n\varphi\}$ we get $b_1 = E_0 a^2$ and all other a's and b's vanish ($a_0 = 0$, $b_n = 0$, and $a_n = 0$ for $n \geq 2$). We obtain

$$\phi(s,\varphi) \;=\; E_0\big(\frac{a^2}{s} - s\big)\cos\phi$$

Problem 4. The potential at the surface of the sphere of radius R is given by

$$\phi(r,\theta,\varphi) \;=\; V_0 \sin^2\theta$$

Find the surface charge density on the sphere. (Assume there are no other charges inside or outside the sphere).

Solution
Asymuthal symmetry $\Rightarrow$

$$\Phi(r,\theta) \;=\; \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) \qquad\qquad r < R$$
$$\Phi(r,\theta) \;=\; \sum_{l=0}^{\infty} B_l R^{-l-1} P_l(\cos\theta) \qquad\qquad r > R$$

At the surface

$$\phi(R,\theta) \;=\; V \sin^2\theta \;=\; \frac{2}{3}V - \frac{2}{3}V P_2(\cos\theta)$$

and therefore

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) \;=\; \sum_{l=0}^{\infty} B_l R^{-l-1} P_l(\cos\theta) \;=\; \frac{2}{3}V - \frac{2}{3}V P_2(\cos\theta)$$

From the orthogonality of Legendre polynomials we get

$$A_0 \;=\; \frac{B_0}{R} \;=\; \frac{2}{3}V, \qquad A_2 \;=\; -\frac{2V}{3R^2}, \quad B_2 \;=\; \frac{2VR^3}{3}$$

so

$$\Phi(r,\theta) \;=\; \frac{2}{3}V - \frac{2V}{3R^2} r^2 P_2(\cos\theta) \qquad\qquad r < R$$
$$\Phi(r,\theta) \;=\; \frac{2VR}{3r} - \frac{2VR^3}{3r^3} P_2(\cos\theta) \qquad\qquad r > R$$

and therefore

$$\sigma(\theta) \;=\; \epsilon_0 \Big\{ \frac{\partial\phi(r,\theta)}{\partial r}\Big|_{r\to R+} - \frac{\partial\phi(r,\theta)}{\partial r}\Big|_{r\to R-} \Big\} \;=\; -\,\epsilon_0 \frac{V}{R}\sin^2\theta$$

Problem 5.

Consider the vector potential $\vec{A}(\vec{r}) \;=\; \vec{r}\times(\vec{r}\times\hat{e}_3)$.

a) Calculate the magnetic field.

b) Does the vector potential satisfy Coulomb gauge condition $\vec{\nabla}\cdot\vec{A}=0$? If not, modify $\vec{A}$ such that the magnetic field remains unchanged but the Coulomb condition is satisfied.

Solution

a)

Using formula $\vec{r}\times(\vec{r}\times\hat{e}_3)=z\vec{r}-r^2\hat{e}_3$ and $\vec{\nabla}\times(f\vec{A})=\vec{\nabla}f\times\vec{A}+f(\vec{\nabla}\times\vec{A})$ we get

$$\begin{aligned}\vec{B} \;&=\; \vec{\nabla}\times\left(z\vec{r}-r^2\hat{e}_3\right) \;=\; \vec{\nabla}z\times\vec{r}+z\vec{\nabla}\times\vec{r}-\vec{\nabla}r^2\times\hat{e}_3 \\ &=\; \vec{\nabla}z\times\vec{r}-\vec{\nabla}r^2\times\hat{e}_3 \;=\; \hat{e}_3\times\vec{r}-2\vec{r}\times\hat{e}_3 \;=\; 3\hat{e}_3\times\vec{r} \;=\; 3(x\hat{e}_2-y\hat{e}_1)\end{aligned}$$

b)

Using $\vec{\nabla}\cdot(f\vec{A}) \;=\; \vec{\nabla}f\cdot\vec{A}+f\vec{\nabla}\cdot\vec{A}$ we obtain

$$\vec{\nabla}\cdot(z\vec{r}-r^2\hat{e}_3) \;=\; \vec{\nabla}z\cdot\vec{r}+z\vec{\nabla}\cdot\vec{r}-\vec{\nabla}r^2\cdot\hat{e}_3 = \; z+3z-2z \;=\; 2z$$

so Coulomb gauge condition is not satisfied. To satisfy it, we should add $\vec{A} \;\to\; \vec{A}'=\vec{A}+\vec{\nabla}\Lambda$ such that $\nabla^2\Lambda \;=\; -2z$. It is easy to guess that $\Lambda=-\frac{z^3}{3}$ does the job, so

$$\vec{A}' -= \; \vec{A}-\vec{\nabla}\frac{z^3}{3} \;=\; \vec{A}-z^2\hat{e}_3$$

Problem 6.

A current I flows down the long cylindrical wire (radius a) made of linear material with susceptibility χ_m. The current is distributed uniformly through the cross section of the wire. Find the magnetic field (inside and outside of the wire) and all bound currents.

Solution

Amperian loop with $r>a$

$$\int\vec{H}\cdot\vec{dl} \;=\; 2\pi H(s) \;=\; I \quad\Rightarrow\quad \vec{H} \;=\; \frac{I}{2\pi s}\hat{e}_\phi \;\Rightarrow\; \vec{B} \;=\; \frac{\mu_0 I}{2\pi s}\hat{e}_\phi$$

Amperian loop with $r<a$

$$\int\vec{H}\cdot\vec{dl} \;=\; 2\pi H(s) \;=\; I_{\rm enc} \;=\; I\frac{s^2}{a^2} \quad\Rightarrow\quad \vec{H} \;=\; \frac{Is}{2\pi a^2}\hat{e}_\phi \;\Rightarrow\; \vec{B} \;=\; \frac{\mu I s}{2\pi a^2}\hat{e}_\phi$$

Surface bound current

$$\vec{K} \;=\; \vec{M}\times\hat{n} \;=\; \chi_m\vec{H}\times\hat{e}_s \;=\; \chi_m\frac{I}{2\pi a}\hat{e}_3$$

Volume bound current

$$\vec{J} \;=\; \vec{\nabla}\times\vec{M} \;=\; \chi_m\vec{\nabla}\times\vec{H} \;=\; \chi_m J_f \;=\; \chi_m\frac{I}{\pi a^2}\hat{e}_3$$

Check: total bound current is

$$K(2\pi a)+J(\pi a^2) \;=\; -\,\chi_m\frac{I}{2\pi a}2\pi a\hat{e}_3+\chi_m\frac{I}{\pi a^2}\pi a^2\hat{e}_3 \;=\; 0$$