HW11 Due Mon Nov 22 at 7 p.m. in my mailbox or by email

Problem

Dielectric material with permittivity ϵ fills the whole space except for the spherical cavity with radius a. A pure dipole p is placed in the center of the cavity. Find the potential inside and outside the cavity.

Solution

First, assume $\vec{\nabla}\cdot\vec{P}=0$ inside the dielectric, to be checked *a posteriori*. We will solve the Laplace equation at $\vec{r}\neq 0$ and check thhat the soluton satisfies $\vec{\nabla}\cdot\vec{P}=0$.

The expansion in Legendre polynomials has the form:

$$\phi(r,\theta) \stackrel{r\leq a}{=} \frac{p\cos\theta}{4\pi\epsilon_0 r^2}+\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)$$
$$\phi(r,\theta) \stackrel{r\geq a}{=} \sum_{l=0}^{\infty}B_l r^{-l-1} P_l(\cos\theta)$$

where the term singular as $r\to 0$ corresponds to the potential of the pure dipole $p\hat{e}_3$. From the form of the dipole "input" and orthogonality of the Legendre polynomials it is easy to see that the only non-vanishing terms correspond to $l=1$:

$$\phi(r,\theta) \stackrel{r\leq a}{=} \Big(\frac{p}{4\pi\epsilon_0 r^2}+Ar\Big)\cos\theta, \qquad \phi(r,\theta) \stackrel{r\geq a}{=} \frac{B}{r^2}\cos\theta$$

The boundary conditions at $r=a$ are

$$E_\theta^{\text{above}}=E_\theta^{\text{below}} \quad\Rightarrow\quad \frac{B}{a^2} = \Big(\frac{p}{4\pi\epsilon_0 a^2}+Aa\Big)$$
$$\epsilon E_r^{\text{above}}=\epsilon_0 E_r^{\text{below}} \quad\Rightarrow\quad -2\frac{B}{a^3}\epsilon = \Big(-\frac{p}{2\pi\epsilon_0 a^3}+A\Big)\epsilon_0$$

The solution of these eqs. is $A=-\frac{p}{2\pi\epsilon_0 a^3}\frac{\epsilon-\epsilon_0}{2\epsilon+\epsilon_0}$ and $B=\frac{p}{4\pi\epsilon_0}\frac{3\epsilon_0}{2\epsilon+\epsilon_0}$ so the potential is

$$\phi(r,\theta) \stackrel{r\leq a}{=} \frac{p}{4\pi\epsilon_0}\Big(\frac{1}{r^2}-\frac{2\epsilon-2\epsilon_0}{2\epsilon+\epsilon_0}\,\frac{r}{a^3}\Big)\cos\theta, \qquad \phi(r,\theta) \stackrel{r\geq a}{=} \frac{p}{4\pi\epsilon_0 r^2}\,\frac{3\epsilon_0}{2\epsilon+\epsilon_0}\,\cos\theta$$

Now since the electric field in the dielectric is proportional to the field of pure dipole at the origin, $\vec{\nabla}\cdot\vec{P}\sim\vec{\nabla}\cdot\vec{P}=0$ which justifies our original assumption.