HW4 solution

a) The Green function is

$$G_D(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} - \frac{1}{|\vec{r}_* - \vec{r}'|}$$

where $\vec{r}_* = (x, y, -z)$ is the position of the image charge. In the explicit form

$$G(\vec{r},\vec{r}') = [(x-x')^2+(y-y')^2+(z-z')^2]^{-1/2} - [(x-x')^2+(y-y')^2+(z+z')^2]^{-1/2}$$

The symmetry $\vec{r} \leftrightarrow \vec{r}'$ and the boundary condition $G(\vec{r},\vec{r}')|_{z=0} = 0$ are evident.

b) From the Eq. (2.7) we get

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x'\ G_D(\vec{r},\vec{r}')\rho(\vec{r}') - \frac{1}{4\pi}\int dx'dy'\ \ \phi(\vec{r}')\frac{\partial G_D(\vec{r},\vec{r}')}{\partial n'}\bigg|_{z'=0}$$

In our case $\rho = 0$ and

$$\frac{\partial}{\partial n'}G_D(\vec{r},\vec{r}')\bigg|_{z'=0} = -\frac{\partial}{\partial z'}G_D(\vec{r},\vec{r}')\bigg|_{z'=0} = \frac{-2z}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}$$

so one obtains

$$\begin{aligned}
&\phi(s,\varphi,z)\\
&= \frac{zV}{2\pi}\int_0^a s'ds'\int_0^{2\pi} d\varphi'\ \frac{1}{[s^2+s'^2+z^2-2ss'\cos(\varphi-\varphi')]^{3/2}}\\
&= \frac{zV}{2\pi}\int_0^a s'ds'\int_0^{2\pi} d\varphi'\ \frac{1}{[s^2+s'^2+z^2-2ss'\cos\varphi']^{3/2}}
\end{aligned}$$

c) If $s = 0$ the above eqn. reduces to

$$\phi(s,\varphi,z) = zV\int_0^a s'ds'\ \frac{1}{(s'^2+z^2)^{3/2}} = V\Big(1-\frac{z}{\sqrt{z^2+a^2}}\Big) = \frac{Va^2}{2z^2}\Big(1-\frac{3a^2}{4z^2}+\frac{5a^4}{8z^4}\Big)$$

d) At $s^2+z^2 \gg s'^2$

$$\begin{aligned}
\phi(s,\varphi,z) &= \frac{zV}{2\pi}\int_0^a s'ds'\int_0^{2\pi} d\varphi'\ \frac{1}{(s^2+z^2)^{3/2}}\\
&\times\left[1+\frac{3ss'\cos\varphi'-\frac{3}{2}s'^2}{s^2+z^2}+\frac{15}{2}\frac{[ss'\cos\varphi'-\frac{s'^2}{2}]^2}{(s^2+z^2)^2}+...\right]\\
&= \frac{zV}{(s^2+z^2)^{3/2}}\int_0^a s'ds'\left[1-\frac{3s'^2}{2(s^2+z^2)}+\frac{15(2s^2s'^2+s'^4)}{8(s^2+z^2)^2}+...\right]\\
&= \frac{Vza^2}{2(s^2+z^2)^{3/2}}\left[1-\frac{3a^2}{4(s^2+z^2)}+\frac{15a^2s^2+5a^4}{8(s^2+z^2)^2}+...\right]
\end{aligned}$$